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Analysis of the $M^{[X]}/G/1$ Queues with Second Multi-optional Service and Unreliable Server

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Abstract A bulk-arrival single server queueing system with second multi-optional service and unreliable server is studied in this paper. Customers arrive in batches according to a homogeneous Poisson process, all customers demand the first "essential" service, whereas only some of them demand the second "multi-optional" service. The first service time and the second service all have general distribution and they are independent. We assume that the server has a service-phase dependent, exponentially distributed life time as well as a service-phase dependent, generally distributed repair time. Using a supplementary variable method, we obtain the transient and the steady-state solutions for both queueing and reliability measures of interest.

Keywords Bulk-arrival queue, first essential service, second optional service, reliability
2000 MR Subject Classification 60K25; 90B22

1 Introduction

In a recent paper, Madan[2] considers an $M/G/1$ queue with the second optional service, in which some of the customers may require immediately after completion of the first essential service. Such queueing situations can be found in day to day life. By using a supplementary variable method, Madan[2] studies the time-dependent as well as the steady state behavior of this kind of queueing system. Based on Madan's model, some authors propose various modifications. Medhi[3] considers a more general case of such a queue, where the service times of all service phases are independent and have general distributions. Yin et al.[6] generalize the model with assumptions that customers arrive at the system in batches according to a Poisson process, and the customer may opt for a second multi-optional services when the first essential service is completed. Wang[5] studies Madan's model from the viewpoint of reliability theory with the assumption that the server may be subject to breakdowns and repairs during the service processes.

The present study focuses on generalizing the above works in an integrated way, i.e., the $M^{[X]}/G/1$ queueing system with server breakdowns and repairs is studied in this paper. Such queueing situations are also common in practice. For instance, in flexible manufacturing system, there are versatile, multi-functional machines which can perform several types of operations, e.g., lathing, drilling, milling and so forth. Workpieces arrive in batches with different processing requirements, all need the main essential service, whereas some of them may require further particular type of operation after the main essential service. Meanwhile, the machine may be

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subject to random failures and/or interruptions which have impact on the system's performance. We wish to understand their effects on measures of system performance which influence the efficient operation of the systems.

The rest of this paper is organized as follows. In the next section, we give a relative formal description of the model and introduce supplementary variables to make the process Markovian. In Section 3, we investigate the time-dependent solutions to this model by using the Laplace transforms technique. In Section 4, the steady state solutions for the queueing quantities are obtained and it is shown that some previous works are special cases of our model. In Section 5, we consider the reliability aspect of the model and obtain some important reliability measures of the server, including the server availability, failure frequency, and the reliability function. Finally, in Section 6, we work out some numerical examples to illustrate the effect of the unreliability parameters on the system performance.

2 Basic Assumptions and Description of the Model

We consider the $M^{[X]}/G/1$ queueing systems with the following structure.

1. Customers arrive at the system according to a Poisson process with rate $\lambda > 0$ and arrive in batches such that the batch size X are i.i.d. random variables with distribution $P(X = i) = C_i,\ i = 1, 2, \cdots$, with mean $c = EX$ and $EX^2 < \infty$.

2. The first essential service is needed by all arriving customers. Let $B_0(v)$ and $b_0(v)$, respectively, be the distribution function and the density function of the first service times V_0, with mean $1/\mu_0$, and let $\mu_0(x) = \frac{b_0(x)}{1-B_0(x)}$ be the hazard rate function. It is assumed that $EV_0^2 < \infty$.

3. As soon as the first service of a customer is completed, then with probability $r_k(1 \le k \le m)$ he may opt for the k-type second service, in which case his second service will immediately commence or else with probability $r_0 = 1 - \sum\limits_{k=1}^{m} r_k$ he may opt to leave the system, in which case another customer at the head of queue (if any) is taken up for his first essential service.

4. The k-type second service times V_k are assumed to be generally distributed with distribution function $B_k(v)$, density function $b_k(v)$, hazard rate function $\mu_k(x) = \frac{b_k(x)}{1-B_k(x)}$ with mean service time $1/\mu_k, 1 \le k \le m$. It is assumed that $EV_k^2 < \infty$.

5. We assume that the server's life time has exponential distribution with mean $1/\alpha_0$ in the first essential service. In the k-type second optional service, the server fails at an exponential rate $\alpha_k(1 \le k \le m)$.

6. The server may break down when servicing customers, and when the server breaks down it is sent for repair immediately. The customer just being served before server breakdown waits for the server to complete its remaining service. The repair times, $W_k(0 \le k \le m)$, of both service phases are arbitrarily distributed with probability distribution function $G_k(x)(0 \le k \le m)$. Also, let $g_k(x)$, $\beta_k(x)$ and $1/\beta_k(0 \le k \le m)$, be the corresponding probability density functions, hazard rates functions, and means, repectively. Immediately

after the server is fixed, it starts to serve customers. It is assumed that the service time for a customer is cumulative, and after repair the server is as good as new. Furthermore, we assume that $EW_k^2 < \infty$, $0 \le k \le m$.

7. Various stochastic processes involved in the system are assumed independent of each other.

Let $N(t)$ be the number of customers in the system at time t, and $N_1(t)$ the number of customers in the queue. To make it a Markov process, we introduce supplementary variables. Define $X(t)$ as the elapsed service time of the customer currently being served at time t, and $Y(t)$ the elapsed repair time of the failed server at time t, and define the state probabilities at time t as follows:

1. $I(t)$ is the probability that the server is idle at time t.

2. $P_{n,0}(t,x)dx$ is the joint probability that at time t there are n customers in the queue excluding the one being provided the first essential service, the server is up and a customer is being served with the elapsed service time between x and $x+dx$ $(n \ge 0)$.

3. $P_{n,k}(t,x)dx$, $1 \le k \le m$, is the joint probability that at time t there are n customers in the queue excluding the one being provided the k-type second optional service $(n \ge 0)$.

4. $R_{n,0}(t,x,y)dy$ is the joint probability that at time t there are n customers in the queue excluding the one being provided the first essential service, the elapsed service time for the customer under service is equal to x, and the server is being repaired with the elapsed repair time between y and $y+dy$ $(n \ge 0)$.

5. $R_{n,k}(t,x,y)dy$, $1 \le k \le m$, is the joint probability that at time t there are n customers in the queue excluding the one being provided the k-type second optional service, the elapsed service time for the customer under service is equal to x, and the server is being repaired with the elapsed repair time between y and $y+dy$ $(n \ge 0)$.

