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Adaptive full state hybrid projective synchronization of chaotic systems with the same and different order ☆

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Abstract

This Letter further investigates the full state hybrid projective synchronization (FSHPS) of chaotic and hyper-chaotic systems with fully unknown parameters. Based on the Lyapunov stability theory, a unified adaptive controller and parameters update law can be designed for achieving the FSHPS of chaotic and/or hyper-chaotic systems with the same and different order. Especially, for two chaotic systems with different order, reduced order MFSHPS (an acronym for modified full state hybrid projective synchronization) and increased order MFSHPS are first studied in this Letter. Five groups numerical simulations are provided to verify the effectiveness of the proposed scheme. In addition, the proposed FSHPS scheme is quite robust against the effect of noise.

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Keywords: FSHPS; Adaptive control; MFSHPS; Reduced order; Increased order

1. Introduction

Synchronization is a fundamental phenomenon that enables coherent behavior in coupled systems. In 1990, Pecora and Carroll proposed a successful method to synchronize two identical chaotic systems with different initial conditions [1]. The idea of synchronization is to use the output of a drive (master) system to control a response (slave) system so that the response of the latter follows the output (or the function of output) of the drive system asymptotically. Due to many potential applications in secure communication, biological science, optical science, chemical reaction, social science, and many other fields, the synchronization of coupled chaotic dynamical systems has been one of the most interesting topics in nonlinear science and many theoretical and experimental results have been obtained.

According to the classification [2,3] about the research of chaos synchronization, there are two main directions (i) analysis and (ii) synthesis. The problem of synchronization analysis consists of understanding and/or giving theoretical description of synchronization. So far, there exist many types of synchronization such as complete synchronization [1], phase synchronization [4], anti-synchronization [5], partially synchronization [6], generalized synchronization [7], projective synchronization [8–12], Q–S synchronization [13], etc. The problem of synchronization synthesis concerns on finding or designing a synchronization control signal, such that two coupled chaotic systems exhibit different type of synchronization behaviors. Up to now, a wide variety of

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approaches have been used to design a synchronization control signal, for example, PC method [1,14], impulsive control method [15], active control [16–18], adaptive control [19–24], etc.

Recently, we investigated a new type of synchronization phenomenon—full state hybrid projective synchronization (FSHPS), which bridges a gap from chaos control to chaos synchronization, to generalized synchronization in works [17,18,20]. For two coupled chaotic systems

$$\dot{x}(t) = F(x) \quad \longleftarrow \quad \text{drive system,} \qquad \text{(a)}$$

$$\dot{y}(t) = G(y) + u(x, y) \quad \longleftarrow \quad \text{response system} \qquad \text{(b)}$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ are the n-dimensional state vector, $F, G : R^n \to R^n$ are two continuous vector functions and $u(x, y)$ is a controller, the FSHPS means that there exists constant matrix $H = \mathrm{diag}(h_1, h_2, \ldots, h_n) \in R^{n\times n}$ such that $\lim_{t\to\infty} \|y - Hx\| = 0$ i.e., $\lim_{t\to\infty} |y_i - h_i x_i| = 0$, $i = 1, 2, \ldots, n$. Here H is called scaling matrix, $h_1, h_2, \ldots, h_n$ are called scaling factors. The novelty feature of this synchronization phenomenon is that the scaling factors of the synchronization can be arbitrarily designed to different state variables by means of control. It is believed that, in application to secure communications, this feature could be used to extend binary digital to variety M-nary digital communications for getting more secure and fast communications. The general scheme for achieving FSHPS in continuous and discrete chaotic systems is respectively presented in [17] and [18]. Shortly afterwards, we presented a adaptive FSHPS scheme which guarantees one can achieve simultaneously the FSHPS and parameter identification of coupled identical chaotic systems in [20].

At present, most of theoretical results about synchronization of chaos focus on the systems whose models are identical, similar or with mismatched parameters [25]. However, synchronization of chaos also can be induced even in strictly different systems [2,26] and systems of different order [3,21–23,27,28], especially the systems in biological science and social science [29]. One example is the synchronization that occurs between heart and lung, where one can observe that both circulatory and respiratory systems behave in synchronous way, but their models are essentially different and they have different order. So, the study of synchronization for strictly different dynamical systems and different order dynamical systems is both very important from the perspective of control theory and very necessary from the perspective of practical application. Whereas, this kind of research is just at the beginning stage. To the best of our knowledge, there are few theoretical results about FSHPS of different order chaotic systems.

Motivated by the above discussion, the aim of this Letter is to study the FSHPS of drive and response chaotic (hyper-chaotic) systems with fully unknown parameters (means the parameters of both drive system and response system are all unknown) based on the Lyapunov stability theorem. A general controller and parameters update law is proposed for the FSHPS of chaotic systems which can be with the same and different order, based on active control idea.

The rest of this Letter is organized as follows. In Section 2, the adaptive FSHPS scheme of chaotic systems with the same order is presented. In Section 3, the applications of results of Section 2 to Lorenz and Genesio chaotic systems, hyper-chaotic Chen and Lü systems are considered respectively. In Section 4, the adaptive FSHPS scheme of chaotic systems with different order is given. Especially, for two chaotic systems with different order, reduced order MFSHPS and increased order MFSHPS are first studied in this Letter. In Section 5, the applications of results of Section 4 to generalized Lorenz and Lorenz chaotic systems, Lorenz chaotic system and hyper-chaotic Lü system are considered respectively. The effect of noise is also considered in all numerical simulations, from which we can see that the proposed scheme is quite robust against the effect of noise. Finally, concluding remarks end the Letter.

2. Adaptive FSHPS scheme of chaotic systems with the same order

Consider an n-dimensional chaotic (hyper-chaotic) system in the form of

$$\dot{x} = f(x) + F(x)\Lambda, \tag{1}$$

where $x \in R^n$ is the state vector of the system, $f : R^n \to R^n$ is a continuous vector function, $F : R^n \to R^{n\times d}$ is a matrix function and $\Lambda \in R^d$ is parameter vector.

Remark 1. Nonlinear dynamical system (1) in this Letter depends linearly on the parameters and many well-known chaotic and hyper-chaotic systems belong to (1), such as Lorenz, Genesio chaotic systems and hyper-chaotic Lorenz, Chen, Lü systems.

Eq. (1) is considered as a drive system. A controlled response system is given by

$$\dot{y} = g(y) + G(y)\Theta + U, \tag{2}$$

where $y \in R^n$ is the state vector, $g : R^n \to R^n$ is a vector function, $G : R^n \to R^{n\times k}$ is a matrix function, $\Theta \in R^k$ is parameter vector and $U \in R^n$ is a controller.

Let the vector error state be $e(t) = y(t) - Hx(t)$, where H is a n-order diagonal matrix, i.e. $H = \mathrm{diag}(h_1, h_2, \dots, h_n)$. Thus the error dynamical system between the drive system (1) and the response system (2) is

$$\dot{e}(t) = \dot{y} - H\dot{x} = g(y) + G(y)\Theta - Hf(x) - HF(x)\Lambda + U = R(e,x) + G(e,x)\Theta - HF(x)\Lambda + U, \tag{3}$$

where $R(e,x) = g(e + Hx) - Hf(x)$. The aim of FSHPS is to design a controller U, which is able to synchronize the state of the drive system and the response system up to different arbitrary scaling factors.

