Iterative algorithm for phase extraction from interferograms with random and spatially nonuniform phase shifts

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An advanced iterative algorithm is presented to extract phase distribution from randomly and spatially nonuniform phase-shifted interferograms. The proposed algorithm divides the interferograms into small blocks and retrieves local phase shifts accurately by iterations. Therefore, the phase distribution can be calculated with high precision by eliminating the effect of tilts occurring during phase shifting. Simulated results and experiments demonstrate that the proposed algorithm exhibits high precision and converges faster than previous algorithms even when the tilt errors are up to 27.6% of the normal phase step. © 2008 Optical Society of America

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1. Introduction

Among automated interferogram analysis methods, the temporal phase-shifting (TPS) technique is recognized as the one providing the most accurate wavefront extraction [1,2]. The measurement accuracy of the TPS depends on practical limitations, such as the phase shifter performance and measurement environment instabilities. Therefore, numerous algorithms have been proposed for dealing with phase-retrieving errors. In 1991, Okada *et al.* [3] proposed a least-squares-based iterative algorithm that solves the approximate linear equations iteratively to determine phase-shift amounts and phase distribution simultaneously. Wang and Han [4] proposed an improved iterative algorithm for phase extraction of randomly phase-shifted interferograms that does not require an accurate initial estimation of phase shifts.

All the iterative methods cited above assume that every pixel in an interferogram must have the same amount of shifted phase. However, due to an unbalanced piezoelectric effect in the phase shifter or instability of the optical platform, for example, some phase shifters may introduce a significant tilt. As a consequence the phase-shift steps are no longer the same at all points but vary with a linear function across the field defined as a phase-shift plane.

To eliminate the tilt-shift errors, Hibino *et al.* [5] developed a systematic method by which compensation algorithms can be generated if the phase shifts obey a polynomial behavior. Such algorithms are unable to compensate for the effects of tilts if the angles and orientations of those tilts have random values or do not follow a polynomial rule. Chen *et al.* [6] proposed an algorithm that is immune to both transitional and tilt-shift errors. Its operation is based on an iterative alternate adjustment of phase distribution and local phase shifts by a first-order Taylor series expansion of the phase-shift errors (including both translational and tilt-shift errors). Dobroiu *et al.* [7] proposed an algorithm to globally adjust the phase-shift planes and compensate both translational and tilt-shift errors by using calculated contrast maps. Its operation is based on blockwise processing of interferograms divided into small regions with uniform phase steps. However, the supposed phase shifts should be known to calculate the contrast maps.

In this paper, an improved iterative algorithm is proposed to cope with random and spatially nonuniform phase shifts. First we ignore tilt errors and es-

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timate the phase using the algorithm of Wang and Han, and then divide the interferograms into small blocks and retrieve local phase shifts. Finally, the phase distributions and phase-shift plane with tilt information are determined and updated after iterations. The proposed algorithm needs only four randomly phase-shifted (including both translational and tilt-shift errors) interferograms and gives an accurate phase map in which the effects of translational and tilt-shift errors are both canceled. We first discuss the principles of the algorithm and then give its verification by computer simulations and experiments.

2. Principle

A. Iterative Algorithm Determination of the Phase Distribution

If the piezoelectric transducer (PZT) device of the test optical element in the interferometer has orientation errors during the shift, the test surface of the element will be tilted. In this case the intensity at pixel (x, y) of the nth interferogram can be represented as

$$I_n^t(x, y) = A(x, y) + B(x, y)\cos[\Phi(x, y) + k_{xn}x + k_{yn}y + d_n], \tag{1}$$

where I is the intensity of the interferogram, the superscript t denotes the theoretical value, and x and y denote spatial coordinates in the interferogram. In the equation, $A(x, y)$ is the background or mean intensity; $B(x, y)$ is the modulation of the fringe pattern; $\Phi(x, y)$ is the phase distribution under test; k_{xn}, k_{yn}, and d_n denote the gradients of the phase-shift plane along the x and y directions and the phase shift value of the center pixel in the nth phase-shifted interferogram ($n = 1, 2, \ldots, N$), respectively; and N is the total number of frames.

