A Blichfeldt-type Theorem for *H*-points

Penghao Cao and Liping Yuan

Abstract. Let H be the set of vertices of a tiling of the plane by regular hexagons of unit area. A point of H is called an H-point. Let $[s]$ denote the greatest integer less than or equal to s, and let $\{s\} = s - [s]$. In this paper we prove a Blichfeldt-type theorem for H-points. It is shown that for any bounded set $D \subset \mathbb{R}^2$ of area s, if $0 \le \{s\} < 1/3$, then D can be translated so as to cover at least $2[s] + 1$ H-points; if $1/3 \le \{s\} < 1$, then by a translation D can be made to cover at least $2[s] + 2$ H-points. Furthermore, we show that the results obtained are the best possible.

1. INTRODUCTION. Let $\vec{u}$ and $\vec{v}$ be two linearly independent vectors in $\mathbb{R}^2$. The set of all points $P = m\vec{u} + n\vec{v}$ with $m, n \in \mathbb{Z}$ is called the *lattice* Λ generated by $\vec{u}$ and $\vec{v}$. A point of the lattice Λ is called a *lattice point*. The parallelogram Q induced by the four vertices of the form $m\vec{u} + n\vec{v}$, where $m, n \in \{0, 1\}$, is said to be the *fundamental parallelogram* of the lattice Λ. Let $\det(\Lambda)$ be defined as the area of the fundamental parallelogram of the lattice Λ. In particular, if $\vec{u}$ and $\vec{v}$ are mutually orthogonal unit vectors, then the lattice Λ is called an *integral lattice*, and is denoted by $\mathbb{Z}^2$.

In 1896, Hermann Minkowski proved a famous theorem, and then developed a new research area, namely, the geometry of numbers.

Minkowski's Theorem ([8]). *Let $C \subset \mathbb{R}^2$ be a convex set, centrally symmetric about the origin. If the area of C is greater than 4, then C contains at least one point from $\mathbb{Z}^2$ different from the origin.*

Later, a new principle in the geometry of numbers was discovered. The credit for this breakthrough goes to Hans Frederik Blichfeldt, who in 1914 published a theorem from which a great portion of the geometry of numbers follows.

Blichfeldt's Theorem ([1, 8]). *For any nonnegative integer A, any bounded planar region with area $> A$ can be translated so that the number of points of $\mathbb{Z}^2$ inside the region will be at least $A + 1$.*

Since the integral lattice $\mathbb{Z}^2$ can be considered as the set of vertices of the Archimedean tiling by squares of unit area, it is interesting to extend results from classical geometry of numbers to other Archimedean tilings, especially to that formed by regular hexagons [**3–7**]. Now let $\mathcal{H}$ be the Archimedean tiling formed by regular hexagons of unit area, and let H be the set of vertices of $\mathcal{H}$. A point of H is called an *H-point*. In this paper we prove a Blichfeldt-type theorem for H-points in $\mathbb{R}^2$.

2. MAIN RESULTS. In fact, H can be considered as the union of two disjoint sets H^+ and H^- such that for any two points, either both from H^+ or both from H^-, there exists a translation of the plane which maps one of the two points to the other and H to H. To be specific, all points in H^+ have three tiling edges leaving the points in the same three directions, while all points in H^- have edges which leave in the opposite three directions. A point of H^+ is called an *H^+-point* and a point of H^- is called an *H^--point*, as shown in Figure 1.

http://dx.doi.org/10.4169/amer.math.monthly.118.08.743

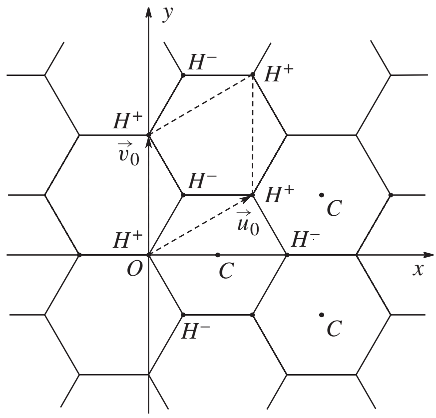

Figure 1. H^+-points, H^--points, and C-points.