Noting that the parameters of both drive and response system are fully unknown, by using adaptive control techniques, the controller can be determined as

$$U = -R(e,x) - G(e,x)\tilde{\Theta} + HF(x)\tilde{\Lambda} + Ae, \tag{4}$$

where the matrix A is determined in the later, $\tilde{\Theta}$ and $\tilde{\Lambda}$ are the estimated values of unknown parameters Θ and Λ, respectively. From (3) and (4), the error dynamics is described by

$$\dot{e} = Ae + HF(x)(\tilde{\Lambda} - \Lambda) - G(e,x)(\tilde{\Theta} - \Theta). \tag{5}$$

Hence the FSHPS problem becomes the stability of error dynamics (5). If it is globally stabilized at the origin, the FSHPS of drive system (1) and response system (2) can be globally realized. If the updating laws of the estimated parameters are chosen by

$$\dot{\tilde{\Theta}} = G^T(e,x)e, \qquad \dot{\tilde{\Lambda}} = -F^T(x)He \tag{6}$$

and construct a Lyapunov function

$$V = \frac{1}{2}\left(e^T e + \hat{\Theta}^T\hat{\Theta} + \hat{\Lambda}^T\hat{\Lambda}\right), \tag{7}$$

where $\hat{\Lambda} = \tilde{\Lambda} - \Lambda$ and $\hat{\Theta} = \tilde{\Theta} - \Theta$.

With the choice of the controller (4) and the updating laws (6), the time derivative of V along the trajectories of Eq. (5) is

$$\frac{dV}{dt} = e^T\dot{e} + \dot{\hat{\Theta}}^T\hat{\Theta} + \dot{\hat{\Lambda}}^T\hat{\Lambda} = e^T Ae. \tag{8}$$

Suppose we select an appropriate matrix A such that $\frac{dV}{dt} < 0$, that is, $\frac{dV}{dt}$ is negative definite. Then, according to the Lyapunov stability theorem, the FSHPS of chaotic (hyper-chaotic) systems (1) and (2) is achieved under the certain chosen feedback controller U (4) and parameters update law $\dot{\tilde{\Theta}}$ and $\dot{\tilde{\Lambda}}$ (6). In fact, the obtained result extend and improve that given in [24].

3. Application of the adaptive FSHPS scheme with same order

In this section we will choose Lorenz and Genesio chaotic systems, Chen and Lü hyper-chaotic systems to illustrate the effectiveness of the adaptive FSHPS scheme with the same order.

3.1. Adaptive FSHPS between Lorenz and Genesio chaotic systems

The nonlinear differential equations that describe the Lorenz system [30] are

$$\begin{cases} \dot{x}_1(t) = a(x_2 - x_1), \\ \dot{x}_2(t) = bx_1 - x_2 - x_1x_3, \\ \dot{x}_3(t) = x_1x_2 - cx_3 \end{cases} \tag{9}$$

and the Genesio system [31] are

$$\begin{cases} \dot{y}_1(t) = y_2, \\ \dot{y}_2(t) = y_3, \\ \dot{y}_3(t) = -a_1y_1 - b_1y_2 - c_1y_3 + y_1^2, \end{cases} \tag{10}$$

where a, b, c and a_1, b_1, c_1 are unknown system parameters. Our purpose is to achieve the FSHPS between system (9) and (10). Let Lorenz chaotic system drive Genesio chaotic system. We rewrite the drive system and controlled response system respectively in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_2 - x_1x_3 \\ x_1x_2 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{11}$$

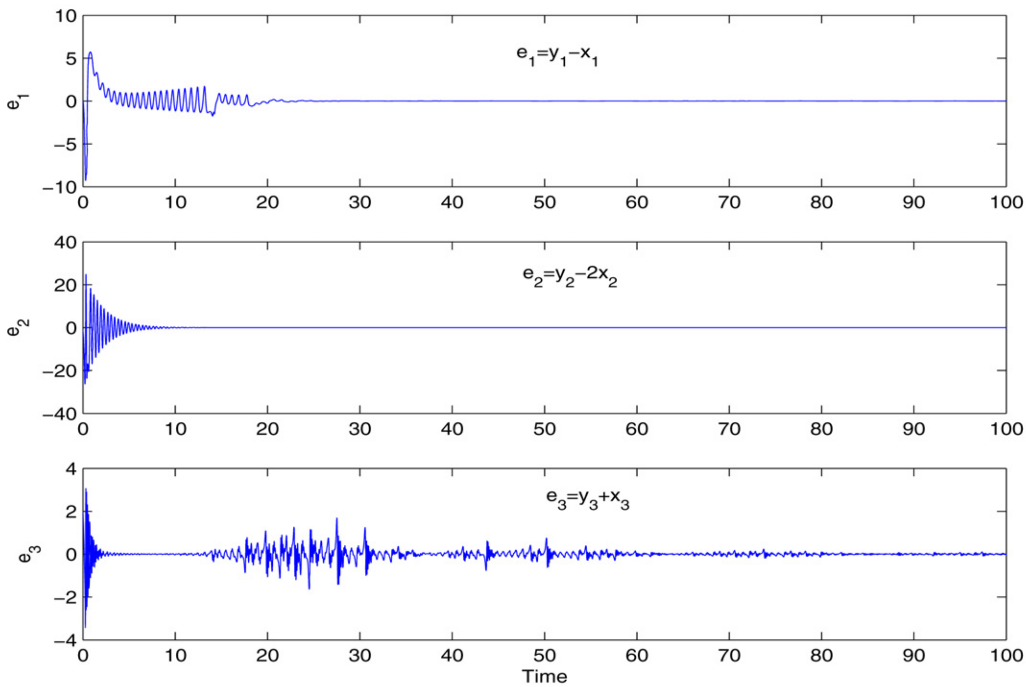

Fig. 1. FSHPS of Lorenz and Genesio chaotic systems with fully unknown parameters.

and

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_1^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -y_1 & -y_2 & -y_3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{12}$$

where $(u_1, u_2, u_3)^T$ is the controller to be determined.

If the matrix A is chosen as $A = \mathrm{diag}(-1, -1, -1)$. Then, according to Eqs. (4) and (6), we get the controller:

$$\begin{cases} u_1 = -(e_2 + h_2 x_2) + h_1(x_2 - x_1)\hat{a} - e_1, \\ u_2 = -(e_3 + h_3 x_3) - h_2(x_2 + x_1 x_3) + h_2 x_1 \hat{b} - e_2, \\ u_3 = h_3 x_1 x_2 - (e_1 + h_1 x_1)^2 + (e_1 + h_1 x_1)\hat{a}_1 + (e_2 + h_2 x_2)\hat{b}_1 + (e_3 + h_3 x_3)\hat{c}_1 - h_3 x_3 \hat{c} - e_3 \end{cases} \tag{13}$$

and the estimates $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{a}_1$, $\hat{b}_1$ and $\hat{c}_1$ obey the following update laws:

$$\begin{cases} \dot{\hat{a}} = -h_1 e_1 (x_2 - x_1), \\ \dot{\hat{b}} = -h_2 e_2 x_1, \\ \dot{\hat{c}} = h_3 e_3 x_3 \end{cases} \tag{14}$$

and

$$\begin{cases} \dot{\hat{a}}_1 = -(e_1 + h_1 x_1) e_1, \\ \dot{\hat{b}}_1 = -(e_2 + h_2 x_2) e_1, \\ \dot{\hat{c}}_1 = -(e_3 + h_3 x_3) e_1. \end{cases} \tag{15}$$

