Auto-Bäcklund transformations and exact solutions for some nonlinear partial differential equations with nonlinear terms of any order

BIAO LI *), YONG CHEN, HONGQING ZHANG

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
and
Key Laboratory of Mathematics Mechanization, Chinese Acad. Sci., Beijing 100080, China

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In this paper, by introducing some proper transformations, the applied range of the homogenous balance (HB) method is extended. With the help of *Mathematica*, we obtain three auto-Bäcklund transformations (BT) for the generalized Fithugh–Nagumo equation, the generalized Burgers–Fisher equation, the generalized Burgers–Huxley equation, respectively, by use of the extended HB method. From these BTs, some exact solutions for these equations are derived.

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1 Introduction

The Bäcklund transformations (BT) of nonlinear partial differential equations (PDEs) play an important role in soliton theory, which is an efficient method to obtain exact solutions of nonlinear PDEs [1,2]. In order to obtain the BT of the given nonlinear PDE, various methods, such as Painlevé method [3], Hirota method [4], homogenous balance (HB) method [5–12], have been presented. The HB method, which is a primary and concise method to seek exact solutions of nonlinear PDEs [5–9], is extended to search for Bäcklund transformation and similarity reductions of nonlinear PDEs by Fan [9,10]. So, more solutions can be obtained by the improved HB method. But they only consider the cases where the balance constants are positive integers. In this paper, by introducing some proper transformations, we can deal with the cases where the balance constants are fractions.

Now we briefly describe the HB method, for a given nonlinear PDE, say, in two variables,

$$H(u, u_x, u_t, u_{xx}, \cdots) = 0. \tag{1.1}$$

We seek the BT of (1.1) in the form

$$u = \partial_x^m \partial_t^n f[w(x,t)] + \widetilde{u}, \tag{1.2}$$

*) E-mail: libiao@student.dlut.edu.cn

where $w(x,t)$, $u = u(x,t)$ are undetermined functions, $\widetilde{u} = \widetilde{u}(x,t)$ is a seed solution of the given PDE and m, n are positive integers determined by balancing the highest derivative term with the nonlinear terms in (1.1) (see Refs. [9,10] for detail). However, we find that the constants m, n need not be restricted to positive integers. In order to apply the HB method to obtain a BT for given PDE when m, n are not equal to positive integers, we must search for some proper transformations. In this paper, we succeed in obtaining three auto-Bäcklund transformations for three nonlinear PDEs with nonlinear terms of any order by introducing three proper transformations. Then, based on these BTs, some exact solutions for these equations are obtained.

The rest of the paper is organized as follows. In Sections 2–4, the BT and some exact solutions of the generalized Fithugh–Nagumo equation, the generalized Burgers–Fisher equation and the generalized Burgers–Huxley equation, respectively, are found. Conclusions will be presented finally.

2 The generalized Fithugh–Nagumo equation

2.1 Auto-Bäcklund transformation

The generalized Fithugh–Nagumo equation [13,14] reads

$$u_t - \alpha u_{xx} = \beta u(1 - u^\delta)(u^\delta - r), \tag{2.1}$$

where α, β, $\delta \geq 0$ and $r \in [-1,1)$. Equation (2.1) is a generation of the Huxley equation for nerve propagation in neurophysics and wall propagation in liquid crystals.

According to the idea of HB method [5–12], by balancing u_{xx} with $u^{2\delta+1}$ in (2.1), we obtain balance constants $m = 1/\delta$, $n = 0$. It is obvious that m may be arbitrary constants. In order to apply the HB method under this condition, we make the transformation

$$u(x,t) = v^{1/\delta}(x,t), \tag{2.2}$$

and substituting it into (2.1) we obtain

$$r\beta\delta^2 v^2 - (1+r)\beta\delta^2 v^3 + \beta\delta^2 v^4 + \alpha(-1+\delta)v_x^2 + \delta v(v_t - \alpha v_{xx}) = 0\,. \tag{2.3}$$

Then, by balancing v^4 with vv_{xx} in (2.3), we get the value of the balance constants $m = 1$, $n = 0$. Therefore we seek the Bäcklund transformation of (2.3) in the form

$$v = f' w_x + \phi. \tag{2.4}$$

Here and in the following context $' := \partial/\partial w$, $f^{(r)} = \partial^r/\partial w^r$, and $f = f(w)$, $w = w(x,t)$ is undetermined function and $\phi = \phi(x,t)$ is a seed solution of (2.3).