Without loss of generality, we establish a cartesian coordinate system of $\mathbb{R}^2$ with an H^+-point as the origin, and the x-axis lying along one edge of a regular hexagonal tile, as shown in Figure 1. Let $\vec{u}_0 = (\sqrt[4]{12}/2, \sqrt[4]{108}/6)$, $\vec{v}_0 = (0, \sqrt[4]{108}/3)$. Then the set H^+ is the lattice generated by $\vec{u}_0$ and $\vec{v}_0$.

Let C denote the set of all centers of the hexagonal tiles which determine $\mathcal{H}$. A point of C is called a *C-point*. Let τ be the transformation defined by $x' = x + \sqrt[4]{12}/3$, $y' = y$. The inverse transformation of τ is denoted by $\tau^{-1} : x' = x - \sqrt[4]{12}/3$, $y' = y$. It is not difficult to see that $H^- = \tau^{-1}(H^+)$ and $C = \tau(H^+)$.

It is clear that H^+, H^-, and C are pairwise disjoint and the union $H^+ \cup H^- \cup C$ is the lattice generated by $\vec{u} = (\sqrt[4]{12}/6, \sqrt[4]{108}/6)$ and $\vec{v} = (\sqrt[4]{12}/3, 0)$, which is denoted by T. A point of T is called a *T-point*. Trivially, $\det(T) = 1/3$.

Let $\mathscr{N}_A(B) = \mathrm{card}(A \cap B)$. We have the following lemma immediately.

Lemma 2.1. *Let $D \subset \mathbb{R}^2$ be a bounded set. Then $\mathscr{N}_D(H^+) = \mathscr{N}_{\tau(D)}(C) = \mathscr{N}_{\tau^{-1}(D)}(H^-)$, $\mathscr{N}_D(H^-) = \mathscr{N}_{\tau(D)}(H^+) = \mathscr{N}_{\tau^{-1}(D)}(C)$, and $\mathscr{N}_D(C) = \mathscr{N}_{\tau(D)}(H^-) = \mathscr{N}_{\tau^{-1}(D)}(H^+)$.*

Let $[s]$ denote the greatest integer less than or equal to s, and let $\{s\} = s - [s]$. We rephrase Blichfeldt's theorem (see [**2**]) in the following way.

Lemma 2.2. *Let $D \subset \mathbb{R}^2$ be a bounded set of area s and $\Lambda \subset \mathbb{R}^2$ an arbitrary lattice. Then D can be translated so as to cover at least $[s/\det(\Lambda)] + 1$ lattice points.*

Now we present a Blichfeldt-type theorem for H-points in $\mathbb{R}^2$.

Theorem 2.3. *Let $D \subset \mathbb{R}^2$ be a bounded set of area s. If $0 \le \{s\} < 1/3$, then D can be translated so as to cover at least $2[s] + 1$ H-points. If $1/3 \le \{s\} < 1$, then by a translation D can be made to cover at least $2[s] + 2$ H-points.*

Proof. Case 1. $0 \le \{s\} < 1/3$.

Since $\det(T) = 1/3$, we have $[s/\det(T)] = 3[s]$ if $0 \le \{s\} < 1/3$. By Lemma 2.2, D can be translated to a position D' so as to cover at least $3[s] + 1$ T-points, i.e., $\mathscr{N}_{D'}(T) \ge 3[s] + 1$. Recalling that $T = H^+ \cup H^- \cup C$, we have $\mathscr{N}_{D'}(T) = \mathscr{N}_{D'}(H^+) + \mathscr{N}_{D'}(H^-) + \mathscr{N}_{D'}(C) \ge 3[s] + 1$. Without loss of generality, we suppose that $\mathscr{N}_{D'}(C)$ is the smallest of the three numbers $\mathscr{N}_{D'}(H^+)$, $\mathscr{N}_{D'}(H^-)$, and $\mathscr{N}'_D(C)$.