As in the conventional phase-shifting algorithm, it is assumed that the background intensity and the modulation amplitude do not have frame-to-frame variation; i.e., they are only functions of pixels. Under the assumption, we define a new set of variables as $a(x, y) = A(x, y)$, $b(x, y) = B(x, y)\cos[\Phi(x, y)]$, and $c(x, y) = -B(x, y)\sin[\Phi(x, y)]$, and Eq. (1) is rewritten as

$$I_n^t(x, y) = a(x, y) + b(x, y)\cos(k_{xn}x + k_{yn}y + d_n) + c(x, y)\sin(k_{xn}x + k_{yn}y + d_n). \tag{2}$$

For the pixel (x, y), if k_{xn}, k_{yn}, and d_n are known, there are three unknowns and N equations. The unknowns can be solved by use of the overdetermined least-squares method if $N \geq 3$. The least-squares error between theoretical and experimental interferogram $S(x, y)$, which is accumulated from all the images described by Eq. (2), can be written as

$$S(x, y) = \sum_{n=1}^{N} \left[I_n^e(x, y) - I_n^t(x, y)\right]^2, \tag{3}$$

where $I_n^e(x, y)$ is the experimentally measured intensity of the interferogram. The least-squares criteria required for three unknowns $[a(x, y), b(x, y)$, and $c(x, y)]$ can be expressed as

$$\frac{\partial S(x, y)}{\partial a(x, y)} = 0, \quad \frac{\partial S(x, y)}{\partial b(x, y)} = 0, \quad \frac{\partial S(x, y)}{\partial c(x, y)} = 0. \tag{4}$$

Defining random and spatially nonuniform phase shifts as $\delta_n = k_{xn}x + k_{yn}y + d_n$ (for notation brevity their dependence on spatial coordinates x, y has been omitted), from Eq. (4) we can obtain that

$$\begin{bmatrix} a(x, y) \\ b(x, y) \\ c(x, y) \end{bmatrix} = \begin{bmatrix} N & \sum_{n=1}^{N} \cos\delta_n & \sum_{n=1}^{N} \sin\delta_n \\ \sum_{n=1}^{N} \cos\delta_n & \sum_{n=1}^{N} \cos^2\delta_n & \sum_{n=1}^{N} \cos\delta_n \sin\delta_n \\ \sum_{n=1}^{N} \sin\delta_n & \sum_{n=1}^{N} \sin\delta_n \cos\delta_n & \sum_{n=1}^{N} \sin^2\delta_n \end{bmatrix}^{-1} \times \begin{bmatrix} \sum_{n=1}^{N} I_n^e(x, y) \\ \sum_{n=1}^{N} I_n^e(x, y)\cos\delta_n \\ \sum_{n=1}^{N} I_n^e(x, y)\sin\delta_n \end{bmatrix}. \tag{5}$$

From Eq. (5), the unknowns $a(x, y)$, $b(x, y)$, and $c(x, y)$ can be solved. Then the phase distribution can be determined as

$$\Phi(x, y) = \tan^{-1}[-c(x, y)/b(x, y)]. \tag{6}$$

B. Iterative Algorithm Determination of the Phase-Shift Plane

Using the universal phase-shifting algorithm, phase distribution can be extracted if the phase-shift planes are known. If phase distribution is known, the phase-shift planes can be determined by a similar method but in an inverse way. In the inverse algorithm, we divide the interferogram space into several blocks and retrieve local phase shifts. If the blocks are sufficiently small, we may consider that the background intensities, modulation amplitudes, and phase shifts in each block do not have pixel-to-pixel variation and can be assumed as constants. In the kth block, defining another set of variables for the nth frame as $a_n(k) = A(k)$, $b_n(k) = B(k)\cos(d_{nk})$, and $c_n(k) = -B(k)\sin(d_{nk})$, Eq. (1) is rewritten as

$$I_{nk}^t(x, y) = a_n(k) + b_n(k)\cos\Phi(x, y) + c_n(k)\sin\Phi(x, y). \tag{7}$$