With the help of Mathematica, substituting (2.4) into (2.3) yields (because the formula is very long, just one part of it is shown here)

$$[\beta\delta^2 f'^4 + \alpha(-1+\delta)f''^2 - \alpha\delta f' f^{(3)}]w_x^4 + \cdots = 0. \tag{2.5}$$

To simplify (2.5), setting the coefficient of w_x^4 to zero yields an ordinary differential equation for f:

$$\beta\delta^2 f'^4 + \alpha(-1+\delta)f''^2 - \alpha\delta f' f^{(3)} = 0. \tag{2.6}$$

Solving (2.6) we obtain a solution

$$f = \pm\sqrt{\frac{\alpha(1+\delta)}{\beta\delta^2}}\ln w. \tag{2.7}$$

Setting $A = \pm\sqrt{\alpha(1+\delta)/(\beta\delta^2)}$ [1]), then substituting (2.7) into (2.5), formula (2.5) can be simplified to a linear polynomial of $1/w$, then setting the coefficients of $1/w^i$ $(i = 0, \cdots, 3)$ to zero yields a set of partial differential equations for $w(x,t)$:

$$r\beta\delta^2\phi^2 - (1+r)\beta\delta^2\phi^3 + \beta\delta^2\phi^4 + \alpha(-1+\delta)\phi_x^2 + \delta\phi(\phi_t - \alpha\phi_{xx}) = 0, \tag{2.8}$$

$$\begin{aligned}-3(1+r)\beta\delta^2\phi^2 w_x + 4\beta\delta^2\phi^3 w_x + \delta\phi(2r\beta\delta w_x + w_{xt} - \alpha w_{xxx})\\ + \delta w_x\phi_t + \alpha(2(-1+\delta)w_{xx}\phi_x - \delta w_x\phi_{xx}) = 0,\end{aligned} \tag{2.9}$$

$$\begin{aligned}6A\beta\delta^2\phi^2 w_x^2 + A\alpha(-1+\delta)w_{xx}^2 - \delta\phi w_x\{w_t + 3[A(1+r)\beta\delta w_x - \alpha w_{xx}]\}\\ + A\delta w_x(w_{xt} - \alpha w_{xxx}) + w_x^2[Ar\beta\delta^2 - 2\alpha(-1+\delta)\phi_x] = 0,\end{aligned} \tag{2.10}$$

$$w_x^2\{A\delta w_t + \alpha[(1+r)(1+\delta) - 2(2+\delta)\phi]w_x - A\alpha(2+\delta)w_{xx}\} = 0. \tag{2.11}$$

Therefore, from (2.2), (2.4) and (2.7), we obtain the desired auto-Bäcklund transformation of (2.1)

$$u = \left[\pm\sqrt{\frac{\alpha(1+\delta)}{\beta\delta^2}}\frac{w_x}{w} + \phi\right]^{1/\delta}, \tag{2.12}$$

where w satisfies (2.9)–(2.11), ϕ is a seed solution of (2.3) (or (2,8)).

2.2 Exact solutions

Now we use the Bäcklund transformation consisting of (2.12) and (2.8)–(2.11) to exploit some explicit exact solutions for (2.1). To solve Eqs. (2.8)–(2.11), we assume that $\phi(x,t)$ and $w(x,t)$ are of the form

$$\phi(x,t) = B, \quad w(x,t) = C + H\exp[k(x - \lambda t)], \tag{2.13}$$

where B, $C \neq 0$, $H \neq 0$, k and λ are constants to be determined.

Substituting (2.13) into (2.8)–(2.11), we find that (2.13) satisfies Eqs. (2.8)–(2.11) under the following cases:

Case 1.

$$B = 0, \quad k = \frac{1}{A}, \quad \lambda = \frac{A\beta\delta(-1+r+r\delta)}{1+\delta}. \tag{2.14}$$

[1]) Note: in the rest of Section 2, A denotes $\pm\sqrt{\alpha(1+\delta)/\beta\delta^2}$.

