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AN ANISOTROPIC REGULARITY CRITERION FOR THE 3D NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we establish an anisotropic regularity criterion for the 3D incompressible Navier-Stokes equations. It is proved that a weak solution u is regular on $[0,T]$, provided $\frac{\partial u_3}{\partial x_3} \in L^{t_1}(0,T;L^{s_1}(\mathbf{R}^3))$, with $\frac{2}{t_1}+\frac{3}{s_1} \le 2$, $s_1 \in (\frac{3}{2},+\infty]$ and $\nabla_h u_3 \in L^{t_2}(0,T;L^{s_2}(\mathbf{R}^3))$, with either $\frac{2}{t_2}+\frac{3}{s_2} \le \frac{19}{12}+\frac{1}{2s_2}$, $s_2 \in (\frac{30}{19},3]$ or $\frac{2}{t_2}+\frac{3}{s_2} \le \frac{3}{2}+\frac{3}{4s_2}$, $s_2 \in (3,+\infty]$. Our result in fact improves a regularity criterion of Zhou and Pokorný [Nonlinearity **23** (2010), 1097–1107].

1. **Introduction.** We consider the following three-dimensional Navier-Stokes equations (NSE) of viscous incompressible fluids in $(0,T)\times\mathbf{R}^3$,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \nu\Delta u, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $u = (u_1(t,x),u_2(t,x),u_3(t,x)) : (0,T)\times\mathbf{R}^3 \to \mathbf{R}^3$ is the velocity field, $p(t,x) : (0,T)\times\mathbf{R}^3 \to \mathbf{R}$ is the scalar pressure, $u_0(x)$ with $\operatorname{div} u_0 = 0$ in the sense of distribution is the initial velocity field, and $\nu > 0$ is the viscosity. Since the value of the viscosity does not play any role in our further considerations, we assume $\nu = 1$ in the sequel.

The existence of a weak solution to the three-dimensional Navier-Stokes equations is well known since the pioneering works by Leray [1] and Hopf [2]. However, its uniqueness and global regularity are still major challenging open problems in applied analysis.

On the other hand, starting from the famous papers of Prodi [3] and of Serrin [4], many sufficient conditions ensuring the smoothness of a weak solution are known. The classical Prodi-Serrin's type criteria (see [3, 4], and for the case $s = 3$, see [5]) say that if a weak solution u additionally belongs to $L^t(0,T;L^s(\mathbf{R}^3))$, with $\frac{2}{t}+\frac{3}{s} = 1$, $s \in [3,+\infty]$, then it is regular and unique in the class of all weak

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solutions satisfying the following energy inequality

$$\|u(t,\cdot)\|_{L^2(\mathbf{R}^3)}^2 + \int_0^t \|\nabla u(\tau,\cdot)\|_{L^2(\mathbf{R}^3)}^2 \, d\tau \le \|u(0,\cdot)\|_{L^2(\mathbf{R}^3)}^2,$$

for all $t \in [0,+\infty)$. Analogous result in terms of the gradient of velocity, i.e., $\nabla u \in L^t(0,T;L^s(\mathbf{R}^3))$, with $\frac{2}{t}+\frac{3}{s}=2$, $s \in [\frac{3}{2},+\infty]$ is established by Beirão da Veiga (see [6]).

Later, some articles were dedicated to providing regularity criteria via one velocity component. The first result in this direction is due to Neustupa, Novotný and Penel [7] (see also Zhou [8]), where they proved that if $u_3 \in L^t(0,T;L^s(\mathbf{R}^3))$, with $\frac{2}{t}+\frac{3}{s}=\frac{1}{2}$, $s \in (6,+\infty]$, then the solution is smooth. A similar criterion in terms of the gradient of one velocity component is independently obtained by Zhou [9] and Pokorný [10].