If $\Phi(x, y)$ is known, there are $3N$ unknowns and XYN/K equations, where X, Y denote the total number of pixels in the x and y directions, respectively, and K is the number of blocks in the full field of the interferogram. Therefore, if the number of pixel in each block XY/K is larger than 3, the unknowns can be solved again by use of the overdetermined least-squares method in the same way as step A (spatial coordinates x, y has been omitted):

$$\begin{bmatrix} a_n(k) \\ b_n(k) \\ c_n(k) \end{bmatrix} = \begin{bmatrix} XY/K & \sum_k \cos\Phi & \sum_k \sin\Phi \\ \sum_k \cos\Phi & \sum_k \cos^2\Phi & \sum_k \sin\Phi\cos\Phi \\ \sum_k \sin\Phi & \sum_k \sin\Phi\cos\Phi & \sum_k \sin^2\Phi \end{bmatrix}^{-1} \times \begin{bmatrix} \sum_k I \\ \sum_k I\cos\Phi \\ \sum_k I\sin\Phi \end{bmatrix}, \tag{8}$$

where $\sum_k$ denotes the sum in the kth block. Then the amount of phase shift in each block can be determined from

$$d_{nk} = \tan^{-1}[-c_n(k)/b_n(k)]. \tag{9}$$

The unwrapped K phase shifts in each nth frame can be fitted to a phase-shift plane by a linear least-squares method; thus, the tilts k_{xn}, k_{yn} and average phase shift amounts d_n are obtained. This procedure can adjust the input of the phase-shift planes in step A and increase the accuracy of phase extractions.

C. Iterative Strategy for Random and Spatial Nonuniform Phase Shifting

The proposed iterative algorithm includes three steps in each iteration cycle. In the ith iteration cycle the steps are step 1—calculating phase distribution based on phase-shift planes obtained in the second step of the previous iteration cycle, step 2—determining the updated phase-shift planes from the phase distribution obtained in the first step of this cycle, step 3—checking to see whether the iteration results satisfy the convergence criteria. It is the relative phase-shift plane that will converge, so the convergence criteria can be expressed as

$$\begin{cases} \left|(d_n{}^i - d_1{}^i) - (d_n{}^{i-1} - d_1{}^{i-1})\right| < \varepsilon_1 \\ \left|(k_{xn}{}^i - k_{x1}{}^i) - (k_{xn}{}^{i-1} - k_{x1}{}^{i-1})\right| \\ \quad + \left|(k_{yn}{}^i - k_{y1}{}^i) - (k_{yn}{}^{i-1} - k_{y1}{}^{i-1})\right| < \varepsilon_2 \end{cases}, \tag{10}$$

where i represents the number of iterations and ε_1 and ε_2 are the predefined threshold of accuracy.

The proposed iterative algorithm contains a nonlinear iterative process. If the estimated initial phase shift planes in step 1 deviate from the actual ones greatly, the phase distribution $\Phi(x, y)$ obtained from the first iteration will be far from the actual phase. Consequently, the phase shift planes obtained in the second iteration, mainly the tilts k_{xn} and k_{yn}, may deviate from the actual ones further and result in instability in the iteration. Therefore, we first ignore the tilt errors and degrade the proposed algorithm similar to the iterative algorithm of Wang and Han. After a few iterations until Eq. (10) meets a given accuracy, e.g., $\varepsilon_1 = 0.5$, the phase distribution is obtained and is approximate to the actual one. Then we substitute the approximate phase into the proposed iterative algorithm. The iteration will not stop until the criteria are met (e.g., $\varepsilon_1 = 10^{-5}$, $\varepsilon_2 = 10^{-5}$). Finally, the phase distributions and phase-shift planes with tilt information are updated and determined after iterations.