Case 2.

$$B=0,\quad k=\frac{r}{A},\quad \lambda=\frac{A\beta\delta(1-r+\delta)}{1+\delta}. \tag{2.15}$$

Case 3.

$$B=1,\quad k=-\frac{1}{A},\quad \lambda=\frac{A\beta\delta(-1+r+r\delta)}{1+\delta}. \tag{2.16}$$

Case 4.

$$B=r,\quad k=-\frac{r}{A},\quad \lambda=\frac{A\beta\delta(1-r+\delta)}{1+\delta}. \tag{2.17}$$

Therefore, from (2.12)–(2.17) we obtain the following solutions of the generalized Fithugh–Nagumo equation (2.1):

Case 1.

$$u_1=\left\{\frac{1}{1+\frac{C}{H}\exp\left[-\frac{1}{A}\left(x-\frac{A\beta\delta(-1+r+r\delta)}{1+\delta}t\right)\right]}\right\}^{1/\delta}, \tag{2.18}$$

Case 2.

$$u_2=\left\{\frac{r}{1+\frac{C}{H}\exp\left[-\frac{r}{A}\left(x-\frac{A\beta\delta(-1+r+r\delta)}{1+\delta}t\right)\right]}\right\}^{1/\delta}, \tag{2.19}$$

Case 3.

$$u_3=\left\{\frac{-1}{1+\frac{C}{H}\exp\left[\frac{1}{A}\left(x-\frac{A\beta\delta(-1+r+r\delta)}{1+\delta}t\right)\right]}+1\right\}^{1/\delta}, \tag{2.20}$$

Case 4.

$$u_4=\left\{\frac{-r}{1+\frac{C}{H}\exp\left[\frac{r}{A}\left(x-\frac{A\beta\delta(1-r+\delta)}{1+\delta}t\right)\right]}+r\right\}^{1/\delta}, \tag{2.21}$$

If we set $C=H$ and $C=-H$ in (2.18)–(2.21), respectively, we can obtain the kink-profile solitary wave solutions and blow-up solitary wave solutions for (2.1):

Family 1. From (2.18) and (2.20), we obtain the following solutions of (2.1):

$$u_{11}=\left\{\frac{1}{2}\pm\frac{1}{2}\tanh\left[\pm\frac{1}{2A}\left(x-\frac{A\beta\delta(-1+r+r\delta)}{1+\delta}t\right)\right]\right\}^{1/\delta}, \tag{2.22}$$

$$u_{12}=\left\{\frac{1}{2}\pm\frac{1}{2}\coth\left[\pm\frac{1}{2A}\left(x-\frac{A\beta\delta(-1+r+r\delta)}{1+\delta}t\right)\right]\right\}^{1/\delta}. \tag{2.23}$$

Family 2. From (2.19) and (2.21), we obtain the following solutions of (2.1):

$$u_{21} = \left\{ \frac{r}{2} \left[1 \pm \tanh \left[\pm \frac{r}{2A} \left(x - \frac{A\beta\delta(1-r+\delta)}{1+\delta} t \right) \right] \right] \right\}^{1/\delta}, \tag{2.24}$$

$$u_{22} = \left\{ \frac{r}{2} \left[1 \pm \coth \left[\pm \frac{r}{2A} \left(x - \frac{A\beta\delta(1-r+\delta)}{1+\delta} t \right) \right] \right] \right\}^{1/\delta}. \tag{2.25}$$

Remark 1:

a) From the solutions (2.22)–(2.25), the solutions obtained in [13,14] can be recovered. Therefore, the solutions obtained in [13,14] are special cases of our solutions (2.18)–(2.21).

b) When we set $\phi(x,t)$ and $w(x,t)$ to be following more general forms:

$$\phi(x,t) = B, \quad w(x,t) = C(t) + H(t) \exp[xp(t) - q(t)], \tag{2.26}$$

where B is an arbitrary constant, $C(t), H(t), p(t)$ and $q(t)$ are arbitrary functions of t, we only obtain the same results as the results obtained in the paper. When we set $\phi(x,t)$ and $w(x,t)$ to be more complex forms than Eq. (2.13), we do not obtain anything besides a system of complex constraint PDEs. In order to make the work feasible, the way how to choose the forms of $\phi(x,t)$ and $w(x,t)$ needs to be further studied.

c) We have a try to use the solutions (2.18)–(2.21) as a new seed solution of (2.3), but we do not obtain any explicit exact solutions of (2.3) besides the complex constraint PDEs.