RK4 method is used to our all simulations with time step being equal to 0.001. In this numerical simulations, we select the “unknown” parameters of the Lorenz system as $a = 10$, $b = 28$, $c = 8/3$ and the “unknown” parameters of the Genesio system as $a_1 = 6$, $b_1 = 2.92$, $c_1 = 1.2$ to ensure the chaotic behavior. The initial states of the drive system and response system are $x_1(0) = 1$, $x_2(0) = 1$, $x_3(0) = 1$, $y_1(0) = 1$, $y_2(0) = 1$, $y_3(0) = 1$, the parameters have initial conditions $a(0) = 0$, $b(0) = 0$, $c(0) = 0$, $a_1(0) = 0$, $b_1(0) = 0$, $c_1(0) = 0$ and scaling factors are $h_1 = 1$, $h_2 = 2$, $h_3 = -1$. Fig. 1 shows the chaotic FSHPS. Since noise is ubiquitous in both man-made and nature systems, FSHPS of concrete models is unavoidably subject to internal and external noise. Therefore, it is important to investigate the noise’s effect in FSHPS between chaotic systems. To consider the robustness of the adaptive FSHPS scheme against noise, the additive noise with the strength 3 in time series x_2 is added, and corresponding numerical simulation is given in Fig. 2. The simulation results show sufficiently that the proposed adaptive FSHPS scheme is very effective and quite robust against the effect of noise.

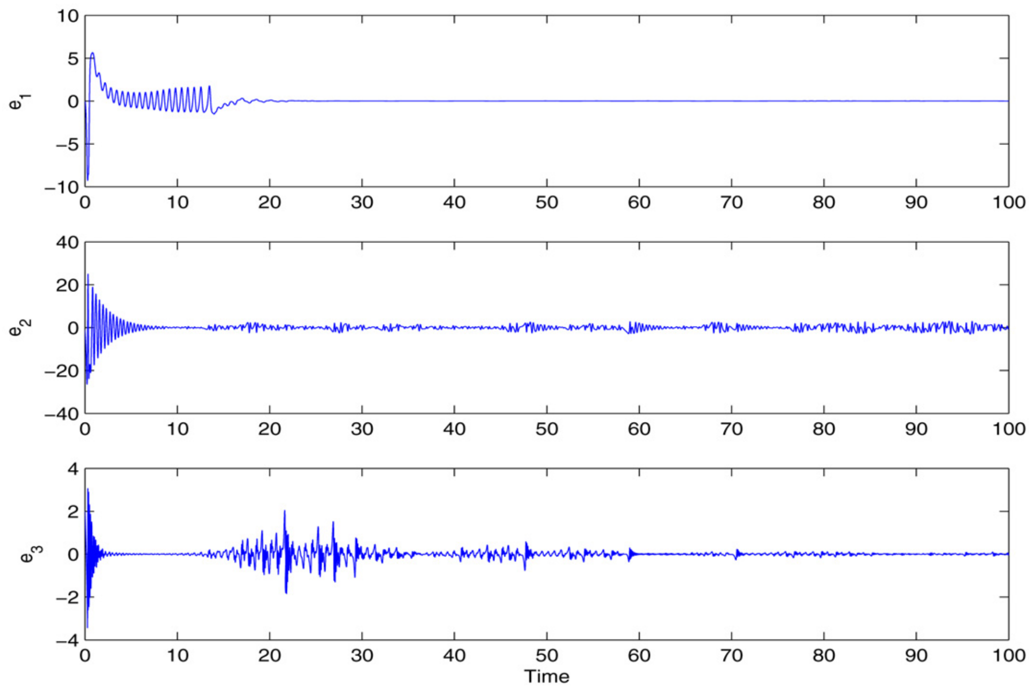

Fig. 2. FSHPS of Lorenz and Genesio chaotic systems under the effect of noise.

3.2. Adaptive FSHPS between hyper-chaotic Chen and Lü systems

For further illustrating the effectiveness of the proposed scheme, in this subsection, we select the hyper-chaotic Chen system [32]

$$\begin{cases} \dot{x}_1(t) = a(x_2 - x_1) + x_4, \\ \dot{x}_2(t) = dx_1 - x_1x_3 + cx_2, \\ \dot{x}_3(t) = x_1x_2 - bx_3, \\ \dot{x}_4(t) = x_2x_3 + rx_4 \end{cases} \tag{16}$$

and the hyper-chaotic Lü system [33]

$$\begin{cases} \dot{y}_1(t) = a_1(y_2 - y_1) + y_4, \\ \dot{y}_2(t) = b_1y_2 - y_1y_3, \\ \dot{y}_3(t) = -c_1y_3 + y_1y_2, \\ \dot{y}_4(t) = d_1y_4 + y_1y_3 \end{cases} \tag{17}$$

as examples, where a, b, c, d, r and a_1, b_1, c_1, d_1 are unknown system parameters. Let hyper-chaotic Chen system drive hyper-chaotic Lü system. We present the drive system and response system respectively in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ -x_1x_3 \\ x_1x_2 \\ x_2x_3 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & x_1 & 0 \\ 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ r \end{pmatrix} \tag{18}$$

and

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} y_4 \\ -y_1y_3 \\ y_1y_2 \\ y_1y_3 \end{pmatrix} + \begin{pmatrix} y_2 - y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & -y_3 & 0 \\ 0 & 0 & 0 & y_4 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{19}$$

where $(u_1, u_2, u_3, u_4)^T$ is the controller to be determined.

According to Eqs. (4) and (6), the matrix A is chosen as $A = \mathrm{diag}(-1, -1, -1, -1)$. Then the control law is:

$$\begin{cases} u_1 = -(e_4 + h_4x_4) + h_1x_4 - (e_2 - e_1 + h_2x_2 - h_1x_1)\hat{a}_1 + h_1(x_2 - x_1)\hat{a} - e_1, \\ u_2 = (e_1 + h_1x_1)(e_3 + h_3x_3) - h_2x_1x_3 - (e_2 + h_2x_2)\hat{b}_1 + h_2x_2\hat{c} + h_2x_1\hat{d} - e_2, \\ u_3 = -(e_1 + h_1x_1)(e_2 + h_2x_2) + h_3x_1x_2 + (e_3 + h_3x_3)\hat{c}_1 - h_3x_3\hat{b} - e_3, \\ u_4 = -(e_1 + h_1x_1)(e_3 + h_3x_3) + h_4x_2x_3 - (e_4 + h_4x_4)\hat{d}_1 + h_4x_4\hat{r} - e_4 \end{cases} \tag{20}$$