Recently, several interesting improvements appeared (see, e.g., [13, 12, 11, 14, 15]). In particular, by applying the multiplicative embedding theorem, Cao and Titi [11] showed the smoothness under the condition $u_3 \in L^t(0,T;L^s(\mathbf{R}^3))$, $\frac{2}{t}+\frac{3}{s} < \frac{2}{3}+\frac{1}{3s}$, $s > \frac{7}{2}$. Based on the method from [11], Zhou and Pokorný [15] proved that the weak solution is regular, provided $u_3 \in L^t(0,T;L^s(\mathbf{R}^3))$, with $\frac{2}{t}+\frac{3}{s} \le \frac{3}{4}+\frac{1}{2s}$, $s > \frac{10}{3}$ or

$$\nabla u_3 \in L^t(0,T;L^s(\mathbf{R}^3)),\ \frac{2}{t}+\frac{3}{s} \le \begin{cases} \dfrac{19}{12}+\dfrac{1}{2s},\ s \in \left(\dfrac{30}{19},3\right], \\ \dfrac{2}{t}+\dfrac{3}{s} \le \dfrac{3}{2}+\dfrac{3}{4s},\ s \in (3,+\infty]. \end{cases} \tag{2}$$

In view of the divergence free condition $\operatorname{div} u = \frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0$, we believe that $\frac{\partial u_i}{\partial x_i}$ may contain more useful information than $\frac{\partial u_i}{\partial x_j}$ $(i \ne j)$ when we estimate the convective term. Therefore, a natural question is whether the criterion of $\frac{\partial u_3}{\partial x_3}$ in (2) can be relaxed to the natural scaling of the Navier-Stokes equations: $\frac{2}{t}+\frac{3}{s}=2$. In this paper, we give a positive answer to this question.

Before we state the main theorem, let us introduce some notations which will be used in what follows. We will use $\|\cdot\|_p$ to represent the norm of the standard Lebesgue spaces $L^p(\mathbf{R}^3)$ and denote by $L^{t,s}$ the spaces $L^t(0,T;L^s(\mathbf{R}^3))$ for fixed T. We set $\nabla_h = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ to be the horizontal gradient, $\nabla_h \nabla u$ to be the tensor $(\frac{\partial^2 u_i}{\partial x_j \partial x_k})_{ijk}$ $(j=1,2;\ k,\ i=1,2,3)$, while $\Delta_h = \frac{\partial^2}{\partial x_1 \partial x_1}+\frac{\partial^2}{\partial x_2 \partial x_2}$ is the horizontal Laplacian. In addition, we denote

$$J^2(t) = \sup_{\tau \in (0,t)} \|\nabla_h u(\tau)\|_2^2 + \int_0^t \|\nabla_h \nabla u(\tau)\|_2^2 \, d\tau,$$

which plays an important role in our proof.

The main result of this paper reads:

Theorem 1.1. *Let $u_0 \in H^1(\boldsymbol{R}^3)$ with $\operatorname{div} u_0 = 0$ and u be a Leray-Hopf weak solution to the NSE (1) corresponding to u_0. Let additionally*

$$\frac{\partial u_3}{\partial x_3} \in L^{t_1,s_1},\ \frac{2}{t_1}+\frac{3}{s_1} \le 2,\ s_1 \in \left(\frac{3}{2},+\infty\right], \tag{3}$$

and

$$\nabla_h u_3 \in L^{t_2,s_2},\ \frac{2}{t_2}+\frac{3}{s_2} \le \begin{cases} \dfrac{19}{12}+\dfrac{1}{2s_2},\ s_2 \in \left(\dfrac{30}{19},3\right], \\ \dfrac{2}{t_2}+\dfrac{3}{s_2} \le \dfrac{3}{2}+\dfrac{3}{4s_2},\ s_2 \in (3,+\infty], \end{cases} \tag{4}$$

then u is regular on the interval $[0,T]$.

Remark 1. Note that although our condition added on $\nabla_h u_3$ is the same to Zhou and Pokorný's (see [15]), the condition added on $\frac{\partial u_3}{\partial x_3}$ reaches to the natural scaling of the Navier-stokes equations, i.e., $\frac{2}{t_1}+\frac{3}{s_1}=2$. Therefore, our result is an improvement of Zhou and Pokorný's result.

Remark 2. It is a quite challenging open problem to prove regularity provided $\nabla u_3 \in L^{t,s}$, with $\frac{2}{t}+\frac{3}{s}\le 2$, $s\in[\frac{3}{2},+\infty]$.

2. Proof of the main result.

In what follows we focus on the strong solution u on its maximal interval of existence $[0,T^*)$. Suppose $T>T^*$, then it is sufficient to show that the H^1 norm of the strong solution is bounded uniformly in time over $[0,T)$.