3 The generalized Burgers–Fisher equation

3.1 Auto-Bäcklund transformation

The generalized Burgers–Fisher equation [13, 15–17] reads:

$$u_t + \alpha u^\delta u_x - \frac{m}{u} u_x^2 - u_{xx} = \beta u(1-u^\delta), \quad (\delta > 0). \tag{3.1}$$

Equation (3.1) includes the following two generalized Fisher equations [15]:

$$u_t - u_{xx} - \frac{m}{u} u_x^2 = u(1-u^\delta), \quad -1 - \frac{1}{2}\delta < m < \infty, \quad 0 < \delta < \infty, \tag{3.2}$$

$$u_t + \alpha u^\delta u_x - u_{xx} = \beta u(1-u^\delta). \tag{3.3}$$

If setting $m = 0$, $\delta = 1$, Eq. (3.2) will become the well-known Fisher equation, which shows a simple model for describing the interaction between reaction mechanism and diffusion transport in corresponding physical and biological systems.

Proceeding as in Section 2, we obtain an auto-Bäcklund transformation of (3.1) as follows. For simplicity, we omit the detailed steps in this section.

$$u = \left[-\frac{1+m+\delta}{\alpha\delta} \frac{w_x}{w} + \phi \right]^{1/\delta}, \tag{3.4}$$

where $w = w(x,t)$ satisfies the following equations (3.5)–(3.8); $\phi = \phi(x,t)$ is a solution of (3.5).

$$\beta\delta^2\phi^3 + (-1 - m + \delta)\phi_x^2 + \delta\phi^2(-\beta\delta + \alpha\phi_x) + \delta\phi(\phi_t - \phi_{xx}) = 0, \tag{3.5}$$

$$\begin{aligned} &- \delta\phi^2(3\beta\delta w_x + \alpha w_{xx}) - \delta w_x\phi_t + 2w_{xx}\phi_x + 2mw_{xx}\phi_x - 2\delta w_{xx}\phi_x \\ &+ \delta\phi(-w_{xt} + w_{xxx} + 2w_x(\beta\delta - \alpha\phi_x)) + \delta w_x\phi_{xx} = 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &- \alpha^2\delta^2\phi^2 w_x^2 + (1 + 2m + m^2 - \delta^2)w_{xx}^2 \\ &- \delta\phi w_x(\alpha\delta w_t + 3\beta\delta(1 + m + \delta)w_x + \alpha(2 + 2m - \delta)w_{xx}) \\ &- \delta(1 + m + \delta)w_x(w_{xt} - w_{xxx}) + \delta w_x^2(\beta\delta(1 + m + \delta) + \alpha(1 + m - 3\delta)\phi_x) = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &w_x^2(\alpha\delta(1 + m + \delta)w_t + \delta(\beta(1 + m + \delta)^2 \\ &+ 2(1 + m)\alpha^2\phi)w_x - (1 + m)\alpha(1 + m + \delta)w_{xx}) = 0. \end{aligned} \tag{3.8}$$

3.2 Exact solutions

Now we use the Bäcklund transformation consisting of (3.4) and (3.5)–(3.8) to exploit some explicit exact solutions for (3.1). To solve Eqs. (3.5)–(3.8), we assume that $\phi(x,t)$ and $w(x,t)$ are of the form

$$\phi(x,t) = B, \quad w(x,t) = C + H\exp[k(x - \lambda t)], \tag{3.9}$$

where B, $C \neq 0$, $H \neq 0$, k and λ are constants to be determined.