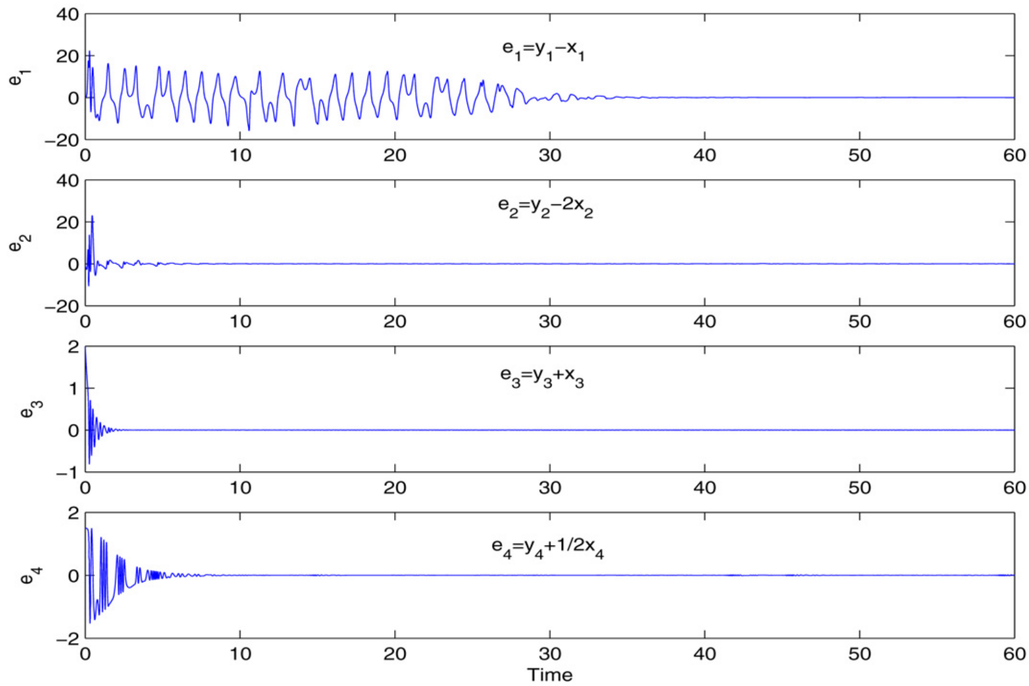

Fig. 3. FSHPS of hyper-chaotic Chen and hyper-chaotic Lü systems with fully unknown parameters.

and the estimates $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{r}$, $\hat{a}_1$, $\hat{b}_1$, $\hat{c}_1$, $\hat{d}_1$ are updated according to the following algorithm:

$$\begin{cases} \dot{\hat{a}} = -h_1 e_1 (x_2 - x_1), \\ \dot{\hat{b}} = h_3 e_3 x_3, \\ \dot{\hat{c}} = -h_2 e_2 x_2, \\ \dot{\hat{d}} = -h_2 e_2 x_1, \\ \dot{\hat{r}} = -h_4 e_4 x_4 \end{cases} \tag{21}$$

and

$$\begin{cases} \dot{\hat{a}}_1 = e_1 (e_2 - e_1 + h_2 x_2 - h_1 x_1), \\ \dot{\hat{b}}_1 = e_2 (e_2 + h_2 x_2), \\ \dot{\hat{c}}_1 = -e_3 (e_3 + h_3 x_3), \\ \dot{\hat{d}}_1 = e_4 (e_4 + h_4 x_4). \end{cases} \tag{22}$$

In this numerical simulations, we select the "unknown" parameters of the 4D Chen system as $a = 35$, $b = 3$, $c = 12$, $d = 7$, $r = 0.5$ and the "unknown" parameters of the 4D Lü system as $a_1 = 36$, $b_1 = 20$, $c_1 = 3$, $d_1 = 1.3$ to ensure the chaotic behavior. The initial states of the drive system and response system are $x_1(0) = 1$, $x_2(0) = 1$, $x_3(0) = 1$, $x_4(0) = 1$, $y_1(0) = 1$, $y_2(0) = 1$, $y_3(0) = 1$, $y_4(0) = 1$, the parameters have initial conditions $a(0) = 0$, $b(0) = 0$, $c(0) = 0$, $d(0) = 0$, $r(0) = 0$, $a_1(0) = 0$, $b_1(0) = 0$, $c_1(0) = 0$, $d_1(0) = 0$ and $h_1 = 1$, $h_2 = 2$, $h_3 = -1$, $h_4 = -1/2$. The simulation results are shown as Figs. 3 and 4. Fig. 3 shows the hyper-chaotic FSHPS and Fig. 4 shows that when an additive noise with the strength 1 is simultaneously added to the signals x_1 and x_2, an additive noise with the strength 5 is added to the signals x_2 and an additive noise with the strength 3 is added to the signals x_3.

The above numerical simulations show that chaotic or hyper-chaotic FSHPS with fully unknown parameters can be well achieved by the proposed adaptive FSHPS scheme. In addition, we find from these simulations that such FSHPS is robust against the effect of noise, namely the expected FSHPS means that the synchronization error is eventually smaller than a threshold value (does not tend to zero), then the present scheme is physically feasible in the noisy situation.

4. Adaptive FSHPS scheme of chaotic systems with different order

The aim of this section is to address the FSHPS of two coupled chaotic systems with different order. For this, Eq. (1) is still viewed as a drive system, controlled response system is given by

$$\dot{y} = g(y) + G(y)\Theta + U, \tag{23}$$

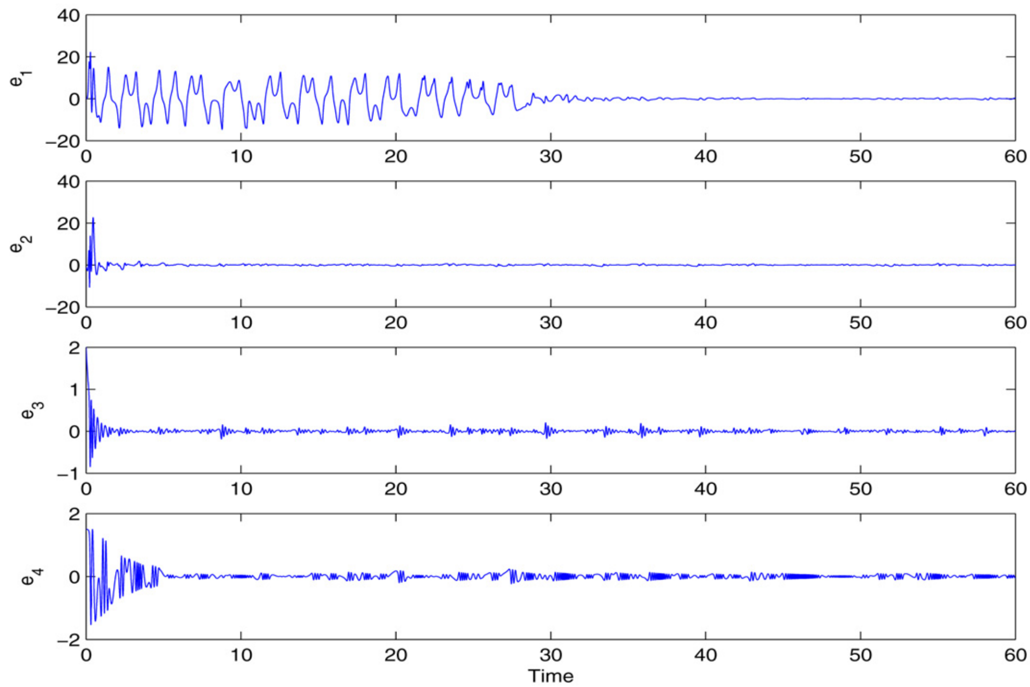

Fig. 4. FSHPS of hyper-chaotic Chen and hyper-chaotic Lü systems under the effect of noise.

where $y \in R^m$ is the state vector, $g : R^m \to R^m$, $G : R^m \to R^{m\times k}$ and $\Theta \in R^k$ is parameter vector and $U \in R^m$ is a controller. Note worthily, the order of controlled response system in this section is different from the one in Section 2. We take two kinds of cases (I) $n > m$ and (II) $n < m$ into consideration.

Remark 2. Since the order of two chaotic system considered in this and the next section is different, the term "full state hybrid projective synchronization" is not accurate. In the present Letter, we consider the synchronization which contains the full states of response system, so we call it "modified full state hybrid projective synchronization (MFSHPS)".