Thus, at an arbitrary time, the state of the system can be characterized by the random variables $N(t)$, $X(t)$ and $Y(t)$. By considering transitions of the process between time t and $t+\Delta t$ and letting $\Delta t \to 0$, we derive the system of forward equations for $n = 0,1,2,\cdots$.

$$\Big(\frac{d}{dt}+\lambda\Big)I(t) = r_0\int_0^\infty P_{0,0}(t,x)\mu_0(x)dx + \sum_{k=1}^m \int_0^\infty P_{0,k}(t,x)\mu_k(x)dx, \tag{2.1}$$

$$\begin{aligned}&\Big[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_k(x)+\lambda+\alpha_k\Big]P_{n,k}(t,x)\\ =&\lambda\sum_{i=1}^n C_iP_{n-i,k}(t,x) + \int_0^\infty R_{n,k}(t,x,y)\beta_k(y)dy,\end{aligned} \tag{2.2}$$

$$\Big[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\lambda+\beta_k(y)\Big]R_{n,k}(t,x,y) = \lambda\sum_{i=1}^n C_iR_{n-i,k}(t,x,y), \tag{2.3}$$

where $0 \le k \le m, n = 0,1,2,\cdots$. These equations are to be solved subject to the following boundary conditions:

$$P_{n,0}(t,0) = r_0 \int_0^\infty P_{n+1,0}(t,x)\mu_0(x)dx + \sum_{k=1}^{m} \int_0^\infty P_{n+1,k}(t,x)\mu_k(x)dx + \lambda C_{n+1}Q(t), \tag{2.4}$$

$$P_{n,k}(t,0) = r_k \int_0^\infty P_{n,0}(t,x)\mu_0(x)dx, \qquad 1 \le k \le m, \tag{2.5}$$

$$R_{n,k}(t,x,0) = \alpha_k P_{n,k}(t,x), \qquad 0 \le k \le m, \tag{2.6}$$

where $n = 0, 1, 2, \cdots$.

We assume that at $t = 0$ there is no customers in the system, and the server is idle. So the initial condition is $I(0) = 1$, where $P_{-1,k}(t,x) = 0$, $R_{-1,k}(t,x,y) = 0$, $0 \le k \le m$.

3 The Model Solutions

To solve the above equations, we define the following generating functions:

$$P_k(t,x,z) = \sum_{n=0}^{\infty} P_{n,k}(t,x)z^n, \quad R_k(t,x,y,z) = \sum_{n=0}^{\infty} R_{n,k}(t,x,y)z^n, \qquad 0 \le k \le m.$$

Also, define the Laplace transform of a given function $f(x)$ as $f^*(s) = \int_0^\infty e^{-st} f(t)dt, \mathrm{Re}(s) > 0$.

Theorem 3.1. *The Laplace transforms of $I(t)$, $P_k(t,x,z)$, $R_k(t,x,y,z)$, $k = 0, 1, \cdots, m$, are given by*

$$I^*(s) = \frac{1}{s + \lambda - \lambda C(z_s)}, \tag{3.1}$$

$$P_0^*(s,x,z) = P_0^*(s,0,z)\exp\{-\Phi_0(s,z)x\}[1 - B_0(x)], \tag{3.2}$$

$$P_k^*(s,x,z) = r_k b_0^*(\Phi_0(s,z))P_0^*(s,0,z)\exp\{-\Phi_k(s,z)x\}[1 - B_k(x)], \qquad 1 \le k \le m. \tag{3.3}$$

$$R_k^*(s,x,y,z) = \alpha_k P_k^*(s,x,z)\exp\{-[s + \lambda - \lambda C(z)]y\}[1 - G_k(y)], \qquad 0 \le k \le m, \tag{3.4}$$

where

$$P_0^*(s,0,z) = \frac{[s + \lambda - \lambda C(z)]I^*(s) - 1}{r_0 b_0^*(\Phi_0(s,z)) + b_0^*(\Phi_0(s,z)) \sum\limits_{k=1}^{m} r_k b_k^*(\Phi_k(s,z)) - z}, \tag{3.5}$$

$\Phi_k(s,z) := s + \lambda + \alpha_k - \lambda C(z) - \alpha_k g_k^*(s + \lambda - \lambda C(z)),\ 0 \le k \le m,\ C(z) = \sum\limits_{i=1}^{\infty} C_i z^i$ *and z_s is the root of the equation:*

$$x = r_0 b_0^*(\Phi_0(s,x)) + b_0^*(\Phi_0(s,x)) \sum_{k=1}^{m} r_k b_k^*(\Phi_k(s,x)).$$

Proof. Taking the Laplace transforms with respect to t for equations (2.1)–(2.3), using the initial condition, we have

$$(s+\lambda)I^*(s)-1=r_0\int_0^\infty P^*_{0,0}(s,x)\mu_0(x)dx+\sum_{k=1}^m\int_0^\infty P^*_{0,k}(s,x)\mu_k(x)dx, \tag{3.6}$$

$$\begin{aligned}&\frac{\partial}{\partial x}P^*_{n,k}(s,x)+[s+\lambda+\alpha_k+\mu_k(x)]P^*_{n,k}(s,x)\\ =&\lambda\sum_{i=1}^n C_iP^*_{n-i,k}(s,x)+\int_0^\infty R^*_{n,k}(s,x,y)\beta_k(y)dy,\end{aligned} \tag{3.7}$$

$$\frac{\partial}{\partial y}R^*_{n,k}(s,x,y)+[s+\lambda+\beta_k(y)]R^*_{n,k}(s,x,y)=\lambda\sum_{i=1}^n C_iR^*_{n-i,k}(s,x,y). \tag{3.8}$$

Similarly, taking the Laplace transform of the boundary condition (2.4)–(2.6), we get for $n\ge 0$:

$$\begin{aligned}P^*_{n,0}(s,0)=r_0&\int_0^\infty P^*_{n+1,0}(s,x)\mu_0(x)dx\\ &+\sum_{k=1}^m\int_0^\infty P^*_{n+1,k}(s,x)\mu_k(x)dx+\lambda C_{n+1}I^*(s),\end{aligned} \tag{3.9}$$

$$P^*_{n,k}(s,0)=r_k\int_0^\infty P^*_{n,0}(s,x)\mu_0(x)dx, \qquad 1\le k\le m, \tag{3.10}$$

$$R^*_{n,k}(s,x,0)=\alpha_kP^*_{n,k}(s,x), \qquad 0\le k\le m. \tag{3.11}$$

Multiplying equation (3.7) by suitable powers of z, summing over n and use the generating function defined above, and after simplifying, we obtain for $0\le k\le m$:

$$\begin{aligned}\frac{\partial}{\partial x}P^*_k(s,x,z)=&-[s+\lambda+\alpha_k+\mu_k(x)-\lambda C(z)]P^*_k(s,x,z)\\ &+\int_0^\infty R^*_k(s,x,y,z)\beta_k(y)dy.\end{aligned} \tag{3.12}$$