Substituting (3.9) into (3.5)–(3.8), we find that (3.9) satisfies Eqs. (3.5)–(3.8) under the following two cases:

Case 1.

$$B = 0, \quad k = -\frac{\alpha\delta}{1 + m + \delta}, \quad \lambda = \frac{(1 + m)\alpha^2 + \beta(1 + m + \delta)^2}{\alpha(1 + m + \delta)}. \tag{3.10}$$

Case 2.

$$B = 1, \quad k = \frac{\alpha\delta}{1 + m + \delta}, \quad \lambda = \frac{(1 + m)\alpha^2 + \beta(1 + m + \delta)^2}{\alpha(1 + m + \delta)}. \tag{3.11}$$

Therefore, from (3.4), (3.9)–(3.11), we obtain two solutions of Eq. (3.1):

$$u_1 = \left\{ \frac{1}{1 + \frac{C}{H}\exp\left[\frac{\alpha\delta}{1 + m + \delta}\left(x - \frac{(1 + m)\alpha^2 + \beta(1 + m + \delta)^2}{\alpha(1 + m + \delta)}t\right)\right]} \right\}^{1/\delta}, \tag{3.12}$$

$$u_2 = \left\{ \frac{-1}{1 + \frac{C}{H}\exp\left[-\frac{\alpha\delta}{1 + m + \delta}\left(x - \frac{(1 + m)\alpha^2 + \beta(1 + m + \delta)^2}{\alpha(1 + m + \delta)}t\right)\right]} + 1 \right\}^{1/\delta}, \tag{3.13}$$

where $C \neq 0$, $H \neq 0$ are arbitrary constants.

If we set $C = H$ and $C = -H$ in (3.12) and (3.13), respectively, we obtain the following exact solitary wave solutions and blow-up solitary wave solutions of (3.1):

$$u_{1,2} = \left\{\frac{1}{2} \pm \frac{1}{2}\tanh\left[\mp\frac{\alpha\delta}{2(1+m+\delta)}\left(x - \frac{(1+m)\alpha^2 + \beta(1+m+\delta)^2}{\alpha(1+m+\delta)}t\right)\right]\right\}^{1/\delta}, \tag{3.14}$$

$$u_{3,4} = \left\{\frac{1}{2} \pm \frac{1}{2}\coth\left[\mp\frac{\alpha\delta}{2(1+m+\delta)}\left(x - \frac{(1+m)\alpha^2 + \beta(1+m+\delta)^2}{\alpha(1+m+\delta)}t\right)\right]\right\}^{1/\delta}. \tag{3.15}$$

Remark 2: It is easy to see that (3.14) and (3.15) reproduce the solutions obtained in [13,15–17]. Therefore the solutions of (3.1) obtained in [13,15–17] are special cases of our solutions (3.12) and (3.13).

4 The generalized Burgers–Huxley equation

4.1 Auto-Bäcklund transformation

The generalized Burgers–Huxley equation [13,17] reads:

$$u_t + au^\delta u_x - \frac{m}{u}u_x^2 - Du_{xx} = \beta u(1-u^\delta)(u^\delta - r). \tag{4.1}$$

According to the steps in Section 2, we obtain an auto-Bäcklund transformation of (4.1) as follows:

$$u = \left[\frac{a \pm \sqrt{a^2 + 4\beta(D+m+D\delta)}}{2\beta\delta}\frac{w_x}{w} + \phi\right]^{1/\delta}, \tag{4.2}$$

where $w = w(x,t)$ satisfies the following equations (4.4)–(4.6); $\phi = \phi(x,t)$ is a solution of (4.3): [2])

$$\delta\phi\phi_t + a\delta\phi^2\phi_x + (-D-m+D\delta)\phi_x^2 + \delta\phi\left(-\beta\delta(r-\phi)(-1+\phi)\phi - D\phi_{xx}\right) = 0, \tag{4.3}$$

$$\begin{aligned}&\delta\phi w_{xt} + a\delta\phi^2 w_{xx} - D\delta\phi w_{xxx} - 2Dw_{xx}\phi_x - 2mw_{xx}\phi_x + 2D\delta w_{xx}\phi_x\\ &+ \delta w_x(2r\beta\delta\phi - 3\beta\delta\phi^2 - 3r\beta\delta\phi^2 + 4\beta\delta\phi^3 + \phi_t + 2a\phi\phi_x - D\phi_{xx}) = 0,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&-\delta\phi w_t w_x + A(-D-m+D\delta)w_{xx}^2 + \delta w_x[Aw_{xt} + (2aA+3D)\phi w_{xx} - ADw_{xxx}]\\ &+ w_x^2\{\delta[-a\phi^2 + A\beta\delta(r - 3\phi - 3r\phi + 6\phi^2)] + [2m - 2D(-1+\delta) + aA\delta]\phi_x\} = 0,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&w_x^2\{-A\delta w_t - \delta[A^2\beta\delta(1+r-4\phi) + 2aA\phi + 2D\phi]w_x\\ &+ A[2m + aA\delta + D(2+\delta)]w_{xx}\} = 0,\end{aligned} \tag{4.6}$$

