Darboux and binary Darboux transformations for discrete integrable systems I. Discrete potential KdV equation

Darboux and binary Darboux transformations for discrete integrable systems I. Discrete potential KdV equation

Ying Shi[1,2], Jonathan J C Nimmo[2] and Da-jun Zhang[1]

[1] Department of Mathematics, Shanghai University, Shanghai 200444,
People's Republic of China
[2] Department of Mathematics, University of Glasgow, Glasgow G12 8QQ, UK

E-mail: y.shi.math@gmail.com

Abstract

The Hirota–Miwa equation can be written in 'nonlinear' form in two ways:
the discrete KP equation and, by using a compatible continuous variable, the
discrete potential KP equation. For both systems, we consider the Darboux
and binary Darboux transformations, expressed in terms of the continuous
variable, and obtain exact solutions in Wronskian and Grammian form. We
discuss reductions of both systems to the discrete KdV and discrete potential
KdV equation, respectively, and exploit this connection to find the Darboux
and binary Darboux transformations and exact solutions of these equations.

1. Introduction

The Hirota–Miwa equation [1, 2] is the three-dimensional discrete integrable system

$$(a_1 - a_2)\tau_{12}\tau_3 + (a_2 - a_3)\tau_{23}\tau_1 + (a_3 - a_1)\tau_{31}\tau_2 = 0, \tag{1.1}$$

where *lattice parameters* a_k are constants, $k = 1, 2, 3$, and for $\tau = \tau(n_1, n_2, n_3)$ each
subscript i denotes a forward shift in the corresponding discrete variable n_i. There are many
papers on the Hirota–Miwa equation describing a number of important results. These include
finding its exact solutions by direct methods and Bäcklund/Darboux transformations [3–7];
its extension to nonautonomous [8–11] and noncommutative forms [12, 13], and reductions
to lower dimensional discrete integrable systems [14–16]. In this paper, we also discuss in
detail the reduction to the discrete Korteweg de Vries (dKdV) equation [17] and the discrete
potential KdV (dpKdV) equation [18]. We exploit these connections to get the Lax pairs, and

the Darboux and binary Darboux transformations of these systems in a natural way. The dpKdV equation is also known as the H1 equation in the Adler–Bobenko–Suris (ABS) classification [19]. Its integrability is understood in the sense of the multidimensional consistency property [20, 21], and gives a Lax pair directly. However, this Lax pair is not suitable for the application of classical Darboux transformations [22, 23]. So, one motivation for this work is to obtain a Lax pair through reduction of the linear system of the Hirota–Miwa equation. This paper is the first of a planned series which will explore the equations in the ABS list, their Lax pairs and Darboux transformations as reductions of the Hirota–Miwa equation.

The Hirota–Miwa equation (1.1) can also be written in terms of 'nonlinear' variables rather than τ-function in two distinct ways[12, 13], when using variables $u^{ij} := \tau_{ij}\tau/\tau_i\tau_j$ and the linear system

$$a_i\phi_i - a_j\phi_j = (a_i - a_j)u^{ij}\phi, \quad 1 \leqslant i < j \leqslant 3, \tag{1.2}$$

where for $\phi = \phi(n_1, n_2, n_3)$ each subscript i denotes a forward shift in the corresponding discrete variable n_i. This linear system (1.2) is compatible if and only if

$$(a_1 - a_2)u^{12} + (a_2 - a_3)u^{23} + (a_3 - a_1)u^{13} = 0, \tag{1.3a}$$

$$(u^{ij})_k u^{ik} = (u^{ik})_j u^{ij}. \tag{1.3b}$$

Each of the variables u^{ij} relates to the solution $u = (\log \tau)_{xx}$ of the KP equation in the continuum limit, so we call (1.3a) the discrete KP (dKP) equation. Note that when one uses the formula $u^{ij} = \tau_{ij}\tau/\tau_i\tau_j$, (1.3a) gives (1.1) and (1.3b) is satisfied identically. A second way is to suppose $u^{ij} = (v_j - v_i + (a_i - a_j))/(a_i - a_j)$, where $v = (\log \tau)_x$. This ansatz solves (1.3a) exactly and (1.3b) becomes the discrete potential KP (dpKP) equation (see also [24]).