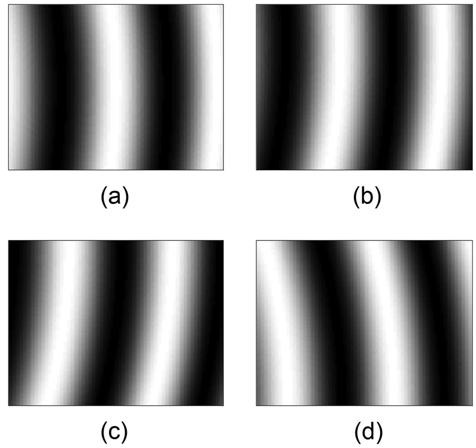

Fig. 1. Interferograms with different phase-shift values and tilts: (a) $d_1 = 0$, $k_{x1} = 0$, $k_{y1} = 0$; (b) $d_2 = 1.5$, $k_{x2} = 0.167$, $k_{y2} = 0.67$; (c) $d_3 = 3.1$, $k_{x3} = 0.167$, $k_{y3} = 1.33$; (d) $d_4 = 4.8$, $k_{x4} = 0.167$, $k_{y4} = -1.33$.

During iteration, the proposed method converges faster than the algorithm of Wang and Han because the local phase shifts are compensated in every pixel. However, the algorithm described is limited to the number of blocks K, which affects the accuracy of the tilt values. For interferograms with small tilts and high signal-to-noise ratios, dividing the field into 2×2 blocks is sufficient, whereas larger tilts and low signal-to-noise ratios require dividing the field into more blocks; otherwise, the calculated tilts deviate

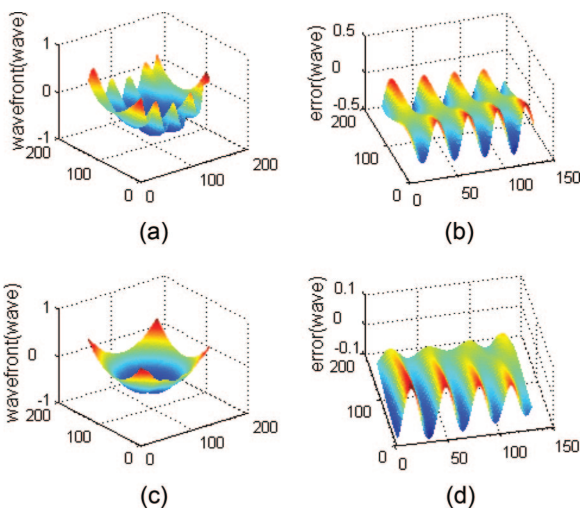

Fig. 2. (Color online) Simulation results: (a) Phase extracted by the iterative algorithm of Wang and Han, (b) residual error of (a), (c) phase extracted by the proposed iterative algorithm, (d) residual error of (c).

Table 1. Iterative Results of the Phase-Shift Planes for Four Different Sets and Corresponding Residual Errors[a]