Otherwise, by Lemma 2.1 we have $\mathcal{N}_{\tau(D')}(C)$ is the smallest of the three numbers $\mathcal{N}_{\tau(D')}(H^+)$, $\mathcal{N}_{\tau(D')}(H^-)$, and $\mathcal{N}_{\tau(D')}$ or $\mathcal{N}_{\tau^{-1}(D')}(C)$ is the smallest of the three numbers $\mathcal{N}_{\tau^{-1}(D')}(H^+)$, $\mathcal{N}_{\tau^{-1}(D')}(H^-)$, and $\mathcal{N}_{\tau^{-1}(D')}$. It follows that $\mathcal{N}_{D'}(H^+) + \mathcal{N}_{D'}(H^-) \geq (2/3)(3[s]+1) = 2[s] + 2/3$, and since $\mathcal{N}_{D'}(H^+) + \mathcal{N}_{D'}(H^-)$ is an integer, it follows that it must be at least $2[s]+1$.

Case 2. $1/3 \leq \{s\} < 1$.

Then $[s/\det(T)] \geq 3[s]+1$, where $\det(T) = 1/3$. Lemma 2.2 implies that D can be translated to a position D' covering at least $3[s]+2$ T-points, i.e., $\mathcal{N}_{D'}(T) \geq 3[s]+2$. By a method similar to that described in Case 1, we suppose without loss of generality that $\mathcal{N}_{D'}(C)$ is the smallest of the three numbers $\mathcal{N}_{D'}(H^+)$, $\mathcal{N}_{D'}(H^-)$, and $\mathcal{N}'_D(C)$. It follows that $\mathcal{N}_{D'}(H^+) + \mathcal{N}_{D'}(H^-) \geq (2/3)(3[s]+2) = 2[s] + 4/3$, and since $\mathcal{N}_{D'}(H^+) + \mathcal{N}_{D'}(H^-)$ is an integer, it follows that it must be at least $2[s]+2$.

The proof is complete. ∎

Remark 2.1. There are many applications of Blichfeldt's theorem, and one of them is to use it to prove Minkowski's theorem [**8**]. In fact, we also obtain a Minkowski-type theorem for H-points in $\mathbb{R}^2$.

Minkowski-type Theorem. *Let $D \subset \mathbb{R}^2$ be a compact convex set which is centrally symmetric about an H-point. If the area of D is greater than or equal to $4/3$, then D contains at least one other H-point.*

By a method similar to that used in [**8**], the Blichfeldt-type theorem for H-points presented in this paper can also be used to prove the Minkowski-type theorem for H-points in $\mathbb{R}^2$. We omit the details of the proof here.

3. TWO EXAMPLES.

Now we give two examples to show that both of the bounds obtained in Theorem 2.3 are the best possible.

Example 3.1. Let $D = efgh$ be a parallelogram such that $k_{eh} = k_{fg} = 0$ (here k_{eh} denotes the slope of the line determined by e and h), $k_{ef} = k_{gh} = \sqrt{3}$, $|eh| = |fg| = (3n+1)\frac{\sqrt[4]{12}}{3} - \varepsilon_1$ (here $|eh|$ denotes the length of the line-segment eh), and $|ef| = |gh| = \frac{\sqrt[4]{12}}{3} - \frac{\varepsilon_1}{2(3n+1)}$ (see Figure 2), where $n \in \mathbb{Z}^+$ and ε_1 is a positive number which can be made as small as we wish. Then the area of D is $s = \sin\frac{\pi}{3} \cdot (\frac{\sqrt[4]{12}}{3} - \frac{\varepsilon_1}{2(3n+1)})((3n+1)\frac{\sqrt[4]{12}}{3} - \varepsilon_1) = n + 1/3 - \varepsilon$, where $\varepsilon > 0$ and $\varepsilon \to 0$ as $\varepsilon_1 \to 0$. By Lemma 2.2, the parallelogram D can be translated to a po-