The outline of this paper is as follows. In section 2, we recall important definitions and properties of the Hirota–Miwa equation. In particular, we write the Hirota–Miwa equation in 'nonlinear' form in two ways: the discrete KP equation and, by using a compatible continuous variable, the dpKP equation. For both equations, we give two different associated linear systems and their corresponding auxiliary linear systems in differential–difference form. So their Darboux and binary Darboux transformations are given in *differential*, rather than the more usual difference form [5, 6]. The differential form uses the first continuous flow x of the KP hierarchy which is compatible with the discrete flows in the Hirota–Miwa equation. These transformations are used to derive exact solutions in Wronskian and Grammian form, respectively. In section 3, we discuss the 2-reduction of the Hirota–Miwa equation, and in particular of the dKP to the dKdV equation and of the dpKP to the dpKdV equation (H1 in the ABS classification [19]). Then, by taking appropriate reductions of the results in section 2, we derive the Lax pairs for the dKdV and dpKdV equations, and their Darboux and binary Darboux transformations and exact solutions.

2. Hirota–Miwa equation

2.1. Wronskian and Grammian solutions of the Hirota–Miwa equation

The Hirota–Miwa equation (1.1) is the compatibility condition for the linear system [5, 12]

$$a_i\phi_i - a_j\phi_j = (a_i - a_j)\frac{\tau_{ij}\tau}{\tau_i\tau_j}\phi, \tag{2.1}$$

for $1 \leqslant i < j \leqslant 3$. It is invariant with respect to the reversal of all lattice directions $n_i \to -n_i$. On the other hand, the linear system (2.1) does not have such invariance and the reflections $n_i \to -n_i$ acting on (2.1) give a new linear system

$$a_i\psi_{\bar{i}} - a_j\psi_{\bar{j}} = (a_i - a_j)\frac{\tau_{\bar{i}\bar{j}}\tau}{\tau_{\bar{i}}\tau_{\bar{j}}}\psi, \tag{2.2}$$

for all $1 \leqslant i < j \leqslant 3$. The subscript $\bar{i}$ denotes a backward shift with respect to n_i, for example, $\psi_{\bar{1}} := \psi(n_1 - 1, n_2, n_3)$. In [5, 12], the difference form of the Darboux and binary Darboux transformations were derived for the Hirota–Miwa equation (1.1) and these were used to construct exact solutions in the form of Casoratian and discrete Grammian determinants. Here, we will express the Darboux and binary Darboux transformations in differential form instead, using the lowest order continuous flow x of the KP hierarchy, and then the solutions obtained will be expressed as Wronskian and (continuous) Grammian determinants. The differential–difference linear equations for ϕ and ψ are

$$\phi_x = a_i \phi_i + \left(\left(\frac{\tau_x}{\tau} \right)_i - \frac{\tau_x}{\tau} - a_i \right) \phi, \tag{2.3}$$

and

$$\psi_x = -a_i \psi_{\bar{i}} + \left(\left(\frac{\tau_x}{\tau} \right)_{\bar{i}} - \frac{\tau_x}{\tau} + a_i \right) \psi, \tag{2.4}$$

where the subscript x denotes the derivative, with τ satisfying the semi-discrete KP equation

$$(a_i - a_j)(\tau_{ij}\tau - \tau_i \tau_j) + \tau_{i,x}\tau_j - \tau_i \tau_{j,x} = 0. \tag{2.5}$$

It is straightforward to check that (2.3) and (2.4) are compatible with (2.1) and (2.2). Note that the reflection symmetry which relates (2.1) and (2.2) may be extended by adding $x \to -x$ to relate (2.3) and (2.4).

The basic Darboux transformation for the Hirota–Miwa equation is stated in the following proposition.

Proposition 2.1. *Let θ be a non-zero solution of the linear system (2.1) and (2.3) for some τ. Then the transformations*

$$\phi \to \widetilde{\phi} = a_i(\phi_i - \theta_i \theta^{-1}\phi) = \phi_x - \theta_x \theta^{-1}\phi, \quad \tau \to \widetilde{\tau} = \theta\tau, \tag{2.6}$$

leave (2.1) and (2.3) invariant, where $i = 1, 2, 3$.

The proof is a straightforward computation. Note that there are four expressions for $\widetilde{\phi}$ in the Darboux transformation (2.6). These are equivalent because of the linear equations (2.1) and (2.3).