Our proof is under the framework of [15], but we estimate the convective term and $J^2(t)$ more carefully by a new decomposition method and viewpoint.

2.1. Estimates for $J^2(t)$.

Taking the inner product of the equation (1) with $-\Delta_h u$ in $L^2(\mathbf{R}^3)$, integrating by parts and taking the divergence free condition $\operatorname{div} u=0$ into account, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h u\|_2^2+\|\nabla_h\nabla u\|_2^2=\int_{\mathbf{R}^3}(u\cdot\nabla u)\cdot\Delta_h u\,dx. \tag{5}$$

Note that

$$\begin{aligned} &\int_{\mathbf{R}^3}(u\cdot\nabla u)\cdot\Delta_h u\,dx \\ =\ &\sum_{i,\,j=1}^{2}\int_{\mathbf{R}^3}u_i\frac{\partial u_j}{\partial x_i}\Delta_h u_j\,dx+\sum_{i=1}^{3}\int_{\mathbf{R}^3}u_i\frac{\partial u_3}{\partial x_i}\Delta_h u_3\,dx+\sum_{j=1}^{2}\int_{\mathbf{R}^3}u_3\frac{\partial u_j}{\partial x_3}\Delta_h u_j\,dx \\ =\ &:J_1+J_2+J_3. \end{aligned}$$

As in [13], we have

$$J_1=\frac{1}{2}\sum_{i,\,j=1}^{2}\int_{\mathbf{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\,dx-\int_{\mathbf{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\,dx+\int_{\mathbf{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\,dx.$$

Furthermore, by integration by parts, we find that

$$\begin{aligned} J_2 &= -\sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbf{R}^3}\frac{\partial u_i}{\partial x_k}\frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_k}\,dx \\ &= -\sum_{k=1}^{2}\left(\int_{\mathbf{R}^3}\frac{\partial u_3}{\partial x_k}\frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_k}\,dx+\sum_{i=1}^{2}\int_{\mathbf{R}^3}\frac{\partial u_i}{\partial x_k}\frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_k}\,dx\right), \\ J_3 &= \sum_{j,\,k=1}^{2}\left(\frac{1}{2}\int_{\mathbf{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}\,dx-\int_{\mathbf{R}^3}\frac{\partial u_3}{\partial x_k}\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_k}\,dx\right). \end{aligned}$$

Therefore, by Hölder's, Gagliardo-Nirenberg's and Young's inequalities, we have

$$\begin{aligned}
&\int_{\mathbf{R}^3} (u\cdot\nabla u)\cdot\Delta_h u\,dx \\
&\le C\int_{\mathbf{R}^3}\left|\frac{\partial u_3}{\partial x_3}\right| |\nabla_h u|^2\,dx + C\int_{\mathbf{R}^3}|\nabla_h u_3|\,|\nabla_h u|\,|\nabla u|\,dx \\
&\le C\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}\|\nabla_h u\|^2_{\frac{2s_1}{s_1-1}} + C\int_{\mathbf{R}^3}|\nabla_h u_3|\,|\nabla_h u|\,|\nabla u|\,dx \\
&\le C\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla_h u\|_2^2 + \varepsilon\|\nabla_h\nabla u\|_2^2 + C\int_{\mathbf{R}^3}|\nabla_h u_3|\,|\nabla_h u|\,|\nabla u|\,dx, \quad (6)
\end{aligned}$$

here and in what follows, $C>0$ denotes a sufficiently large universal constant which is allowed to change from line to line, while $\varepsilon>0$ is a sufficiently small universal constant which is also allowed to change from line to line.

Next, we estimate $J^2(t)$ separately for $s_2\in(\frac{3}{2},3]$ and $s_2\in(3,+\infty]$.

case (i) $s_2\in(\frac{3}{2},3]$.