	Case 1		Case 2		Case 3		Case 4	
	Real	Calculated	Real	Calculated	Real	Calculated	Real	Calculated
d_2	1	1.0185	1.5	1.5169	1.5	1.5122	1.5	1.5396
d_3	2	2.0514	2	2.0401	2	2.0381	2	2.0738
d_4	3	3.0857	2.5	2.5669	2.5	2.5672	2.5	2.6106
k_{x2}	−0.0011	−0.0105	−0.1	−0.0921	−0.2	−0.1818	−0.4	−0.3753
k_{x3}	0.005	0.0051	0.05	0.0553	0.1	0.1147	0.2	0.2094
k_{x4}	0.015	0.0156	0.15	0.1553	0.3	0.3141	0.4	0.4205
k_{y2}	−0.012	−0.0119	−0.1	−0.0968	−0.2	−0.1957	−0.4	−0.3951
k_{y3}	0.016	0.0162	0.16	0.1681	0.3	0.3143	0.5	0.5049
k_{y4}	0.002	0.0021	0.02	0.0217	0.1	0.1067	0.2	0.2090
Error	PV	rms	PV	rms	PV	rms	PV	rms
Ours	0.0359	0.0124	0.0760	0.0181	0.1280	0.0229	0.2087	0.0298
Wang and Han	0.0506	0.0128	0.4979	0.0698	1.4333	0.1598	1.8546	0.1780

[a]Assuming that the first phase-shift planes are zero, e.g., $d_1 = 0$, $k_{x1} = 0$, and $k_{y1} = 0$, the units for d_n are radians and the units for tilts k_{xn} and k_{yn} are $2/X$ rad/pixel, where $X = 128$.

from the actual ones so much that the iterative criteria will not converge.

3. Performance of the Iterative Algorithm

Since the exact expression of an actual object surface is hard to know due to many practical factors, a series of computer simulations have been carried out to verify the effectiveness of the proposed algorithm. To test its accuracy, we define the actual phase map as $F(x, y) = 0.1\pi(x^2 + y^2) + 2\pi y$, the background as $A(x, y) = 150 \exp[-0.2(x^2 + y^2)]$, and the modulation amplitude as $B(x, y) = 100 \exp[-0.1(x^2 + y^2)]$, where $-1 \le x \le 1$, $-1 \le y \le 1$, and the total number of pixels in the x and y directions both equal 128. By changing the phase shift value d_n and tilts k_{xn}, k_{yn}, a set of four phase-shifted interferograms is obtained as shown in Fig. 1. Let the predefined thresholds ε_1 and ε_2 both equal to 10^{-5}, and then the phase is extracted by the proposed iterative algorithm. Figure 2 shows the phases extracted by the algorithm of Wang and Han and the proposed iterative algorithm and their residual errors. It is evident that the proposed iterative algorithm effectively reduces the waviness existing in the iterative algorithm of Wang and Han caused by tilt-shift errors. On the other hand, the iterative results are shown in Table 1, including phase-shift planes and residual phase errors [peak-to-valley (PV) and rms values]. It shows that the residual errors of the proposed algorithm increase with increasing tilt-shift errors. Compared with the algorithm of Wang and Han, our algorithm can calculate the phase-shift planes and phase distribution with higher accuracy. The residual errors of our algorithm are less than 0.21 rad (PV) and 0.03 rad (rms) even when the tilt error is up to 20% of the nominal phase step (in case 4), which is much smaller than that of the algorithm of Wang and Han (1.8546 and 0.1780 rad).

Obviously, our algorithm makes the iteration converge faster than the algorithm of Wang and Han because the tilt errors are compensated. To compare with the algorithm of Chen *et al.*, the common conditions are met, including the same interferograms and the same initial estimated phase-shift planes. After 15 iterations, the residual errors are recorded every iteration and presented in Fig. 3. The curves show that the proposed algorithm also converges much faster than the algorithm of Chen *et al.*