Next we write down the formulae for iterated Darboux transformations, which give solutions in Wronskian and Casoratian determinant form. The Wronskian determinant is the determinant of the $N \times N$ matrix with columns $\mathbf{\Theta}^{(j)} = \mathbf{\Theta}^{(j)}(x, y, t)$, for $j = 0, 1, \ldots, N - 1$, where $\mathbf{\Theta}^{(0)} = (\theta_1(x, y, t), \theta_2(x, y, t), \ldots, \theta_N(x, y, t))^T$ and $\mathbf{\Theta}^{(j)} = \partial^j \mathbf{\Theta}^{(0)}/\partial x^j$. It is written as

$$W(\theta_1, \theta_2, \ldots, \theta_N) = |\mathbf{\Theta}^{(0)}, \mathbf{\Theta}^{(1)}, \ldots, \mathbf{\Theta}^{(N-1)}|,$$

or in a more compact notation [25]

$$W(\theta_1, \theta_2, \ldots, \theta_N) = |\widehat{N - 1}|.$$

The Casoratian determinant can be seen as a discrete analogue of the Wronskian determinant. It is the determinant of the $N \times N$ matrix with columns $\mathbf{\Theta}^{(j)} = \mathbf{\Theta}^{(j)}(n_1, n_2, n_3)$, for $j = 0, 1, \ldots, N - 1$, where $\mathbf{\Theta}^{(0)} = (\theta_1(n_1, n_2, n_3), \theta_2(n_1, n_2, n_3), \ldots, \theta_N(n_1, n_2, n_3))^T$, and $\mathbf{\Theta}^{(j)}$ is defined by the forward shifts, i.e. $\mathbf{\Theta}^{(j)} = T_{n_1}^j(\mathbf{\Theta}^{(0)}) = \mathbf{\Theta}^{(0)}(n_1 + j, n_2, n_3)$. It is written as

$$C(\theta_1, \theta_2, \ldots, \theta_N) = |\mathbf{\Theta}^{(0)}, \mathbf{\Theta}^{(1)}, \ldots, \mathbf{\Theta}^{(N-1)}| = |0, 1, \ldots, N - 1| = |\widehat{N - 1}|. \tag{2.7}$$

We can also use $\mathbf{\Theta}^{(\bar{j})}$, which is defined by the backward shifts, to replace the $\mathbf{\Theta}^{(j)}$, where $\mathbf{\Theta}^{(\bar{j})} = T_{\overline{n_1}}^j(\mathbf{\Theta}^{(0)}) = \mathbf{\Theta}^{(0)}(n_1 - j, n_2, n_3)$.

Below, we use the subscript $[i]$ to designate that the shifts of the Casoratian determinant are with respect to the variable n_i. For example, the Casoratian determinant in (2.7) could be denoted as $C_{[1]}(\theta_1, \theta_2, \ldots, \theta_N)$.

Proposition 2.2. *Let $\theta_1, \theta_2, \ldots, \theta_N$ be non-zero, independent solutions of the linear system (2.1) and (2.3) for some τ. Then N applications of the above Darboux transformations give the transformations*

$$\phi \to \phi[N] = \frac{a_i^N C_{[i]}(\theta_1, \theta_2, \ldots, \theta_N, \phi)}{C_{[i]}(\theta_1, \theta_2, \ldots, \theta_N)} = \frac{W(\theta_1, \theta_2, \ldots, \theta_N, \phi)}{W(\theta_1, \theta_2, \ldots, \theta_N)}, \qquad (2.8)$$

and

$$\tau \to \tau[N] = a_i^{\frac{N(N-1)}{2}} C_{[i]}(\theta_1, \theta_2, \ldots, \theta_N)\tau = W(\theta_1, \theta_2, \ldots, \theta_N)\tau, \qquad (2.9)$$

which leave (2.1) and (2.3) invariant. Here $C_{[i]}$ denotes the Casoratian determinant in forward shifts with respect to the discrete variable n_i, for $i = 1, 2, 3$.

For example, by using the results (2.9), the N-soliton solutions of the Hirota–Miwa equation can be expressed in both Casoratian and Wronskian form in terms of the eigenfunctions,

$$\theta_k(n_1, n_2, n_3) = e^{p_k x} \prod_{i=1}^{3} \left(1 + \frac{p_k}{a_i}\right)^{n_i} + e^{q_k x} \prod_{i=1}^{3} \left(1 + \frac{q_k}{a_i}\right)^{n_i}. \qquad (2.10)$$

Here θ_k, $k = 1, 2, \ldots, N$, is obtained from (2.1) and (2.3), by choosing the trivial solution $\tau = 1$.

Now we can apply the reflections $n_i \to -n_i$ and $x \to -x$ to the above results to deduce corresponding results for the second linear system (2.2) and (2.4).