We have

$$\begin{aligned}
&C\int_{\mathbf{R}^3}|\nabla_h u_3|\,|\nabla_h u|\,|\nabla u|\,dx \\
&\le C\|\nabla_h u_3\|_{s_2}\|\nabla_h u\|_6\|\nabla u\|_{\frac{6s_2}{5s_2-6}} \\
&\le C\|\nabla_h u_3\|_{s_2}\|\nabla_h\nabla u\|_2\|\nabla u\|_2^{\frac{2s_2-3}{s_2}}\|\nabla u\|_6^{\frac{3-s_2}{s_2}} \\
&\le C\|\nabla_h u_3\|_{s_2}\|\nabla_h\nabla u\|_2^{1+\frac{2}{3}\frac{3-s_2}{s_2}}\|\nabla u\|_2^{\frac{2s_2-3}{s_2}}\|\Delta u\|_2^{\frac{1}{3}\frac{3-s_2}{s_2}} \\
&\le C\|\nabla_h u_3\|_s^{\frac{6s_2}{5s_2-6}}\|\nabla u\|_2^{\frac{6(2s_2-3)}{5s_2-6}}\|\Delta u\|_2^{\frac{2(3-s_2)}{5s_2-6}} + \varepsilon\|\nabla_h\nabla u\|_2^2, \quad (7)
\end{aligned}$$

where we used Hölder's inequality, Young's inequality and the following multiplicative Gagliardo-Nirenberg inequality (see [11])

$$\|\nabla u\|_6 \le C\left\|\frac{\partial\nabla u}{\partial x_1}\right\|_2^{1/3}\left\|\frac{\partial\nabla u}{\partial x_2}\right\|_2^{1/3}\left\|\frac{\partial\nabla u}{\partial x_3}\right\|_2^{1/3}.$$

Combining (5), (6) with (7) yields

$$\begin{aligned}
&\frac{d}{dt}\|\nabla_h u\|_2^2 + \|\nabla_h\nabla u\|_2^2 \\
&\le C\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla_h u\|_2^2 + C\|\nabla_h u_3\|_{s_2}^{\frac{6s_2}{5s_2-6}}\|\nabla u\|_2^{\frac{6(2s_2-3)}{5s_2-6}}\|\Delta u\|_2^{\frac{2(3-s_2)}{5s_2-6}}.
\end{aligned}$$

Thus, by Gronwall's inequality and (3), we get

$$\begin{aligned}
&J^2(t) = \sup_{\tau\in(0,t)}\|\nabla_h u(\tau)\|_2^2 + \int_0^t\|\nabla_h\nabla u(\tau)\|_2^2\,d\tau \\
&\le C + C\int_0^t\|\nabla_h u_3\|_{s_2}^{\frac{6s_2}{5s_2-6}}\|\nabla u\|_2^{\frac{6(2s_2-3)}{5s_2-6}}\|\Delta u\|_2^{\frac{2(3-s_2)}{5s_2-6}}\,d\tau \\
&\le C + C\left(\int_0^t\|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\|\nabla u\|_2^2\,d\tau\right)^{\frac{3(2s_2-3)}{5s_2-6}}\left(\int_0^t\|\Delta u\|_2^2\,d\tau\right)^{\frac{3-s_2}{5s_2-6}}. \quad (8)
\end{aligned}$$

case (ii) $s_2 \in (3, +\infty]$.
By Hölder's, Gagliardo-Nirenberg's and Young's inequalities, we obtain

$$\begin{aligned} C\int_{\mathbf{R}^3} |\nabla_h u_3|\,|\nabla_h u|\,|\nabla u|\,dx &\leq C\,\|\nabla_h u_3\|_{s_2}\,\|\nabla u\|_2\,\|\nabla_h u\|_{\frac{2s_2}{s_2-2}} \\ &\leq C\,\|\nabla_h u_3\|_{s_2}\,\|\nabla u\|_2\,\|\nabla_h u\|_2^{1-\frac{3}{s_2}}\,\|\nabla_h u\|_6^{\frac{3}{s_2}} \\ &\leq \varepsilon\,\|\nabla_h \nabla u\|_2^2 + C\,\|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\,\|\nabla u\|_2^2. \end{aligned} \tag{9}$$

Similarly, combining (5), (6) with (9) yields

$$\frac{d}{dt}\|\nabla_h u\|_2^2 + \|\nabla_h \nabla u\|_2^2 \leq C\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla_h u\|_2^2 + C\,\|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\,\|\nabla u\|_2^2.$$