[2]) Note: in the rest of Section 4, $A = \left[a \pm \sqrt{a^2 + 4\beta(D+m+D\delta)}\right]/(2\beta\delta)$.

4.2 Exact solutions

Now we use the Bäcklund transformation consisting of (4.2) and (4.4)–(4.6) to search for some explicit exact solutions for (4.1). To solve Eqs. (4.3)–(4.6), we assume that $\phi(x,t)$ and $w(x,t)$ are of the form

$$\phi(x,t) = B, \quad w(x,t) = C + H\exp[k(x-\lambda t)], \tag{4.7}$$

where B, $C \neq 0$, $H \neq 0$, k and λ are constants to be determined.

Substituting (4.7) into (4.3)–(4.6), we find that (4.7) satisfies Eqs. (4.3)–(4.6) under the following cases

Case 1.

$$B = 0, \quad \lambda = \frac{A^2 r\beta\delta^2 - (D+m))}{A\delta}, \quad k = \frac{1}{A}, \tag{4.8}$$

Case 2.

$$B = 0, \quad \lambda = \frac{A^2\beta\delta^2 - (D+m)r}{A\delta}, \quad k = \frac{r}{A}, \tag{4.9}$$

Case 3.

$$B = 1, \quad \lambda = a + \frac{D + A^2(-1+r)\beta\delta}{A}, \quad k = -\frac{1}{A}, \tag{4.10}$$

Case 4.

$$B = r, \quad \lambda = ar + \frac{Dr - A^2(-1+r)\beta\delta}{A}, \quad k = -\frac{r}{A}, \tag{4.11}$$

Therefore, from (4.2), (4.7)–(4.11), we obtain four solutions of the equation (4.1):

$$u_{1,2} = \left\{ \frac{Ak}{1 + \frac{C}{H}\exp[-k(x-\lambda t)]} \right\}^{1/\delta}, \tag{4.12}$$

$$u_3 = \left\{ \frac{Ak}{1 + \frac{C}{H}\exp[-k(x-\lambda t)]} + 1 \right\}^{1/\delta}, \tag{4.13}$$

$$u_4 = \left\{ \frac{Ak}{1 + \frac{C}{H}\exp[-k(x-\lambda t)]} + r \right\}^{1/\delta}, \tag{4.14}$$

where k, λ in (4.12), (4.13), and (4.14) are determined by (4.8)–(4.9), (4.10), and (4.11), respectively, and C, H are arbitrary constants.

If we set $C = H$ and $C = -H$ in (4.12)–(4.14), respectively, we obtain the following exact solitary wave solutions and blow-up solitary wave solutions of the equation (4.1):

$$u_{11} = \left\{ \frac{1}{2} - \frac{1}{2}\tanh\left[-\frac{1}{2A}\left(x - \frac{r\beta\delta^2 A^2 - (D+M)}{A\delta} t \right) \right] \right\}^{1/\delta}, \tag{4.15}$$

$$u_{12} = \left\{\frac{1}{2} - \frac{1}{2}\coth\left[-\frac{1}{2A}\left(x - \frac{r\beta\delta^2 A^2 - (D+M)}{A\delta}t\right)\right]\right\}^{1/\delta}, \tag{4.16}$$

$$u_{21} = \left\{\frac{r}{2} - \frac{r}{2}\tanh\left[-\frac{r}{2A}\left(x - \frac{\beta\delta^2 A^2 - r(D+M)}{A\delta}t\right)\right]\right\}^{1/\delta}, \tag{4.17}$$