Similarly, we get the following equations from (3.8)–(3.11).

$$\frac{\partial}{\partial y}R^*_k(s,x,y,z)=-[s+\lambda+\beta_k(y)-\lambda C(z)]R^*_k(s,x,y,z), \qquad 0\le k\le m, \tag{3.13}$$

$$\begin{aligned}zP^*_0(s,0,z)=r_0&\int_0^\infty P^*_0(s,x,z)\mu_0(x)dx\\ &+\sum_{k=1}^m\int_0^\infty P^*_k(s,x,z)\mu_k(x)dx+\lambda C(z)I^*(s)\\ &-\Big[r_0\int_0^\infty P^*_{0,0}(s,x)\mu_0(x)dx+\sum_{k=1}^m\int_0^\infty P^*_{0,k}(s,x)\mu_k(x)dx\Big],\end{aligned} \tag{3.14}$$

$$P^*_k(s,0,z)=r_k\int_0^\infty P^*_0(s,x,z)\mu_0(x)dx, \qquad 1\le k\le m, \tag{3.15}$$

$$R^*_k(s,x,0,z)=\alpha_kP^*_k(s,x,z), \qquad 0\le k\le m. \tag{3.16}$$

From (3.13) we get

$$R^*_k(s,x,y,z)=R^*_k(s,x,0,z)\exp\{-[s+\lambda-\lambda C(z)]y\}[1-G_k(y)], \qquad 0\le k\le m. \tag{3.17}$$

With the help of (3.16), we get the following equation from (3.17):

$$\int_0^\infty R_k^*(s,x,y,z)\beta_k(y)dy = \alpha_k g_k^*(s+\lambda-\lambda C(z))P_k^*(s,x,z), \qquad 0 \le k \le m. \tag{3.18}$$

Substituting (3.18) into (3.12), after simplifying, we obtain

$$P_k^*(s,x,z) = P_k^*(s,0,z)\exp\{-\Phi_k(s,z)x\}[1-B_k(x)], \qquad 0 \le k \le m, \tag{3.19}$$

where $\Phi_k(s,z) = s+\lambda+\alpha_k-\lambda C(z)-\alpha_k g_k^*(s+\lambda-\lambda C(z)),\ 0 \le k \le m.$

Taking $k=0$ in (3.19), we get

$$P_0^*(s,x,z) = P_0^*(s,0,z)\exp\{-\Phi_0(s,z)x\}[1-B_0(x)], \tag{3.20}$$

further, we have

$$\int_0^\infty P_0^*(s,x,z)\mu_0(x)dx = b_0^*(\Phi_0(s,z))P_0^*(s,0,z). \tag{3.21}$$

So, from (3.15) we get

$$P_k^*(s,0,z) = r_k b_0^*(\Phi_0(s,z))P_0^*(s,0,z), \qquad 1 \le k \le m. \tag{3.22}$$

Substituting (3.22) into (3.19), we obtain

$$P_k^*(s,x,z) = r_k b_0^*(\Phi_0(s,z))P_0^*(s,0,z)\exp\{-\Phi_k(s,z)x\}[1-B_k(x)], \qquad 1 \le k \le m. \tag{3.23}$$

Substituting (3.16) and (3.23) into (3.17), we get:

$$R_0^*(s,x,y,z) = \alpha_0 P_0^*(s,x,z)\exp\{-[s+\lambda-\lambda C(z)]y\}[1-G_0(y)], \qquad k=0, \tag{3.24}$$

$$R_k^*(s,x,y,z) = \alpha_k P_k^*(s,x,z)\exp\{-[s+\lambda-\lambda C(z)]y\}[1-G_k(y)], \quad 1 \le k \le m. \tag{3.25}$$

From equation (3.23) we have

$$\int_0^\infty P_k^*(s,x,z)\mu_k(x)dx = r_k b_0^*(\Phi_0(s,z))b_k^*(\Phi_k(s,z))P_0^*(s,0,z), \qquad 1 \le k \le m, \tag{3.26}$$

with (3.21) and (3.6), (3.14) becomes

$$P_0^*(s,0,z) = \frac{[s+\lambda-\lambda C(z)]I^*(s)-1}{r_0 b_0^*(\Phi_0(s,z)) + b_0^*(\Phi_0(s,z))\sum\limits_{k=1}^m r_k b_k^*(\Phi_k(s,z)) - z}. \tag{3.27}$$

Integrating (3.20) with regard to x by parts, we get

$$P_0^*(s,z) = \frac{[1-b_0^*(\Phi_0(s,z))]P_0^*(s,0,z)}{\Phi_0(s,z)}. \tag{3.28}$$

Integrating (3.23) with regard to x by parts, we get

$$P_k^*(s,z) = \frac{r_k b_0^*(\Phi_0(s,z))[1-b_k^*(\Phi_k(s,z))]P_0^*(s,0,z)}{\Phi_k(s,z)}, \qquad 1 \le k \le m. \tag{3.29}$$

Integrating (3.24) with regard to x and y by parts, respectively, we get

$$R_0^*(s,z) = \frac{\alpha_0[1-b_0^*(\Phi_0(s,z))][1-g_0^*(s+\lambda-\lambda C(z))]P_0^*(s,0,z)}{\Phi_0(s,z)[s+\lambda-\lambda C(z)]}. \tag{3.30}$$

Substituting (3.22) into (3.25), integrating with regard to x and y by parts, respectively, we get

$$R_k^*(s,z) = \frac{\alpha_k r_k b_0^*(\Phi_0(s,z))[1-b_k^*(\Phi_k(s,z))][1-g_k^*(s+\lambda-\lambda C(z))]P_0^*(s,0,z)}{\Phi_k(s,z)[s+\lambda-\lambda C(z)]}, \tag{3.31}$$

where $1 \le k \le m$.