Thanks again to Gronwall's inequality and (3), we obtain

$$\begin{aligned} & J^2(t) = \sup_{\tau\in(0,t)}\|\nabla_h u(\tau)\|_2^2 + \int_0^t \|\nabla_h \nabla u(\tau)\|_2^2\,d\tau \\ \leq\ & C\exp\left(\int_0^t \left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\,d\tau\right)\left(\|\nabla_h u(0)\|_2^2 + \int_0^t \|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\,\|\nabla u\|_2^2\,d\tau\right) \\ \leq\ & C + C\int_0^t \|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\,\|\nabla u\|_2^2\,d\tau. \end{aligned} \tag{10}$$

2.2. **Uniform H^1 estimates for u.** This section is devoted to prove that the H^1 norm of the strong solution u is bounded uniformly in time over $[0, T^*)$. Similarly, by adding the inner product of the equation (1) with $-\Delta u$ in $L^2(\mathbf{R}^3)$, integrating by parts and taking the divergence free condition into account, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \|\Delta u\|_2^2 = \int_{\mathbf{R}^3}(u\cdot\nabla u)\cdot\Delta u\,dx \\ =\ & \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbf{R}^3} u_i\frac{\partial u_j}{\partial x_i}\Delta u_j\,dx + \sum_{j=1}^{3}\int_{\mathbf{R}^3} u_3\frac{\partial u_j}{\partial x_3}\Delta u_j\,dx \\ =\ & -\sum_{i=1}^{2}\sum_{j,\,k=1}^{3}\int_{\mathbf{R}^3}\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}\,dx + \frac{1}{2}\sum_{i=1}^{2}\sum_{j,\,k=1}^{3}\int_{\mathbf{R}^3}\frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}\,dx \\ & -\sum_{j=1}^{3}\int_{\mathbf{R}^3}\left(\sum_{k=1}^{2}\frac{\partial u_3}{\partial x_k}\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_k} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_j}{\partial x_3}\frac{\partial u_j}{\partial x_3}\right)dx \\ & +\frac{1}{2}\sum_{j,\,k=1}^{3}\int_{\mathbf{R}^3}\frac{\partial u_3}{\partial x_3}\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_k}dx \\ \leq\ & C\int_{\mathbf{R}^3}|\nabla_h u||\nabla u|^2\,dx + C\int_{\mathbf{R}^3}\left|\frac{\partial u_3}{\partial x_3}\right||\nabla u|^2\,dx =: I_1 + I_2. \end{aligned} \tag{11}$$

As before, by Hölder's, Gagliardo-Nirenberg's and Young's inequalities, we get

$$\begin{aligned} I_2 &\leq C\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}\|\nabla u\|^2_{\frac{2s_1}{s_1-1}} \\ &\leq C\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}\|\nabla u\|_2^{\frac{2s_1-3}{s_1}}\|\Delta u\|_2^{\frac{3}{s_1}} \\ &\leq C\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2+\varepsilon\|\Delta u\|_2^2, \end{aligned} \tag{12}$$

and by Hölder's and the multiplicative Gagliardo-Nirenberg inequalities, we obtain

$$\begin{aligned} I_1 &\leq C\|\nabla_h u\|_2\|\nabla u\|_4^2 \\ &\leq C\|\nabla_h u\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\nabla u\|_6^{\frac{3}{2}} \\ &\leq C\|\nabla_h u\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\nabla_h\nabla u\|_2\|\Delta u\|_2^{\frac{1}{2}}. \end{aligned} \tag{13}$$

Integrating (11) over the time interval $(0,t)$, $t<T^*$, and using the estimates (12) and (13) yields

$$\begin{aligned} &\|\nabla u(t)\|_2^2+\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau \\ \leq\;& C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau+C\int_0^t\|\nabla_h u\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\nabla_h\nabla u\|_2\|\Delta u\|_2^{\frac{1}{2}}\,d\tau \\ \leq\;& C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau \\ &+C\sup_{\tau\in(0,t)}\|\nabla_h u\|_2\left(\int_0^t\|\nabla_h\nabla u\|_2^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|\Delta u\|_2^2\,d\tau\right)^{\frac{1}{4}} \\ \leq\;& C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau+CJ^2(t)\left(\int_0^t\|\Delta u\|_2^2\,d\tau\right)^{\frac{1}{4}}. \end{aligned} \tag{14}$$

Now, by employing (3), (4) and the estimates of $J^2(t)$ from section 2.1, we can bound the H^1 norm of u as follows.

case (i) $s_2\in(\frac{3}{2},3]$.