Proposition 2.3. *Let ρ be a non-zero solution of the linear system (2.2) and (2.4) for some τ. Then the transformations*

$$\psi \to \widetilde{\psi} = a_i(\psi_{\bar{i}} - \rho_{\bar{i}}\rho^{-1}\psi) = \psi_x - \rho_x\rho^{-1}\psi, \quad \tau \to \widetilde{\tau} = \rho\tau, \qquad (2.11)$$

leave (2.2) and (2.4) invariant, for all $i = 1, 2, 3$.

In the statement of this proposition the linearity of (2.2) and (2.4) allows us to omit a minus sign.

Proposition 2.4. *Let $\rho_1, \rho_2, \ldots, \rho_N$ be non-zero, independent solutions of the linear system (2.2) and (2.4) for some τ. Then N applications of the above Darboux transformations give the transformations*

$$\psi \to \psi[N] = \frac{a_i^N C_{[\bar{i}]}(\rho_1, \rho_2, \ldots, \rho_N, \psi)}{C_{[\bar{i}]}(\rho_1, \rho_2, \ldots, \rho_N)} = \frac{W(\rho_1, \rho_2, \ldots, \rho_N, \psi)}{W(\rho_1, \rho_2, \ldots, \rho_N)}, \qquad (2.12)$$

and

$$\tau \to \tau[N] = a_i^{\frac{N(N-1)}{2}} C_{[\bar{i}]}(\rho_1, \rho_2, \ldots, \rho_N)\tau = W(\rho_1, \rho_2, \ldots, \rho_N)\tau, \qquad (2.13)$$

which leave (2.2) and (2.4) invariant. Here $C_{[\bar{i}]}$ denotes the Casoratian determinant in backward shifts with respect to the discrete variable n_i, for $i = 1, 2, 3$.

The N-soliton solutions of the Hirota–Miwa equation are given by (2.13). From the linear system (2.2) and (2.4), if we choose the seed solution as $\tau = 1$, then the Casoratian and Wronskian determinants are defined by eigenfunctions,

$$\rho_k(n_1, n_2, n_3) = e^{-p_k x} \prod_{i=1}^{3} \left(1 + \frac{p_k}{a_i}\right)^{-n_i} + e^{-q_k x} \prod_{i=1}^{3} \left(1 + \frac{q_k}{a_i}\right)^{-n_i}, \qquad (2.14)$$

for $k = 1, 2, \ldots, N$.

To construct a binary Darboux transformation, we introduce the potential $\omega = \omega(\phi, \psi)$, defined by the relations

$$\Delta_i \omega(\phi, \psi) = \phi \psi_i, \qquad (2.15)$$

$$\omega_x(\phi, \psi) = \phi \psi, \qquad (2.16)$$

where $\Delta_i = a_i(T_{n_i} - 1)$ and T_{n_i} is the forward shift operator in variable n_i, for $i = 1, 2, 3$. If ϕ and ψ satisfy the linear systems (2.1), (2.3) and (2.2), (2.4), respectively, then (2.15) and (2.16) are compatible. So the potential ω is well-defined.

The following proposition gives the binary Darboux transformation of the Hirota–Miwa equation.

Proposition 2.5. *Suppose θ and ϕ are non-zero solutions of the linear system (2.1) and (2.3), ρ and ψ are non-zero solutions of the linear system (2.2) and (2.4), then the transformations*

$$\phi \to \widehat{\phi} = \phi - \theta \omega(\theta, \rho)^{-1} \omega(\phi, \rho), \qquad (2.17)$$

$$\psi \to \widehat{\psi} = \psi - \rho \omega(\theta, \rho)^{-1} \omega(\theta, \psi), \qquad (2.18)$$

leave (2.1), (2.3) and (2.2), (2.4) invariant, respectively, with τ changing to

$$\widehat{\tau} = \omega(\theta, \rho)\tau. \qquad (2.19)$$

The N-fold iteration of the binary Darboux transformation is as follows.