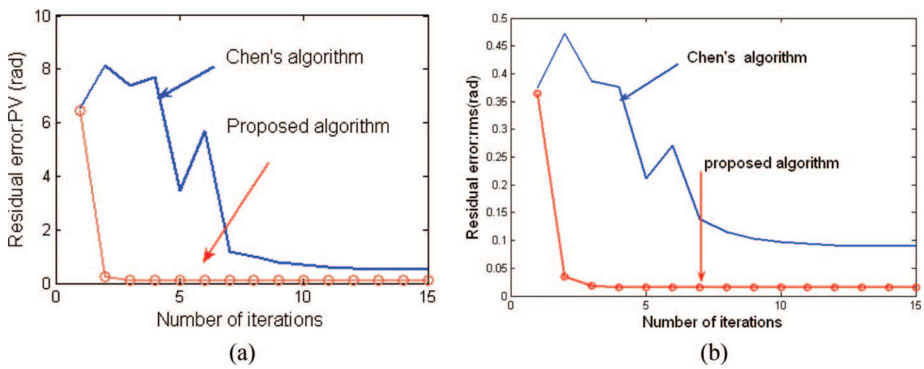

Fig. 3. (Color online) Relationship between residual errors and number of iterations: (a) PV of the residual errors, (b) rms of the residual errors.

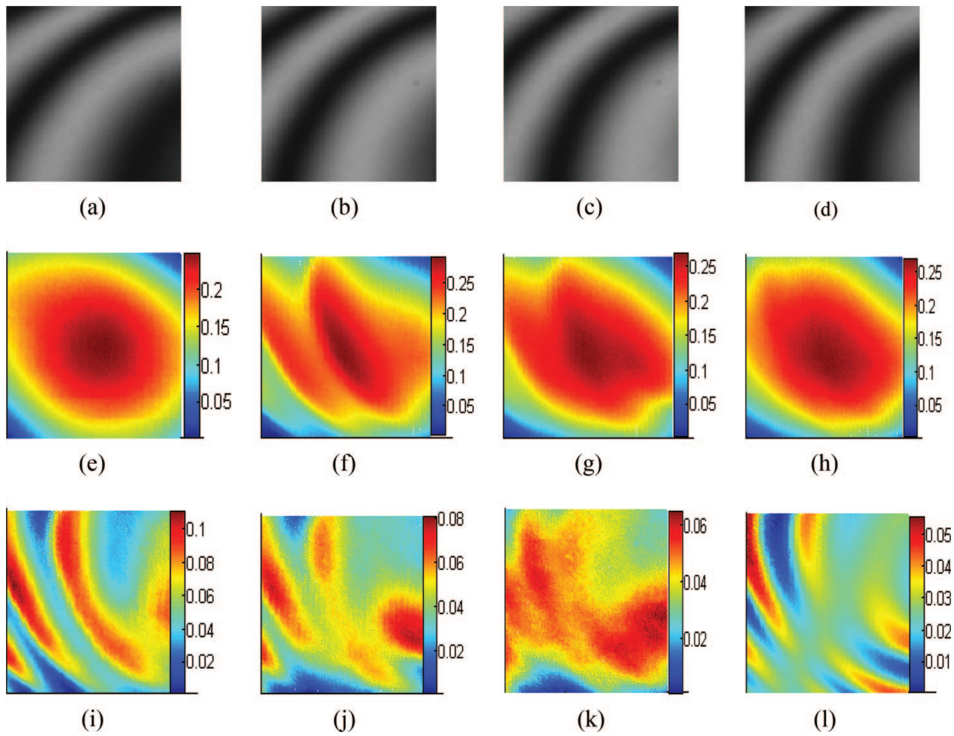

Fig. 4. (Color online) Comparison among the phases extracted by Zygo's software, the conventional four-frame algorithm, the algorithm of Wang and Han, and the proposed algorithm. (a)–(d) Four randomly and spatially nonuniform phase-shifted interferograms. (e)–(h) Phase map extracted by Zygo's software, the conventional four-frame algorithm, the algorithm of Wang and Han, and the proposed algorithm, respectively. (i)–(k) Residual errors of the conventional four-frame algorithm, the algorithm of Wang and Han, and the proposed algorithm compared with Zygo's software. (1) The difference in result between the algorithm of Wang and Han and the proposed algorithm.