Case I. $n > m$, that is, the order of the drive system is greater than that of the response system, so the synchronization (MFSHPS) is only attained in reduced order. (If scaling matrix $H = I$, I denotes the identical matrix, this kind synchronization phenomenon is called reduced order synchronization which has been investigated by the author of [2,21–23,27,28].) We call it reduced order MFSHPS in this Letter. For achieving reduced order MFSHPS, we divide the drive system into two parts.

$$\dot{x}_p(t) = f_p(x) + F_p(x)\Lambda, \tag{24}$$

$$\dot{x}_r(t) = f_r(x) + F_r(x)\Lambda, \tag{25}$$

where $x_p \in R^m$, $x_r \in R^l$, $f_p : R^n \to R^m$, $f_r : R^n \to R^l$, $F_p : R^n \to R^{m\times d}$, $F_r : R^n \to R^{l\times d}$ and $m + l = n$. Let the vector error state be $e(t) = y(t) - Hx_p(t)$, where H is a m-order diagonal matrix, i.e. $H = \mathrm{diag}(h_1, h_2, \ldots, h_m)$.

Now, the reduced order MFSHPS problem of (1) and (23) becomes the FSHPS of (24) and (23) with the same order m. Hence, according to the results in Section 2, if we let

$$U = -R(e, x) - G(e, x)\tilde{\Theta} + HF_p(x)\tilde{\Lambda} + Ae \tag{26}$$

where $R(e, x) = g(e + Hx) - Hf_p(x)$ and

$$\dot{\tilde{\Theta}} = G^T(e, x)e, \qquad \dot{\tilde{\Lambda}} = -F_p^T(x)He \tag{27}$$

then, we can achieve the reduced order MFSHPS between the partial states of drive system and the full states of response system.

Case II. $n < m$, that is, the order of the drive system is lower than that of the response system, so the MFSHPS is only attained in increased order. We call this type of synchronization increased order MFSHPS. For achieving increased order MFSHPS, we must "creat order". One doable way is to construct auxiliary state vector which is the function of state x. For instance, provided $l = m - n$, we define $x_{n+1} = \phi_1(x)$, $x_{n+2} = \phi_2(x), \ldots, x_{n+l} = x_m = \phi_l(x)$, then we can get a new m dimension state vector $X = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m)$. New drive system can write

$$\dot{X} = f_e(x) + F_e(x)\Lambda, \tag{28}$$

where $x \in R^n$, $X \in R^m$, $f_e : R^n \to R^m$, $F : R^n \to R^{m\times d}$ and $\Lambda \in R^d$ is parameter vector. Response system is still Eq. (23). Now, the increased order MFSHPS problem of (1) and (23) becomes the FSHPS of (28) and (23) with the same order m. Hence, according to the results in Section 2, we let

$$U = -R(e,x) - G(e,x)\tilde{\Theta} + HF_e(x)\tilde{\Lambda} + Ae \tag{29}$$

where $R(e,x) = g(e + Hx) - Hf_e(x)$ and

$$\dot{\tilde{\Theta}} = G^T(e,x)e, \qquad \dot{\tilde{\Lambda}} = -F_e^T(x)He \tag{30}$$

then, we can achieve the increased order MFSHPS between the all states $(x_1, x_2, \ldots, x_n)$ and auxiliary states $(x_{n+1}, x_{n+1}, \ldots, x_m)$ of drive system and the full states of response system.

5. Application of the adaptive FSHPS scheme with different order

In this section we will choose generalized Lorenz system, Lorenz system, Lü hyper-chaotic systems to illustrate the effectiveness of the adaptive FSHPS scheme with different order.

5.1. Reduced order MFSHPS of generalized Lorenz system and Lorenz system

In this subsection generalized Lorenz chaotic system and Lorenz chaotic system are chosen to illustrate the effectiveness of reduced order MFSHPS. Generalized Lorenz system [34] are

$$\begin{cases} \dot{x}_1(t) = a(x_2 - x_1) + dx_4, \\ \dot{x}_2(t) = bx_1 - x_1x_3 - x_2, \\ \dot{x}_3(t) = x_1x_2 - cx_3, \\ \dot{x}_4(t) = -x_1 - ax_4, \end{cases} \tag{31}$$

and the Lorenz system are:

$$\begin{cases} \dot{y}_1(t) = a_1(y_2 - y_1), \\ \dot{y}_2(t) = by_1 - y_1y_3 - y_2, \\ \dot{y}_3(t) = y_1y_2 - cy_3. \end{cases} \tag{32}$$

Our purpose is to archive the FSHPS of Lorenz system and the former three states of generalized Lorenz system. Therefore, we need only present the states (x_1, x_2, x_3) of generalized Lorenz system in the form of

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_1x_3 - x_2 \\ x_1x_2 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & -x_3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{33}$$

Similarly, the controlled response system is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -y_1y_3 - y_2 \\ y_1y_2 \end{pmatrix} + \begin{pmatrix} y_2 - y_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & -y_3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{34}$$

where $(u_1, u_2, u_3)^T$ is the controller to be determined. According to (26) and (27), the matrix A is chosen as $A = \mathrm{diag}(-1, -1, -1)$, then we get the controller

$$\begin{cases} u_1 = -(e_2 - e_1 + h_2x_2 - h_1x_1)\hat{a}_1 + h_1(x_2 - x_1)\hat{a} + h_1x_4\hat{d} - e_1, \\ u_2 = -(e_1 + h_1x_1)\hat{b}_1 + h_2x_1\hat{b} - h_2(x_1x_3 + x_2) + (e_2 + h_2x_2) + (e_1 + h_1x_1)(e_3 + h_3x_3) - e_2, \\ u_3 = (e_3 + h_3x_3)\hat{c}_1 + h_3x_1x_2 - h_3x_3\hat{c} - (e_1 + h_1x_1)(e_2 + h_2x_2) - e_3 \end{cases} \tag{35}$$

and the estimates $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{a}_1$, $\hat{b}_1$, $\hat{c}_1$ obey the updating laws:

$$\begin{cases} \dot{\hat{a}} = -h_1(x_2 - x_1)e_1, \\ \dot{\hat{b}} = -h_2x_1e_2, \\ \dot{\hat{c}} = h_3x_3e_3, \\ \dot{\hat{d}} = -h_1x_4e_1 \end{cases} \tag{36}$$