Let $P_q^*(s,z) = \sum\limits_{k=0}^{m}[P_k^*(s,z) + R_k^*(s,z)]$ denote the generating function of the number in the queue irrespective of the type of service being provided. Then adding equations (3.28)–(3.31), with (3.27) we have

$$P_q^*(s,z) = \frac{\left[1 - r_0 b_0^*(\Phi_0(s,z)) - b_0^*(\Phi_0(s,z))\sum\limits_{k=1}^{m} r_k b_k^*(\Phi_k(s,z))\right]\left[I^*(s) - \frac{1}{s+\lambda-\lambda C(z)}\right]}{r_0 b_0^*(\Phi_0(s,z)) + b_0^*(\Phi_0(s,z))\sum\limits_{k=1}^{m} r_k b_k^*(\Phi_k(s,z)) - z}. \tag{3.32}$$

By Rouché's theorem, the denominator of the right hand of (3.32) has one zero z_s inside the unit circle $|z| = 1$ for $\mathrm{Re}(s) > 0$, and it is also the zero point for the numerator of (3.32). This is sufficient to determine the only unknown $I^*(s)$ appearing in the numerator:

$$I^*(s) = \frac{1}{s+\lambda-\lambda C(z_s)}, \tag{3.33}$$

where z_s is the root of the equation

$$x = r_0 b_0^*\big(\Phi_0(s,x)\big) + b_0^*\big(\Phi_0(s,x)\big)\sum_{k=1}^{m} r_k b_k^*\big(\Phi_k(s,x)\big).$$

Thus, $P_q^*(s,z), P_k^*(s,z), R_k^*(s,z), P_k^*(s,x,z), R_k^*(s,x,y,z)$, $0 \le k \le m$, can be completely determined by (3.32), (3.29), (3.31), (3.23), (3.17) together with (3.16), (3.27). □

4 The Queueing Quantities in Steady-state

In this section, we investigate the queueing quantities of our model in steady state. To this end, it should be noted that the "true" service time of a customer is not the length of time from the point when a customer begins to be served to the point until the service is completed because of possible breakdowns. We define the *generalized service time* as the length of time from when a customer begins to be served until the service is completed.

Let $T_{n,0}$ be the generalized service time of the nth customer when he receives the first essential service. We note that it may include some possible down times of the server due to server failures during the service period of the nth customer, since the nth customer begins to be served until the service is completed. Define $\widehat{B}_{n,0}^{[l]}(t) = P\{T_{n,0} \le t$ and the server just fails l times during the first service of the nth customer$\}$, where $n \ge 1,\ l \ge 0,\ t \ge 0$.

It was shown in Cao and Cheng[1] that $T_{n,0}$'s are i.i.d. random variables with the distribution function

$$\widehat{B}_0(t) \equiv \widehat{B}_{n,0}(t) = P\{T_{n,0} \le t\} = \sum_{l=0}^{\infty} \int_0^t G_0^{(l)}(t-u) e^{-\alpha_0 u} \frac{(\alpha_0 u)^l}{l!} dB_0(u),$$

which is independent of n. Its Laplace-Stieltjes transform is

$$\widehat{b}_0^*(s) = \int_0^{\infty} e^{-st} d\widehat{B}_0(t) = b_0^*(s + \alpha_0 - \alpha_0 g_0^*(s)), \qquad \mathrm{Re}(s) > 0,$$

and its first two moments are given by

$$\begin{aligned} ET_{n,0} &= -\frac{d\widehat{b}_0^*(s)}{ds}\Big|_{s=0} = \frac{1}{\mu_0}\Big(1 + \frac{\alpha_0}{\beta_0}\Big), \\ ET_{n,0}^2 &= \frac{\alpha_0}{\mu_0} EW_0^2 + \Big(1 + \frac{\alpha_0}{\beta_0}\Big)^2 EV_0^2. \end{aligned}$$

In a similar manner, we can define $T_{n,k}$ as the generalized service time of the nth customer when he receives the k-type second optional service. We have

$$\begin{aligned} ET_{n,k} &= -\frac{d\widehat{b}_k^*(s)}{ds}\Big|_{s=0} = \frac{1}{\mu_k}\Big(1 + \frac{\alpha_k}{\beta_k}\Big), \\ ET_{n,k}^2 &= \frac{\alpha_k}{\mu_k} EW_k^2 + \Big(1 + \frac{\alpha_k}{\beta_k}\Big)^2 EV_k^2, \qquad 1 \le k \le m. \end{aligned}$$

Thus, the mean generalized service time for a customer is given by $\frac{1}{\mu_0}(1+\frac{\alpha_0}{\beta_0}) + \sum\limits_{k=1}^{m} \frac{r_k}{\mu_k}(1+\frac{\alpha_k}{\beta_k})$. During this period, $c[\rho_0(1+\frac{\alpha_0}{\beta_0}) + \sum\limits_{k=1}^{m} r_k\rho_k(1+\frac{\alpha_k}{\beta_k})]$ customers will arrive at the system in average ($\rho_k := \frac{\lambda}{\mu_k}$). To make the system stable, it requites that $c[\rho_0(1+\frac{\alpha_0}{\beta_0}) + \sum\limits_{k=1}^{m} r_k\rho_k(1+\frac{\alpha_k}{\beta_k})] < 1$.

Under the stability condition, the steady state probabilities and the corresponding probability generating functions can be obtained by applying the well-known $Tauberian$ property:

$$\lim_{s\to 0} sf^*(s) = \lim_{t\to\infty} f(t),$$

and thus we have

$$\begin{aligned} I &= \lim_{t\to\infty} I(t) = \lim_{s\to 0} sI^*(s), \\ P_q(z) &= \lim_{t\to\infty} P_q(t,z) = \lim_{s\to 0} sP_q^*(s,z). \end{aligned}$$

Multiplying both sides of equation (3.32) by s, taking $s \to 0$ and simplifying, we have

$$P_q(z) = \lim_{s\to 0} sP_q^*(s,z) = \frac{[1 - r_0 b_0^*(\Phi_0(0,z)) - b_0^*(\Phi_0(0,z)) \sum\limits_{k=1}^{m} r_k b_k^*(\Phi_k(0,z))] I}{r_0 b_0^*(\Phi_0(0,z)) + b_0^*(\Phi_0(0,z)) \sum\limits_{k=1}^{m} r_k b_k^*(\Phi_k(0,z)) - z}. \tag{4.1}$$

Letting $z = 1$ in equation (4.1), $P_q(z)$ in the above equation is indeterminate of the 0/0 form. Applying L'Hopital's rule, we can obtain after simplifying that

$$P_q(1) = \lim_{z\to 1} P_q(z) = \frac{[-c\rho_0(1+\frac{\alpha_0}{\beta_0}) - c\sum_{k=1}^{m} r_k\rho_k(1+\frac{\alpha_k}{\beta_k})]I}{c\rho_0(1+\frac{\alpha_0}{\beta_0}) + c\sum_{k=1}^{m} r_k\rho_k(1+\frac{\alpha_k}{\beta_k}) - 1}. \tag{4.2}$$

Since $I + P_q(1) = 1$, we get

$$I = 1 - c\rho_0\Big(1+\frac{\alpha_0}{\beta_0}\Big) - c\sum_{k=1}^{m} r_k\rho_k\Big(1+\frac{\alpha_k}{\beta_k}\Big), \tag{4.3}$$

where $c\rho_0(1+\frac{\alpha_0}{\beta_0}) - c\sum_{k=1}^{m} r_k\rho_k(1+\frac{\alpha_k}{\beta_k}) < 1$ turns out to be the stability condition.