4. Experiments and Discussion

For further verification of the performance of the proposed algorithm, we apply it to the practical interferograms presented in Figs. 4(a)–4(d), the phase shifts of which are introduced by moving the PZT of the test optical element rotating around an axis on the test surface. Four randomly and spatially nonuniform phase-shifted interferograms are recorded, and the conventional four-frame algorithm [8], the algorithm of Wang and Han, and the proposed algorithm are used to extract the phase distribution, the results of which are shown in Figs. 4(f)–4(h). On the other hand, the test surface is also measured by Zygo's interferometer with a vibration-isolating platform and a calibrated PZT, and the phase of the test surface is extracted by Zygo's software, the result of which is shown in Fig. 4(e). Therefore, the differences in the results between Zygo's software and the other algorithms are defined as their relative errors shown in Figs. 4(i)–4(k). Furthermore, the difference in the results between the algorithm of Wang and Han and the proposed algorithm is also presented in Fig. 4(l). The PV and rms of the phase extracted by all four algorithms and the relative errors between them are listed in Table 2. According to Fig. 4 and Table 2, the following can be concluded: (1) The conventional four-frame algorithm exhibits larger error and evident waviness in the phase map because of the translation and tilt-shift errors of the phase-shift plane shown in Table 3. (2) The algorithm of Wang and Han calculates the phase-shifting amounts by a least-squares-based iterative algorithm and thus deduces the error, but there is visible waviness existing in the phase map because of tilt-shift errors of the phase-shift planes. (3) The proposed algorithm exhibits high precision with very weak waviness in the phase map by compensating most of the translation and tilt-shift errors and relative error between that and Zygo's MetroPro software mainly caused by the test environment, such as temperature variety, air turbulence, and so on. Table 3 also shows that the maximum local phase shifts of interferomgrams

Table 2. PV and rms of the Phase Extracted by All Four Algorithms and Relative Errors between Them[a]

Errors	Fig. 4(e)	Fig. 4(f)	Fig. 4(g)	Fig. 4(h)	Fig. 4(i)	Fig. 4(j)	Fig. 4(k)	Fig. 4(g)–(h)	Fig. 4(l)
PV	0.2455	0.2907	0.2679	0.2678	0.1107	0.0811	0.0650	0.0841	0.0550
rms	0.0444	0.0535	0.0515	0.0507	0.0963	0.0127	0.0102	0.0168	0.0081

[a]Units are radians.

Table 3. Translation and Tilt Errors of the Phase-Shift Plane[a]

$d_1 k_{x1} k_{y1}$	d_2	d_3	d_4	k_{x2}	k_{x3}	k_{x4}	k_{y2}	k_{y3}	k_{y4}
0	1.7906	2.8642	5.2429	−0.2910	−0.5229	−0.8613	0.3352	0.4975	0.5854

[a]Units for d_n are radians and units for tilts k_{xn} and k_{yn} are $2/X$ rad/pixel, where $X = 256$.

caused by tilt-shift errors are 0.6262, 1.0204, and 1.4467, which are 35.0%, 35.6%, and 27.6% of the normal phase step, respectively.

5. Conclusion

To conclude, we have proposed a new generalized iterative algorithm for extracting phase distribution from randomly and spatially nonuniform phase-shifted interferograms. The proposed algorithm needs only four randomly phase-shifted (including both translational and tilt-shift errors) interferograms and gives an accurate phase map in which the effects of transitional and tilt-shift errors are both calculated and reduced. This is the reason why our algorithm is advantageous over the conventional four-frame algorithm and the algorithm of Wang and Han. Simulated results and experiments demonstrate the effectiveness of the proposed algorithm. Simulations show that the proposed algorithm does not require an accurate initial estimation of the phase-shift plane and makes the iteration converge faster than Chen's algorithm. The proposed iterative algorithm works well with a large-aperture interferometer shifting phase by PZT, especially for real-time and dynamic measurements in an environment with low frequency and high amplitude vibration. With this method, costly and accurate phase-shifting devices are no longer required for steady-state measurement.

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