Inserting (8) into (14), we get

$$\begin{aligned} &\|\nabla u(t)\|_2^2+\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau \\ \leq\;& C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau+C\left(\int_0^t\|\Delta u\|_2^2\,d\tau\right)^{\frac{1}{4}} \\ &+C\left(\int_0^t\|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\|\nabla u\|_2^2\,d\tau\right)^{\frac{3(2s_2-3)}{5s_2-6}}\left(\int_0^t\|\Delta u\|_2^2\,d\tau\right)^{\frac{3-s_2}{5s_2-6}+\frac{1}{4}}. \end{aligned}$$

If we restrict that $\frac{3-s_2}{5s_2-6}+\frac{1}{4}<1$, i.e., $s_2>\frac{30}{19}$, then by Hölder's and Young's inequalities, we have

$$\begin{aligned}
&\|\nabla u(t)\|_2^2+\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau\\
\leq\ & C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau+\varepsilon\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau\\
&+C\left(\int_0^t\|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\|\nabla u\|_2^2\,d\tau\right)^{\frac{3(2s_2-3)}{5s_2-6}\times\frac{20s_2-24}{19s_2-30}}\\
\leq\ & C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau+\varepsilon\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau\\
&+C\left(\int_0^t\|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}\times\frac{12(2s_2-3)}{19s_2-30}}\|\nabla u\|_2^2\,d\tau\right)^{\frac{3(2s_2-3)}{5s_2-6}\times\frac{20s_2-24}{19s_2-30}\times\frac{19s_2-30}{12(2s_2-3)}}\\
\leq\ & C+C\int_0^t\left(\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}+\|\nabla_h u_3\|_{s_2}^{\frac{24s_2}{19s_2-30}}\right)\|\nabla u\|_2^2\,d\tau+\varepsilon\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\nabla u(t)\|_2^2+\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau\\
\leq\ & C+C\int_0^t\left(\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}+\|\nabla_h u_3\|_{s_2}^{\frac{24s_2}{19s_2-30}}\right)\|\nabla u\|_2^2\,d\tau.
\end{aligned}$$

According to Gronwall's inequality, (3) and (4), we get

$$\|\nabla u(t)\|_2^2+\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau\leq C$$

uniformly in time t over $[0,T^*)$.

For the remaining case, $s_2\in(3,+\infty]$, we use the similar approach.

case (ii) $s_2\in(3,+\infty]$.

We have

$$\begin{aligned}
&\|\nabla u(t)\|_2^2+\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau\\
\leq\ & C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau\\
&+C\left[1+\int_0^t\|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\|\nabla u\|_2^2\,d\tau\right]\left(\int_0^t\|\Delta u\|_2^2\,d\tau\right)^{\frac{1}{4}}\\
\leq\ & C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau+\varepsilon\int_0^t\|\Delta u\|_2^2\,d\tau\\
&+C\left(\int_0^t\|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}}\|\nabla u\|_2^2\,d\tau\right)^{\frac{4}{3}}\\
\leq\ & C+C\int_0^t\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}\|\nabla u\|_2^2\,d\tau+\varepsilon\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t \|\nabla_h u_3\|_{s_2}^{\frac{2s_2}{2s_2-3}\times\frac{4}{3}}\|\nabla u\|_2^2\,d\tau \\
\leq\ & C+C\int_0^t\left(\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}+\|\nabla_h u_3\|_{s_2}^{\frac{8s_2}{6s_2-9}}\right)\|\nabla u\|_2^2\,d\tau+\varepsilon\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau.
\end{aligned}
$$

This is,

$$
\|\nabla u(t)\|_2^2+\int_0^t\|\Delta u(\tau)\|_2^2\,d\tau\leq C+C\int_0^t\left(\left\|\frac{\partial u_3}{\partial x_3}\right\|_{s_1}^{\frac{2s_1}{2s_1-3}}+\|\nabla_h u_3\|_{s_2}^{\frac{8s_2}{6s_2-9}}\right)\|\nabla u\|_2^2\,d\tau.
$$

Again, using Gronwall's inequality, (3) and (4) yields the desired bounds.

Thus, we complete the proof of Theorem 1.1.

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