Consequently, we can get the probability that the server is busy and under repair in the steady state:

$$P = \lim_{z\to 1}\lim_{s\to 0} s\sum_{k=0}^{m} P_k^*(s,z) = c\Big(\rho_0 + \sum_{k=1}^{m} r_k\rho_k\Big), \tag{4.4}$$

$$R = \lim_{z\to 1}\lim_{s\to 0} s\sum_{k=0}^{m} R_k^*(s,z) = c\Big(\rho_0\frac{\alpha_0}{\beta_0} + \sum_{k=1}^{m} r_k\rho_k\frac{\alpha_k}{\beta_k}\Big). \tag{4.5}$$

Furthermore, let $P(z)$ denote the number of customers in the system, we have

$$P(z) = I + zP_q(z),$$

and thus

$$\begin{aligned} P(z) =& \Big\{1 + \frac{z[1 - r_0 b_0^*(\Phi_0(0,z)) - b_0^*(\Phi_0(0,z))\sum_{k=1}^{m} r_k b_k^*(\Phi_k(0,z))]}{r_0 b_0^*(\Phi_0(0,z)) + b_0^*(\Phi_0(0,z))\sum_{k=1}^{m} r_k b_k^*(\Phi_k(0,z)) - z}\Big\} \\ &\times \Big[1 - c\rho_0\Big(1+\frac{\alpha_0}{\beta_0}\Big) - c\sum_{k=1}^{m} r_k\rho_k\Big(1+\frac{\alpha_k}{\beta_k}\Big)\Big]. \end{aligned} \tag{4.6}$$

4.1 The Mean Queue Length in the Steady State

Define L_q as the mean number of customers who wait for service in the queue in the steady state, then $L_q = \frac{d}{dz}P_q(z)|_{z=1}$. Let $P_q(z) = N(z)/D(z)$, where $N(z)$ and $D(z)$, respectively, denote the numerator and denominator of the right hand of (4.1). Applying L'Hopital's rule twice, we get

$$L_q = \lim_{z\to 1} P_q^{'}(z) = \frac{N^{''}(1)D^{'}(1) - D^{''}(1)N^{'}(1)}{2(D^{'}(1))^2}.$$

Carrying out the desired derivatives, after simplifying we have

$$L_q = \frac{1}{2\big[1 - c\rho_0(1+\frac{\alpha_0}{\beta_0}) - c\sum_{k=1}^{m} r_k\rho_k(1+\frac{\alpha_k}{\beta_k})\big]}$$

$$
\begin{aligned}
&\times\Big\{\sum_{i=1}^{\infty} i(i-1)C_i\Big[\rho_0\Big(1+\frac{\alpha_0}{\beta_0}\Big)+\sum_{k=1}^{m} r_k\rho_k\Big(1+\frac{\alpha_k}{\beta_k}\Big)\Big]\\
&+\lambda^2c^2\Big[\Big(1+\frac{\alpha_0}{\beta_0}\Big)^2EV_0^2+\sum_{k=1}^{m} r_k\Big(1+\frac{\alpha_k}{\beta_k}\Big)^2EV_k^2\Big]\\
&+2c^2\rho_0\Big(1+\frac{\alpha_0}{\beta_0}\Big)\sum_{k=1}^{m} r_k\rho_k\Big(1+\frac{\alpha_k}{\beta_k}\Big)\\
&+\lambda c^2\Big[\alpha_0\rho_0EW_0^2+\sum_{k=1}^{m} r_k\alpha_k\rho_kEW_k^2\Big]\Big\}. \qquad (4.7)
\end{aligned}
$$

Denote by L the mean number of customers in the system, then we have

$$
L = L_q + (1-I) = L_q + c\rho_0\Big(1+\frac{\alpha_0}{\beta_0}\Big) + c\sum_{k=1}^{m} r_k\rho_k\Big(1+\frac{\alpha_k}{\beta_k}\Big). \qquad (4.8)
$$

4.2 Mean Waiting Time

In this subsection, we will study the customers' mean waiting time W_q and mean sojourn time W in the system under the FCFS discipline. In statistical equilibrium, we assume that upon a batch of customers' arrival there are N customers in the system. If $N = 0$, one customer in the batch will receive the service immediately. If $N \geq 1$, the first one who will receive his service must wait for a period in which all those N customers have finished their services. Therefore, his waiting time will be the sum of these N service times. Here we denote the remaining service time of the customer in service by $\widehat{U}$. The other $N-1$ customers' service times are $U_1, U_2, \cdots, U_{N-1}$. Thus, the first customer's waiting time is

$$
U_{1q} = U_1 + U_2 + \cdots + U_{N-1} + \widehat{U},
$$

the mean waiting time of this customer is

$$
\begin{aligned}
W_{1q} =& E(U_{1q}) = \sum_{n=1}^{\infty} P_nE[U_{1q}|N=n]\\
=& \sum_{n=1}^{\infty} P_nE[U_1+U_2+\cdots+U_{n-1}+\widehat{U}]\\
=& \sum_{n=1}^{\infty} P_n(n-1)EU_1 + \sum_{n=1}^{\infty} P_nE\widehat{U}\\
=& L_qEU_1 + (1-I)E\widehat{U} \qquad (4.9)
\end{aligned}
$$

where P_n is the steady state distribution of queue length, $\sum\limits_{n=1}^{\infty} P_n = 1-I$.

By the previous results, we have

$$
EU_1 = ET_{n,0} + ET_{n,k} = \frac{1}{\mu_0}\Big(1+\frac{\alpha_0}{\beta_0}\Big) + \sum_{k=1}^{m}\frac{r_k}{\mu_k}\Big(1+\frac{\alpha_k}{\beta_k}\Big). \qquad (4.10)
$$