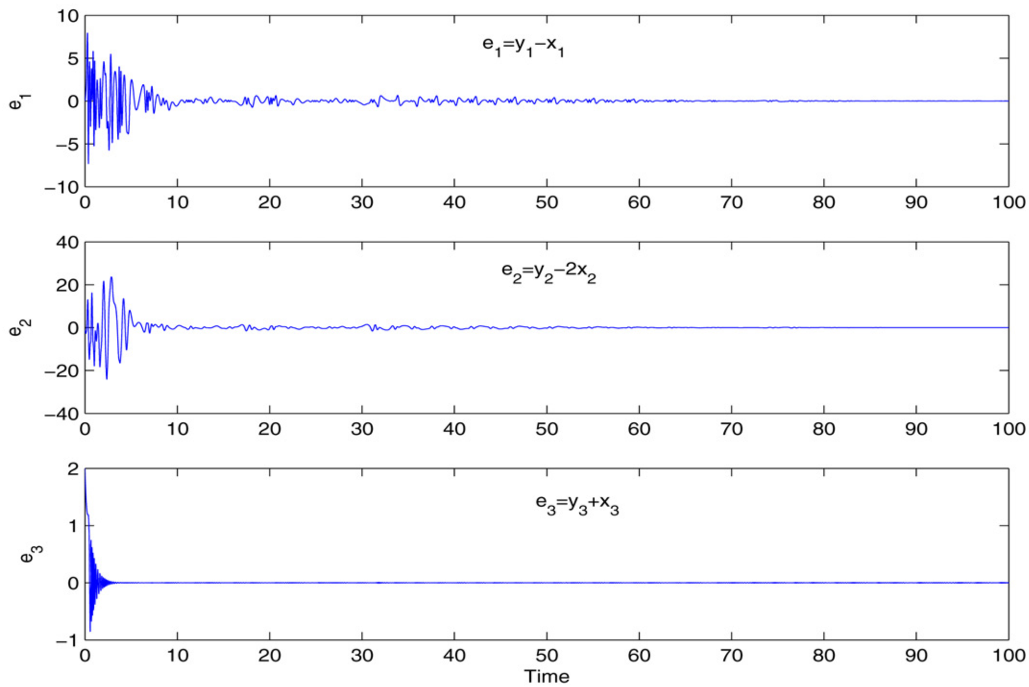

Fig. 5. Reduced order MFSHPS of generalized Lorenz and Lorenz systems with fully unknown parameters.

and

$$\begin{cases} \dot{\hat{a}}_1 = (e_1 - e_2 + h_1x_1 - h_2x_2)e_1, \\ \dot{\hat{b}}_1 = (e_1 + h_1x_1)e_2, \\ \dot{\hat{c}}_1 = -(e_3 + h_3x_3)e_3. \end{cases} \tag{37}$$

In this numerical simulations, we select the "unknown" parameters of the 4D generalized system as $a = 1$, $b = 26$, $c = 0.7$, $d = 1.5$ and the "unknown" parameters of the Lorenz system as $a_1 = 10$, $b_1 = 28$, $c_1 = 8/3$ to ensure the chaotic behavior. The initial states of the drive system and response system are $x_1(0) = 1$, $x_2(0) = 1$, $x_3(0) = 1$, $x_4(0) = 1$, $y_1(0) = 1$, $y_2(0) = 1$, $y_3(0) = 1$, the parameters have initial conditions $a(0) = 0$, $b(0) = 0$, $c(0) = 0$, $a_1(0) = 0$, $b_1(0) = 0$, $c_1(0) = 0$ and $h_1 = 1$, $h_2 = 2$, $h_3 = -1$. The simulation results are shown in Figs. 5 and 6. Fig. 5 shows the reduced order MFSHPS between 4D hyper-chaotic system and 3D chaotic system. Fig. 6 shows the slight effect of a noise with the strength 1 which is simultaneously added to the signals x_1, x_2 and x_3.

5.2. Increased order MFSHPS of Lorenz system and hyper-chaotic Lü system

In this subsection, let Lorenz chaotic system (9) drive hyper-chaotic Lü system (17). Since the order of drive system is 3 but the order of response system is 4, so we need construct an auxiliary state variable $x_4 = \phi(x)$. In what follows, we choose two case to study.

(1) If we let $x_4 = \phi(x) = x_1 + x_2 + x_3$, then $\dot{x}_4 = a(x_2 - x_1) + bx_1 - cx_3 - x_2 - x_1x_3 + x_1x_2$. Let $e_i = y_i - h_ix_i$ $(i = 1, 2, 3, 4)$, where $e_4 = y_4 - h_4(x_1 + x_2 + x_3)$, then according to (29) and (30), we get the controller

$$\begin{cases} u_1 = -(e_2 - e_1 + h_2x_2 - h_1x_1)\hat{a}_1 + h_1(x_2 - x_1)\hat{a} - e_4 - h_4x_4 - e_1, \\ u_2 = -(e_2 + h_2x_2)\hat{b}_1 + h_2x_1\hat{b} - h_2(x_1x_3 + x_2) + (e_1 + h_1x_1)(e_3 + h_3x_3) - e_2, \\ u_3 = (e_3 + h_3x_3)\hat{c}_1 + h_3x_1x_2 - h_3x_3\hat{c} - (e_1 + h_1x_1)(e_2 + h_2x_2) - e_3, \\ u_4 = -(e_4 + h_4x_4)\hat{d}_1 + h_4(x_2 - x_1)\hat{a} + h_4x_1\hat{b} - h_4x_3\hat{c} + h_4(-x_2 - x_1x_3 + x_1x_2) - (e_1 + h_1x_1)(e_3 + h_3x_3) \end{cases} \tag{38}$$

and the estimates $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{a}_1$, $\hat{b}_1$, $\hat{c}_1$, $\hat{d}_1$ obey the updating laws:

$$\begin{cases} \dot{\hat{a}} = -h_1(x_2 - x_1)e_1 - h_4(x_2 - x_1)e_4, \\ \dot{\hat{b}} = -h_2x_1e_2 - h_4x_1e_4, \\ \dot{\hat{c}} = h_3x_3e_3 + h_4x_3e_4 \end{cases} \tag{39}$$

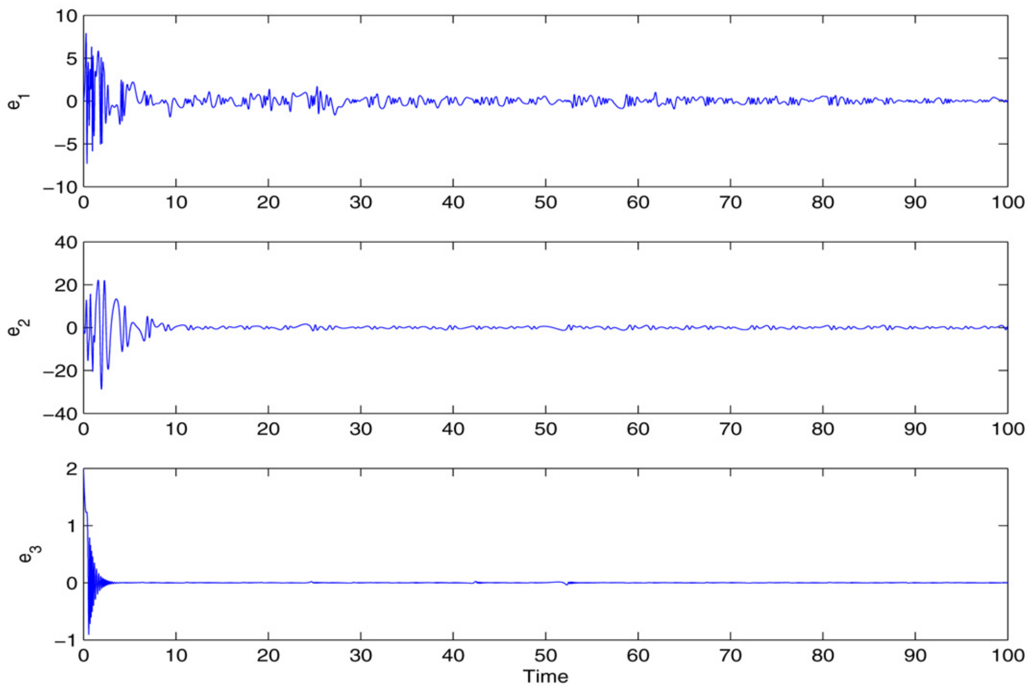

Fig. 6. Reduced order MFSHPS of generalized Lorenz and Lorenz systems under the effect of noise.