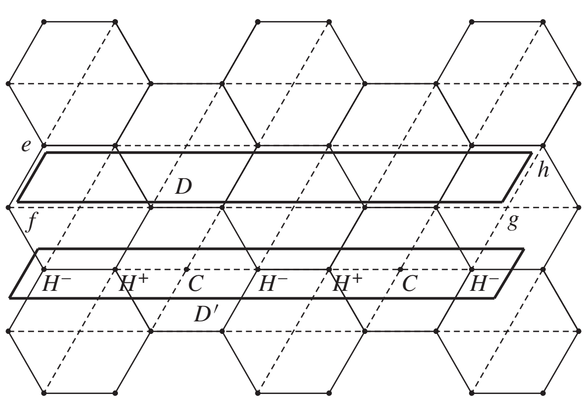

Figure 2. The sets D and D'.

sition D' such that it contains $[\frac{n+(1/3-\varepsilon)}{1/3}]+1=3n+1$ or more T-points. However, according to our choice of D, it is clear that D' cannot contain more than $3n+1$ T-points. Thus there is a translation of D, also denoted by D', containing exactly $3n+1$ T-points. Since $T=H^+\cup H^-\cup C$, and H^-, H^+, and C appear periodically on each horizontal line containing T-points, it is not hard to see that we can chose a position for D' such that it contains exactly $2n+1$ H-points, but no translation of D contains more. That is to say, the bound in the first part of Theorem 2.3 is tight.

Example 3.2. Let $D=efgh$ be the parallelogram described in Example 1. Now we only replace the length of the sides eh and fg by $|eh|=|fg|=(3n+1)\sqrt[4]{12}/3+\varepsilon_1$ and obtain a parallelogram D^*. Then by an argument similar to that presented in Example 3.1, the area s^* of D^* satisfies $1/3\le\{s^*\}<1$, and there is a translation of D^* covering exactly $2n+2$ H-points, whereas no translation of D^* covers more. Hence, the bound in the second part of Theorem 2.3 is also tight.

ACKNOWLEDGMENTS. The authors would like to thank the referees for their careful reading and valuable comments.

Liping Yuan gratefully acknowledges financial supports by NSF of China (10701033, 10426013); WUS Germany (Nr. 2161); SRF for ROCS, SEM; SRF for ROCS, Hebei Province; the Plan of Prominent Personnel Selection and Training for the Higher Education Disciplines in Hebei Province; and NSF of Hebei Normal University. She is also indebted to the Abdus Salam SMS, GC University, Lahore, Pakistan, where part of this work was carried out.

REFERENCES

1. H. F. Blichfeldt, A new principle in the geometry of numbers, with some applications, *Trans. Amer. Math. Soc.* **15** (1914) 227–235. http://dx.doi.org/10.1090/S0002-9947-1914-1500976-6
2. J. W. S. Cassles, *An Introduction to the Geometry of Numbers*, Springer-Verlag, Berlin, 1959.
3. R. Ding, K. Kolodziejczyk, G. Murphy, and J. R. Reay, A Pick-type approximation for areas of H-polygons, *Amer. Math. Monthly* **100** (1993) 669–673. http://dx.doi.org/10.2307/2323889
4. R. Ding and J. R. Reay, The boundary characteristic and Pick's theorem in the Achimedean planar tilings, *J. Combin. Theory Ser. A* **44** (1987) 110–119. http://dx.doi.org/10.1016/0097-3165(87)90063-X
5. R. Ding, J. R. Reay, and J. Zhang, Areas of generalized H-polygons, *J. Combin. Theory Ser. A* **77** (1997) 304–317. http://dx.doi.org/10.1006/jcta.1997.2748
6. K. Kolodziejczyk, Hex-triangles with one interior H-point, *Ars Combin.* **70** (2004) 33–45.
7. ———, Realizable quadruples for hex-polygons, *Graphs Combin.* **23** (2007) 61–72. http://dx.doi.org/10.1007/s00373-006-0688-6
8. C. D. Olds, A. Lax, and G. P. Davidoff, *The Geometry of Numbers*, Mathematical Association of America, Washington, DC, 2000.

College of Mathematics and Information Science, Hebei Normal University, 050016 Shijiazhuang, China

College of Mathematics and Information Science, Hebei Normal University, 050016 Shijiazhuang, China
lpyuan88@yahoo.com