In addition, it is easy to see that

$$\begin{aligned}P\{U_1 \le t\} =& r_0 P\{U_1 \le t | \text{ the customer leaves without second service}\}\\ &+\sum_{k=1}^{m} r_k P\{U_1 \le t | \text{ the customer selects } k\text{-type second service}\}\\ =& r_0 \widehat{B}_0(t) + \sum_{k=1}^{m} r_k (\widehat{B}_0 * \widehat{B}_k)(t),\end{aligned}\tag{4.11}$$

where "$*$" represents convolution of distributions. The mean remaining service time is given by

$$\begin{aligned}E\widehat{U} =& \frac{1}{\frac{1}{\mu_0}(1+\frac{\alpha_0}{\beta_0}) + \sum\limits_{k=1}^{m} \frac{r_k}{\mu_k}(1+\frac{\alpha_k}{\beta_k})} \int_0^\infty t\Big[1 - r_0\widehat{B}_0(t) - \sum_{k=1}^{m} r_k(\widehat{B}_0 * \widehat{B}_k)(t)\Big] dt\\ =& \frac{1}{\frac{1}{\mu_0}(1+\frac{\alpha_0}{\beta_0}) + \sum\limits_{k=1}^{m} \frac{r_k}{\mu_k}\big(1+\frac{\alpha_k}{\beta_k}\big)}\\ &\cdot \Big\{ \int_0^\infty r_0 t\Big[1-\widehat{B}_0(t)\Big]dt + \sum_{k=1}^{m}\int_0^\infty r_k t[1-(\widehat{B}_0 * \widehat{B}_k)(t)]dt\Big\}\\ =& \frac{1}{\frac{1}{\mu_0}(1+\frac{\alpha_0}{\beta_0}) + \sum_{k=1}^{m} \frac{r_k}{\mu_k}(1+\frac{\alpha_k}{\beta_k})}\Big\{\frac{r_0}{2}ET_{n,0}^2 + \sum_{k=1}^{m}\frac{r_k}{2}E[T_{n,0}+T_{n,k}]^2\Big\}\\ =& \frac{1}{\frac{1}{\mu_0}\big(1+\frac{\alpha_0}{\beta_0}\big) + \sum\limits_{k=1}^{m} \frac{r_k}{\mu_k}\big(1+\frac{\alpha_k}{\beta_k}\big)}\Big\{\frac{1}{2}\Big[\frac{\alpha_0}{\mu_0}EW_0^2 + \Big(1+\frac{\alpha_0}{\beta_0}\Big)^2 EV_0^2\Big]\\ &+\sum_{k=1}^{m}\frac{r_k}{2}\Big[\frac{\alpha_k}{\mu_k}EW_k^2 + \Big(1+\frac{\alpha_k}{\beta_k}\Big)^2 EV_k^2 + \frac{2}{\mu_0\mu_k}\Big(1+\frac{\alpha_0}{\beta_0}\Big)\Big(1+\frac{\alpha_k}{\beta_k}\Big)\Big]\Big\}.\end{aligned}\tag{4.12}$$

Substituting (4.3), (4.7), (4.10), (4.12) into (4.9), after simplifying, we get the mean waiting time of the customer:

$$\begin{aligned}W_{1q} =& L_q EU_1 + (1-I)E\widehat{U}\\ =& \frac{\frac{1}{\mu_0}(1+\frac{\alpha_0}{\beta_0}) + \sum\limits_{k=1}^{m}\frac{r_k}{\mu_k}(1+\frac{\alpha_k}{\beta_k})}{2\big[1-c\rho_0(1+\frac{\alpha_0}{\beta_0}) - c\sum\limits_{k=1}^{m} r_k\rho_k(1+\frac{\alpha_k}{\beta_k})\big]}\\ &\Big\{\sum_{i=1}^{\infty} i(i-1)C_i\Big[\rho_0\Big(1+\frac{\alpha_0}{\beta_0}\Big) + \sum_{k=1}^{m} r_k\rho_k\Big(1+\frac{\alpha_k}{\beta_k}\Big)\Big]\\ &+\lambda^2c^2\Big[\Big(1+\frac{\alpha_0}{\beta_0}\Big)^2 EV_0^2 + \sum_{k=1}^{m} r_k\Big(1+\frac{\alpha_k}{\beta_k}\Big)^2 EV_k^2\Big]\\ &+2c^2\rho_0\Big(1+\frac{\alpha_0}{\beta_0}\Big)\sum_{k=1}^{m} r_k\rho_k\Big(1+\frac{\alpha_k}{\beta_k}\Big)\\ &+\lambda c^2\Big[\alpha_0\rho_0 EW_0^2 + \sum_{k=1}^{m} r_k\alpha_k\rho_k EW_k^2\Big]\Big\}\\ &+\lambda c\Big\{\frac{1}{2}\Big[\frac{\alpha_0}{\mu_0}EW_0^2 + \Big(1+\frac{\alpha_0}{\beta_0}\Big)^2 EV_0^2\Big]\end{aligned}$$

$$+\sum_{k=1}^{m}\frac{r_k}{2}\Big[\Big(\frac{\alpha_k}{\mu_k}EW_k^2+\Big(1+\frac{\alpha_k}{\beta_k}\Big)^2EV_k^2\Big)$$
$$+\frac{2}{\mu_0\mu_k}\Big(1+\frac{\alpha_0}{\beta_0}\Big)\Big(1+\frac{\alpha_k}{\beta_k}\Big)\Big]\Big\}. \tag{4.13}$$

The mean waiting time of the j-th customer receiving service in the same batch is given by

$$W_{jq}=W_{1q}+(j-1)EU_1.$$

Furthermore, if the size of this batch is i, and because the service discipline is random service in the same batch, we get the mean waiting time of this batch customers

$$\begin{aligned}W_q^i=&\frac{1}{i}(W_{1q}+W_{2q}+\cdots+W_{iq})\\=&\frac{1}{i}\Big[iW_{1q}+\frac{i(i-1)}{2}EU_1\Big]\\=&W_{1q}+\frac{i-1}{2}EU_1.\end{aligned} \tag{4.14}$$

Therefore, the mean waiting time of any given customer in the queue is

$$\begin{aligned}W_q=&\sum_{i=1}^{\infty}P(W_q^i\mid\text{the customer from a batch whose size is } i)\\&\times P(\text{the customer from a batch whose size is } i)\\=&\sum_{i=1}^{\infty}\frac{iC_i}{EX}\Big[W_{1q}+\frac{i-1}{2}EU_1\Big]\\=&W_{1q}+\frac{EU_1}{2EX}\sum_{i=1}^{\infty}i(i-1)C_i\\=&W_{1q}+\frac{\frac{1}{\mu_0}(1+\frac{\alpha_0}{\beta_0})+\sum\limits_{k=1}^{m}\frac{r_k}{\mu_k}(1+\frac{\alpha_k}{\beta_k})}{2c}[EX^2-c].\end{aligned} \tag{4.15}$$

Finally we get the mean sojourn time of a customer

$$W=W_q+EU_1=W_q+\frac{1}{\mu_0}\Big(1+\frac{\alpha_0}{\beta_0}\Big)+\sum_{k=1}^{m}\frac{r_k}{\mu_k}\Big(1+\frac{\alpha_k}{\beta_k}\Big). \tag{4.16}$$

Remark 4.1. When $\alpha_0=\alpha_k=0,\ 1\le k\le m$, our model becomes the $M^{[X]}/G/1$ queue with second multi-optional services and reliable server. The results obtained in this paper, e.g., (4.3), (4.7), (4.8), (4.15), (4.16), are consistent with equations (32), (41), (42), (48), and (49) in [6] respectively.