and

$$\begin{cases} \dot{\hat{a}}_1 = (e_2 - e_1 + h_2x_2 - h_1x_1)e_1, \\ \dot{\hat{b}}_1 = (e_2 + h_2x_2)e_2, \\ \dot{\hat{c}}_1 = -(e_3 + h_3x_3)e_3, \\ \dot{\hat{d}}_1 = (e_4 + h_4x_4)e_4. \end{cases} \tag{40}$$

In this and the next numerical simulations, the "unknown" parameters are same as Eqs. (9) and (17), respectively and the matrix A is chosen as $A = \mathrm{diag}(-1, -1, -1, -1)$. The initial states of the drive system and response system are $x_1(0) = 1$, $x_2(0) = 1$, $x_3(0) = 1$, $y_1(0) = 1$, $y_2(0) = 1$, $y_3(0) = 1$, $y_4(0) = 1$, the parameters have initial conditions $a(0) = 0$, $b(0) = 0$, $c(0) = 0$, $a_1(0) = 0$, $b_1(0) = 0$, $c_1(0) = 0$, $d_1(0) = 0$ and $h_1 = 1$, $h_2 = 2$, $h_3 = -1$, $h_4 = -1/2$. The simulation results are shown in Figs. 7 and 8. Fig. 7 shows the increased order MFSHPS between 3D Lorenz chaotic system with the auxiliary state variable $x_4 = x_1 + x_2 + x_3$ and 4D hyper-chaotic Lü system. Fig. 8 shows the slight effect of a noise with the strength 3 which is added to the signals x_2.

(2) If we let $x_4 = \phi(x) = x_1^2$, then $\dot{x}_4 = 2ax_1(x_2 - x_1)$. Let $e_i = y_i - h_ix_i$ $(i = 1, 2, 3, 4)$, where $e_4 = y_4 - h_4x_1^2$, then according to (29) and (30), we get the controller

$$\begin{cases} u_1 = -(e_2 - e_1 + h_2x_2 - h_1x_1)\hat{a}_1 + h_1(x_2 - x_1)\hat{a} - e_4 - h_4x_4 - e_1, \\ u_2 = -(e_2 + h_2x_2)\hat{b}_1 + h_2x_1\hat{b} - h_2(x_1x_3 + x_2) + (e_1 + h_1x_1)(e_3 + h_3x_3) - e_2, \\ u_3 = (e_3 + h_3x_3)\hat{c}_1 + h_3x_1x_2 - h_3x_3\hat{c} - (e_1 + h_1x_1)(e_2 + h_2x_2) - e_3, \\ u_4 = -(e_4 + h_4x_4)\hat{d}_1 + 2h_4x_1(x_2 - x_1)\hat{a} - h_4x_3\hat{c} - (e_1 + h_1x_1)(e_3 + h_3x_3) - e_4 \end{cases} \tag{41}$$

and the estimates $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{a}_1$, $\hat{b}_1$, $\hat{c}_1$, $\hat{d}_1$ obey the updating laws:

$$\begin{cases} \dot{a} = -h_1(x_2 - x_1)e_1 - 2h_4x_1(x_2 - x_1)e_4, \\ \dot{b} = -h_2x_1e_2, \\ \dot{c} = h_3x_3e_3 \end{cases} \tag{42}$$

and

$$\begin{cases} \dot{a}_1 = (e_2 - e_1 + h_2x_2 - h_1x_1)e_1, \\ \dot{b}_1 = (e_2 + h_2x_2)e_2, \\ \dot{c}_1 = -(e_3 + h_3x_3)e_3, \\ \hat{d}_1 = (e_4 + h_4x_4)e_4. \end{cases} \tag{43}$$

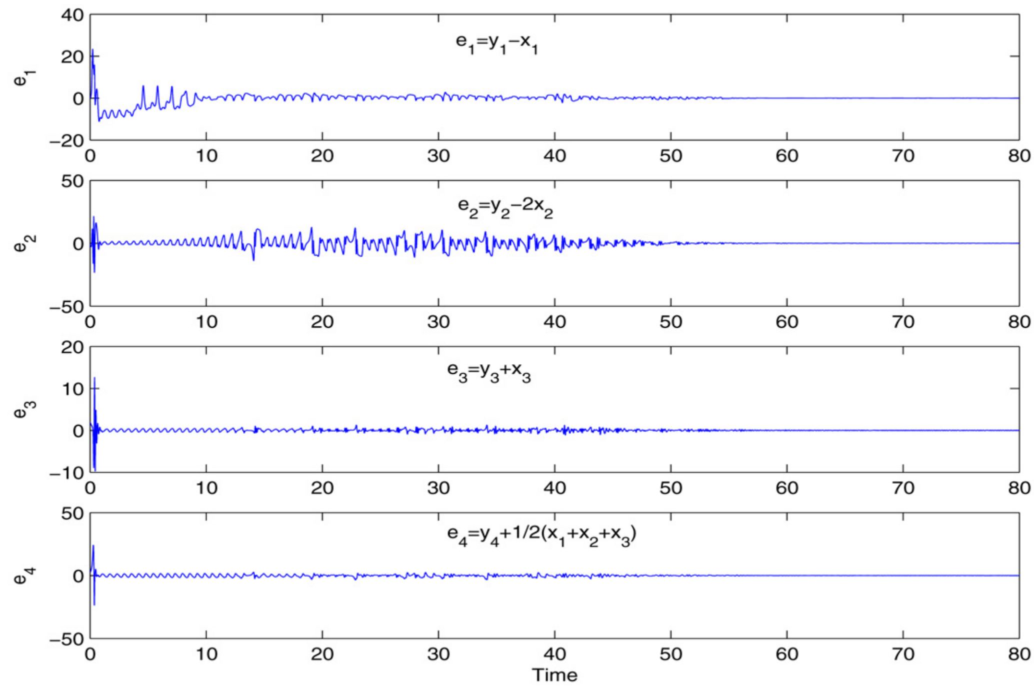

Fig. 7. Increased order MFSHPS of Lorenz and hyper-chaotic Lü systems with fully unknown parameters (auxiliary state variable $x_4 = \phi(x) = x_1 + x_2 + x_3$).

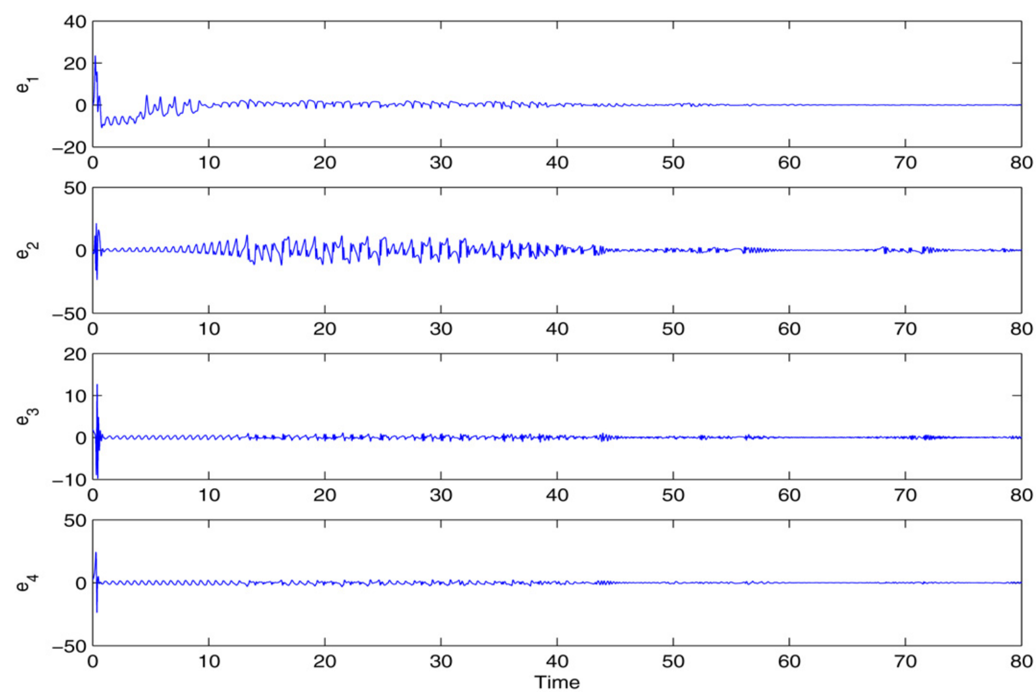

Fig. 8. Increased order MFSHPS of Lorenz and hyper-chaotic Lü systems under the effect of noise with the auxiliary state variable $x_4 = x_1 + x_2 + x_3$.