$$u_{22} = \left\{\frac{r}{2} - \frac{r}{2}\coth\left[-\frac{r}{2A}\left(x - \frac{\beta\delta^2 A^2 - r(D+M)}{A\delta}t\right)\right]\right\}^{1/\delta}, \tag{4.18}$$

$$u_{31} = \left\{\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{1}{2A}\left(x - \frac{D + aA + (-1+r)\beta\delta A^2}{A}t\right)\right]\right\}^{1/\delta}, \tag{4.19}$$

$$u_{32} = \left\{\frac{1}{2} + \frac{1}{2}\coth\left[\frac{1}{2A}\left(x - \frac{D + aA + (-1+r)\beta\delta A^2}{A}t\right)\right]\right\}^{1/\delta}, \tag{4.20}$$

$$u_{41} = \left\{\frac{r}{2} + \frac{r}{2}\tanh\left[\frac{r}{2A}\left(x - \frac{Dr + arA + (1-r)\beta\delta A^2}{A}t\right)\right]\right\}^{1/\delta}, \tag{4.23}$$

$$u_{42} = \left\{\frac{r}{2} + \frac{r}{2}\coth\left[\frac{r}{2A}\left(x - \frac{Dr + arA + (1-r)\beta\delta A^2}{A}t\right)\right]\right\}^{1/\delta}, \tag{4.24}$$

Remark 3: The solutions (4.15)–(4.18) cover the solutions obtained in [13,17]. Therefore the solutions of (4.1) obtained in [13,17] are special cases of our solutions (4.12). To our knowledge, the other solutions obtained here were not found before.

5 Conclusions

In this paper, by introducing some proper transformations, we derive three auto-Bäcklund transformations for three nonlinear PDEs with nonlinear terms of any order by use of the extended HB method. Based on these BTs, several families of exact solutions for equations (2.1), (3.1), and (4.1) are obtained. These special solutions presented in this work can be effectively used to discuss the corresponding phenomena and related problems in physics and biology. This method is computerizable, which allow us to perform complicated and tedious symbolic algebraic calculation on a computer. But in this paper we only consider (1+1)-dimensional nonlinear PDEs and one of the two balance constants is equal to zero by use of the extended HB method. Can we find a Bäcklund transformation for (2+1)-dimensional or (3+1)-dimensional nonlinear PDEs with nonlinear terms of any order using this method? When the balance constants are all fractions, what transformations shall we seek in order that the extended HB method be also effective? These problems will be further studied in the following works.

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References

[1] M.J. Ablowitz and P.A. Clarkson: *Soliton, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, New York, 1991.

[2] Gu CH et al.: *Soliton theory and its application*, Zhejiang Science and Technology Press, Zhejiang, 1990.

[3] J. Weiss, M. Tabor, and G. Carnevale: J. Math. Phys. **24** (1983) 522.

[4] R. Hirota: Phys. Rev. Lett. **27** (1971) 1192.

[5] M.L. Wang, Y.B. Zhou, and Z.B. Li: Phys. Lett. A **216** (1996) 67.

[6] M.L. Wang, Y.B. Zhou, and Z.B. Li: Phys. Lett. A **216** (1996) 67.

[7] M.L. Wang and Y.M. Wang: Phys. Lett. A **287** (2001) 211.

[8] E.G. Fan and H.Q. Zhang: Phys. Lett. A **245** (1998) 389.

[9] E.G. Fan and H.Q. Zhang: Phys. Lett. A **246** (1998) 403.

[10] E.G. Fan: Phys. Lett. A **265** (2000) 353.

[11] E.G. Fan: Phys. Lett. A **294** (2002) 26.

[12] H. Zhang, G.M. Wei, and Y.T. Gao: Czech. J. Phys. **52** (2002) 373.

[13] X. Feng: Int. J. Theor. Phys. **39** (2000) 207.

[14] Z.J. Yang: J. Phys. A **27** (1994) 2837.

[15] X.Y. Wang: Phys. Lett. A **131** (1988) 277.

[16] X.Y. Wang, Z.S. Chu, and Y.K. Lu: J. Phys. A **23** (1990) 271.

[17] B.L. Lu, B.Z. Liu, Z.L. Pang, and X.F. Jiang: Phys. Lett. A **175** (1993) 113.