5 Reliability Analysis

We now consider some reliability quantities of the server in this section. Let $A(t)$ be the system availability at time t, that is, $A(t)=P\{\text{the server is up at time } t\}$, and define the steady-state availability of the server as $A = \lim_{t\to\infty} A(t)$. By definition, its Laplace transform is given by

$$A^*(s) = I^*(s) + \sum_{k=0}^{m} P_k^*(s,1).$$

Based on the results obtained above, we have

Theorem 5.1. *The steady-state availability of the server is*

$$A^*(s) = \frac{1 - b_0^*(\Phi_0(s,1)) + \Phi_0(s,1) b_0^*(\Phi_0(s,1)) \sum\limits_{k=1}^{m} \frac{r_k}{\Phi_k(s,1)}[1 - b_k^*(\Phi_k(s,1))]}{\Phi_0(s,1)[r_0 b_0^*(\Phi_0(s,1)) + b_0^*(\Phi_0(s,1)) \sum\limits_{k=1}^{m} r_k b_k^*(\Phi_k(s,1)) - 1]} \times \Big[\frac{s}{s+\lambda-\lambda C(z_s)} - 1\Big] + \frac{1}{s+\lambda-\lambda C(z_s)}. \tag{5.1}$$

Multiplying both sides of (5.1) by s, and letting $s \to 0$, we get the steady state availability of the server.

Corollary 5.2. *The steady state availability of the server is given by*

$$A = 1 - c\rho_0 \frac{\alpha_0}{\beta_0} - c\sum_{k=1}^{m} r_k \rho_k \frac{\alpha_k}{\beta_k}. \tag{5.2}$$

Proof. The result follows directly from Theorem 5.1 by considering

$$A = \lim_{s\to 0} sA^*(s) = 1 - c\rho_0 \frac{\alpha_0}{\beta_0} - c\sum_{k=1}^{m} r_k \rho_k \frac{\alpha_k}{\beta_k}.$$

□

Let $M_0(t)$ be the expected number of failures of the server in the first "essential" service, and $M_k(t)$ the k-type second "optional" service up to time t, given that the system is initially empty and the server is idle. We have the following results by using the method in [4]:

$$M_i^*(s) = \sum_{n=0}^{\infty} \int_0^{\infty} \alpha_i P_{n,i}^*(s,x)dx = \alpha_i P_i^*(s,1), \qquad 0 \le i \le m.$$

By using the results in Section 3, we have

Theorem 5.3. *The Laplace-Stieltjes transforms of $M_0(t)$ and $M_k(t), 1 \le k \le m$, are given by*

$$M_0^*(s) = \frac{\alpha_0[1 - b_0^*(\Phi_0(s,1))][sI^*(s) - 1]}{\Phi_0(s,1)[r_0 b_0^*(\Phi_0(s,1)) + b_0^*(\Phi_0(s,1)) \sum\limits_{k=1}^{m} b_k^*(\Phi_k(s,1)) - 1]}, \tag{5.3}$$

$$M_k^*(s) = \frac{\alpha_k r_k b_0^*(\Phi_0(s,1))[1 - b_k^*(\Phi_k(s,1))][sI^*(s) - 1]}{\Phi_k(s,1)[r_0 b_0^*(\Phi_0(s,1)) + b_0^*(\Phi_0(s,1)) \sum\limits_{k=1}^{m} b_k^*(\Phi_k(s,1)) - 1]}, \qquad 1 \le k \le m. \tag{5.4}$$

The steady state failure frequency of the server is given by

$$M_f = \lim_{s\to 0} s\sum_{k=0}^{m} M_k^*(s) = c\Big(\alpha_0\rho_0 + \sum_{k=1}^{m} r_k \rho_k \alpha_k\Big). \tag{5.5}$$

Denote by τ the time to the first failure of the server, then the reliability function of the server is

$$R(t) = P(\tau > t)\,.$$

In order to find the reliability of the server, letting the failure state of the server be the absorbing state, then we obtain a new system. In the new system, we use the same notations as in the previous section, then we get the following set of equations:

$$\Big(\frac{d}{dt} + \lambda\Big)I(t) = r_0 \int_0^\infty P_{0,0}(t,x)\mu_0(x)dx + \sum_{k=1}^m \int_0^\infty P_{0,k}(t,x)\mu_k(x)dx, \tag{5.6}$$

$$\Big[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu_k(x) + \lambda + \alpha_k\Big]P_{n,k}(t,x) = \lambda \sum_{i=1}^n C_i P_{n-i,k}(t,x), \qquad 0 \le k \le m, \tag{5.7}$$

with boundary conditions:

$$\begin{aligned} P_{n,0}(t,0) = r_0 \int_0^\infty & P_{n+1,0}(t,x)\mu_0(x)dx \\ &+ \sum_{k=1}^m \int_0^\infty P_{n+1,k}(t,x)\mu_k(x)dx + \lambda C_{n+1} I(t) \end{aligned} \tag{5.8}$$

$$P_{n,k}(t,0) = r_k \int_0^\infty P_{n,0}(t,x)\mu_0(x)dx, \qquad 1 \le k \le m. \tag{5.9}$$

Similar to the method used in Section 3, we get

$$I^*(s) = \frac{1}{s + \lambda + \lambda C(z_m)}, \tag{5.10}$$

$$P_0^*(s,x,z) = \frac{\{[s+\lambda-\lambda C(z)]I^*(s) - 1\}\exp\{-\varphi_0(s,z)\}[1-B_0(x)]}{r_0 b_0^*(\varphi_0(s,z)) + b_0^*(\varphi_0(s,z)) \sum\limits_{k=1}^m r_k b_k^*(\varphi_k(s,z)) - z}, \tag{5.11}$$

$$P_k^*(s,x,z) = \frac{r_k b_0^*(\varphi_0(s,z))\{[s+\lambda-\lambda C(z)]I^*(s) - 1\}\exp\{-\varphi_k(s,z)\}[1-B_k(x)]}{r_0 b_0^*(\varphi_0(s,z)) + b_0^*(\varphi_0(s,z)) \sum\limits_{k=1}^m r_k b_k^*(\varphi_k(s,z)) - z} \tag{5.12}$$

where

$$\varphi_i(s,z) = s + \lambda + \alpha_i - \lambda C(z), \qquad 0 \le i \le m, \quad 1 \le k \le m,$$

and z_m is the root of the following equation inside $|z| = 1$ when $\mathrm{Re}(s) > 0$:

$$x = r_0 b_0^*(\varphi_0(s,x)) + b_0^*\big(\varphi_0(s,x)\big) \sum_{k=1}^m r_k b_k^*\big(\varphi_k(s,x)\big).$$