In this numerical simulations, all the parameters and initial values are same as above. The simulation results are shown in Figs. 9 and 10. Fig. 9 shows the increased order MFSHPS between 3D Lorenz chaotic system with the auxiliary state variable $x_4 = x_1^2$ and 4D hyper-chaotic Lü system. Fig. 10 shows the slight effect of a noise with the strength 1 which is simultaneously added to the signals x_1, x_2 and x_3.

The numerical simulations in Section 5 show that the MFSHPS between chaotic and hyper-chaotic systems with fully unknown parameters can be well achieved by the proposed adaptive FSHPS scheme with different order. In addition, we also find that such synchronization is robust against the effect of noise from these simulations.

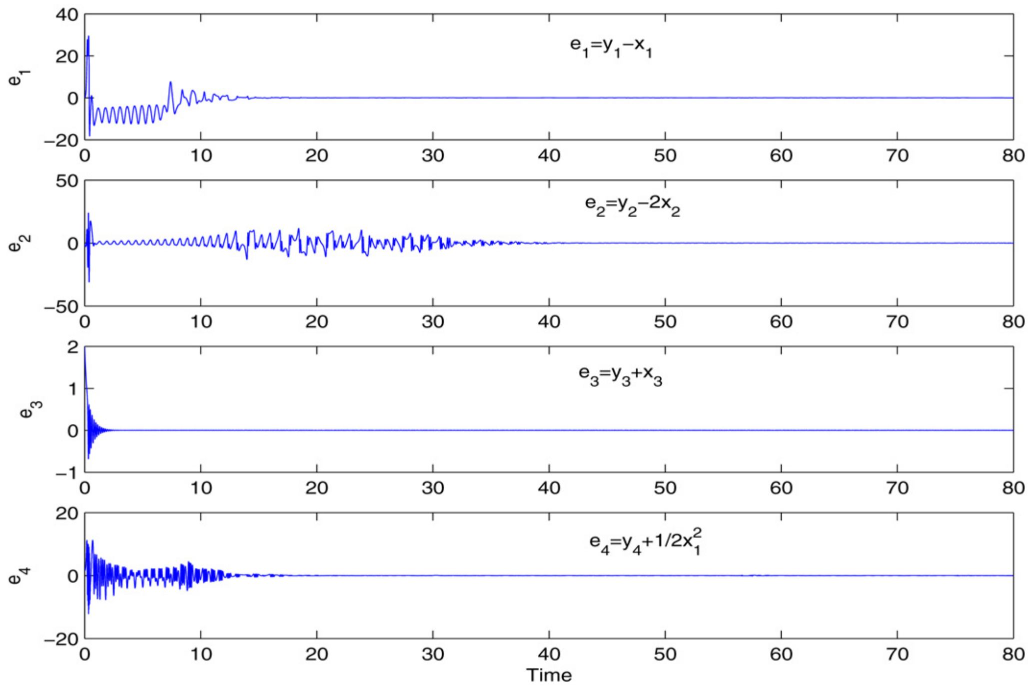

Fig. 9. MFSHPS of Lorenz and hyper-chaotic Lü systems with fully unknown parameters (auxiliary state variable $x_4 = \phi(x) = x_1^2$).

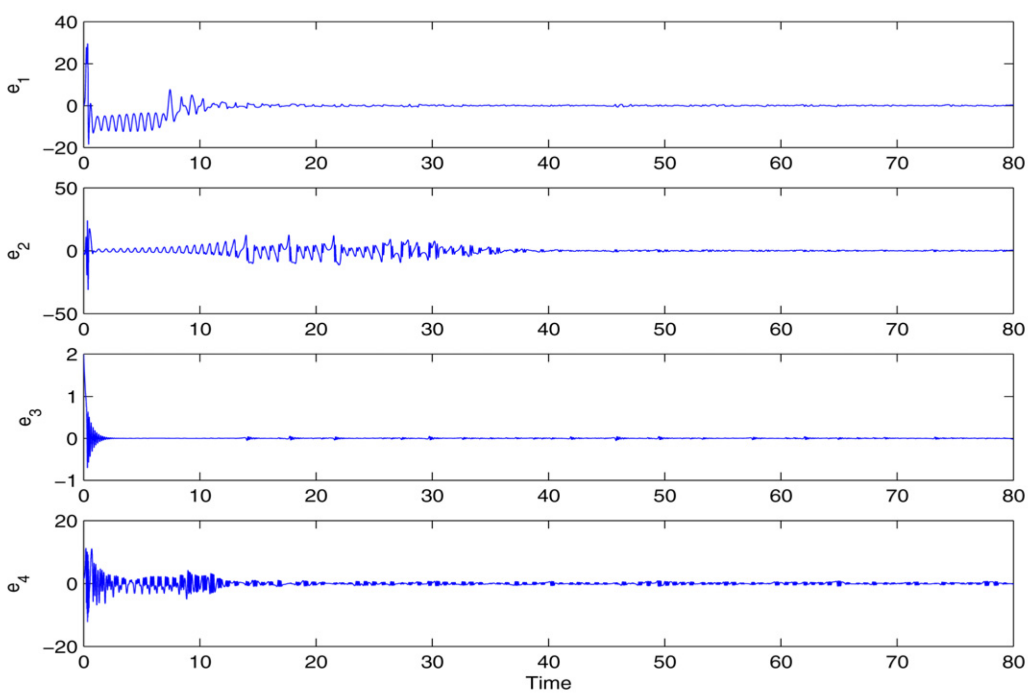

Fig. 10. Increased order MFSHPS of Lorenz and hyper-chaotic Lü systems under the effect of noise with the auxiliary state variable $x_4 = x_1^2$.

6. Conclusions

In this Letter, adaptive FSHPS scheme has been proposed for chaotic (hyper-chaotic) systems with fully unknown parameters. A unified controller and a parameters update law are designed to achieve the FSHPS of two coupled chaotic systems which may be strictly different even with different order, based on the Lyapunov stability theorem. Five groups of numerical simulations are also given to show effectiveness of the proposed scheme.

Because the complete synchronization, anti-synchronization, partial synchronization, projective synchronization are all included in FSHPS, our results contain and extend most existing works. Finally, it is worth noting that although the reduced order synchronization has received some attention, to the best of our knowledge, there are few results on generalized synchronization of different order chaotic dynamical systems, so the proposed reduced order MFSHPS and increased order MFSHPS are worth to further investigate. Continued research would be desirable.

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