By definition, we have

$$R^*(s) = I^*(s) + \lim_{z\to 1^-} \sum_{k=0}^m \int_0^\infty P_k^*(s,x,z)dx.$$

Combining the above equation with (5.10)–(5.12), after simplifying we get

Theorem 5.4. *The Laplace transform of $R(t)$ is given by*

$$R^*(s) = \frac{\frac{1-b_0^*(s+\alpha_0)}{s+\alpha_0} + b_0^*(s+\alpha_0)\sum\limits_{k=1}^{m}\frac{r_k[1-b_k^*(s+\alpha_k)]}{s+\alpha_k}}{r_0 b_0^*(s+\alpha_0) + b_0^*(s+\alpha_0)\sum\limits_{k=1}^{m} r_k b_k^*(s+\alpha_k) - 1} \times \Big(\frac{s}{s+\lambda-\lambda C(z_m)} - 1\Big) + \frac{1}{s+\lambda-\lambda C(z_m)}. \tag{5.13}$$

Based on (5.13), we get the mean time to the first failure (MTTFF) of the server:

$$\mathrm{MTTFF} = \int_0^\infty R(t)dt = R^*(s)|_{s=0}.$$

Corollary 5.5. *The mean time to the first failure (MTTFF) of the server is given by:*

$$\mathrm{MTTFF} = I^*(0) + c\Big(\rho_0 + \sum_{k=1}^{m} r_k\rho_k\Big)\frac{\frac{1-b_0^*(\alpha_0)}{\alpha_0} + b_0^*(\alpha_0)\sum\limits_{k=1}^{m}\frac{r_k[1-\widetilde{b}_k(\alpha_k)]}{\alpha_k}}{1 - r_0 b_0^*(\alpha_0) - b_0^*(\alpha_0)\sum\limits_{k=1}^{m} r_k b_k^*(\alpha_k)}. \tag{5.14}$$

Remark 5.6. (1) If the service times of the second optional service are exponentially distributed, there is only one customer in every batch, and there is only one type of second optional service, then the queueing quantities and reliability indices obtained in this paper are the same as those in [5].

(2) Further simplifying our model by the assumptions that server is reliable based on (1) leads to the original model studied in Madan[2]. It is easy to verify that our results are consistent with those in Madan[2].

6 Numerical Examples

In this section, we present some numerical examples to study the impact of system parameters on the system performance. To this end, we consider a special case when $k = 1$, i.e., the considered queueing system with 1-type second service, to show some sensitivities for α_0, α_1, and r_0.

We assume that the service time distributions and the repair time distributions in the first and second service are all exponentially distributed with parameters μ_0, β_0 and μ_1, β_1, respectively. We also assume that the batch size follows a geometric distribution with mean $c = (1-\sigma)^{-1}$ such that $C_k = (1-\sigma)\sigma^{k-1},\ k \geq 1, \sigma \in [0, 1)$.

Firstly, we investigate the impact of different values of the failure rate α_0 on the steady-state availability A of the server under the ergodicity condition. Without loss of generality, we set the arrival rate $\lambda = 1$ and the mean value of the batch size $c = 1.2$. We select a set of different values of the failure rate α_0 to represent the unreliability of the server during the first service. For seven different values of r_0, from 0 to 1, the steady-state availability A of the server is evaluated as a function of the failure rate α_0, and detailed values are given in Table 1. It is observed that the larger failure rate α_0, the smaller the steady-state availability A of the server.

On the other hand, as we expect, A increases with increasing values of r_0 since there are less customers needing the second optional service (with probability $1 - r_0$).

Similarly, the impact of different values of the failure rate α_1 of the server (in the second optional service phase) on the steady-state availability A of the server is illustrated in Table 2. The steady-state availability A of the server is evaluated as a function of the failure rate α_1 and r_0. It shows that the failure rate α_1 also has a significant impact on the steady-state availability A of the server.

Table 1. The effects of the failure rate α_0 on the availability A

$r_0 =$	0	0.2	0.4	0.5	0.6	0.8	1
α_0	Steady-state availability A of the server						
0	0.3950	0.4360	0.4770	0.4975	0.5180	0.5590	0.6000
0.10	0.3750	0.4160	0.4570	0.4775	0.4980	0.5390	0.5800
0.20	0.3550	0.3960	0.4370	0.4575	0.4780	0.5190	0.5600
0.50	0.2950	0.3360	0.3770	0.3975	0.4180	0.4590	0.5000
0.75	0.2450	0.2860	0.3270	0.3475	0.3680	0.4090	0.4500
1.00	0.1950	0.2360	0.2770	0.2975	0.3180	0.3590	0.4000
1.50	0.0950	0.1360	0.1770	0.1975	0.2180	0.2590	0.3000
2.00		0.0360	0.0770	0.0975	0.1180	0.1590	0.2000
2.25			0.0270	0.0475	0.0680	0.1090	0.1500
2.50					0.0180	0.0590	0.1000

$c = 1.2,\ \lambda = 1,\ \mu_0 = 3,\ \mu_1 = 6,\ \beta_0 = 2,\ \beta_1 = 4,\ \alpha_1 = 0.1$

Table 2. The effects of the failure rate α_1 on the availability A

$r_0 =$	0	0.2	0.4	0.5	0.6	0.8	1
α_1	Steady-state availability A of the server						
0	0.3800	0.4200	0.4600	0.4800	0.5000	0.5400	0.5800
0.10	0.3750	0.4160	0.4570	0.4775	0.4980	0.5390	0.5800
0.20	0.3700	0.4120	0.4540	0.4750	0.4960	0.5380	0.5800
0.50	0.3550	0.4000	0.4450	0.4675	0.4900	0.5350	0.5800
0.75	0.3425	0.3900	0.4375	0.4613	0.4850	0.5325	0.5800
1.00	0.3300	0.3800	0.4300	0.4550	0.4800	0.5300	0.5800
1.50	0.3050	0.3600	0.4150	0.4425	0.4700	0.5250	0.5800
2.00	0.2800	0.3400	0.4000	0.4300	0.4600	0.5200	0.5800
2.25	0.2675	0.3300	0.3925	0.4238	0.4550	0.5175	0.5800
2.50	0.2550	0.3200	0.3850	0.4175	0.4500	0.5150	0.5800

$c = 1.2,\ \lambda = 1,\ \mu_0 = 3,\ \mu_1 = 6,\ \beta_0 = 2,\ \beta_1 = 4,\ \alpha_0 = 0.1$

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