Proposition 2.6. *Let $\theta_1, \ldots, \theta_N$ and $\rho_1, \ldots, \rho_N$ satisfy the linear systems (2.1), (2.3) and (2.2), (2.4), respectively. Define N-vectors $\boldsymbol{\Theta} = (\theta_1, \ldots, \theta_N)^T$ and $\boldsymbol{P} = (\rho_1, \ldots, \rho_N)^T$. Then N applications of the binary Darboux transformation give the transformations*

$$\phi \to \phi[N] = \begin{vmatrix} \boldsymbol{\Omega}(\boldsymbol{\Theta}, \boldsymbol{P}) & \boldsymbol{\Theta} \\ \boldsymbol{\Omega}(\phi, \boldsymbol{P}) & \phi \end{vmatrix} |\boldsymbol{\Omega}(\boldsymbol{\Theta}, \boldsymbol{P})|^{-1}, \quad \psi \to \psi[N] = \begin{vmatrix} \boldsymbol{\Omega}(\boldsymbol{\Theta}, \boldsymbol{P}) & \boldsymbol{P} \\ \boldsymbol{\Omega}(\boldsymbol{\Theta}, \psi) & \psi \end{vmatrix} |\boldsymbol{\Omega}(\boldsymbol{\Theta}, \boldsymbol{P})|^{-1},$$

$$(2.20)$$

which leave (2.1), (2.3) and (2.2), (2.4) invariant, respectively, with τ changing to

$$\tau[N] = |\boldsymbol{\Omega}(\boldsymbol{\Theta}, \boldsymbol{P})|\tau. \qquad (2.21)$$

Here $\boldsymbol{\Omega}(\boldsymbol{\Theta}, \boldsymbol{P}) = (\omega(\theta_i, \rho_j))_{i,j=1,\ldots,N}$ is an $N \times N$ matrix, $\boldsymbol{\Omega}(\phi, \boldsymbol{P}) = (\omega(\phi, \rho_j))_{j=1,\ldots,N}$ and $\boldsymbol{\Omega}(\psi, \boldsymbol{\Theta}) = (\omega(\theta_i, \psi))_{i=1,\ldots,N}$ are N-row vectors.

The proofs of those above propositions are straightforward computation, so we do not give the details. The reader is also referred to the papers [5, 6, 12].

2.2. The discrete potential KP equation

By introducing a potential $v(n_1, n_2, n_3; x) := \tau_x/\tau$, the semi-discrete equation (2.5) gives the relation

$$u^{ij} = \frac{\tau_{ij}\tau}{\tau_i\tau_j} = \frac{v_j - v_i + (a_i - a_j)}{a_i - a_j}, \quad 1 \leqslant i < j \leqslant 3, \qquad (2.22)$$

for v each subscript i denotes a forward shift in the corresponding discrete variable n_i. So (1.3a) is satisfied identically, and (1.3b) becomes

$$\frac{v_2 - v_1 + a_1 - a_2}{(v_2 - v_1 + a_1 - a_2)_3} = \frac{v_3 - v_1 + a_1 - a_3}{(v_3 - v_1 + a_1 - a_3)_2} = \frac{v_3 - v_2 + a_2 - a_3}{(v_3 - v_2 + a_2 - a_3)_1}. \qquad (2.23a)$$

The equation (2.23*a*) can be written in two other equivalent forms as either

$$(v_2 - v_1 + a_1 - a_2)(v_{23} - v_{12} + a_1 - a_3) = (v_3 - v_1 + a_1 - a_3)(v_{23} - v_{13} + a_1 - a_2),$$
(2.23*b*)

or

$$(a_3 + v_{12})(a_1 - a_2 + v_2 - v_1) + (a_2 + v_{13})(a_3 - a_1 + v_1 - v_3)$$
$$+ (a_1 + v_{23})(a_2 - a_3 + v_3 - v_2) = 0.$$
(2.23*c*)

Equation (2.23) is called the dpKP equation. It was first found in [24] as the nonlinear superposition of solutions to the potential KP equation related by the Bäcklund transformations [24]. The trivial solution of the dpKP equation (2.23) could be $v = c$, where c is an arbitrary constant.

The linear systems associated with the dpKP equation, obtained by using relations (2.1) and (2.2), together with (2.22), are

$$a_i \phi_i - a_j \phi_j = (v_j - v_i + a_i - a_j)\phi,$$
(2.24)

and

$$a_i \psi_{\bar{i}} - a_j \psi_{\bar{j}} = (v_{\bar{j}} - v_{\bar{i}} + a_i - a_j)\psi,$$
(2.25)

where $1 \leqslant i < j \leqslant 3$. Its corresponding differential–difference linear systems are (2.3) and (2.4) with $\tau_x / \tau = v$.

Together with the differential–difference linear system (2.3), the Darboux transformation of the linear system (2.24) gives a new solution of the dpKP equation

$$\widetilde{v} = (\log(\theta\tau))_x = v + (\log\theta)_x,$$

where θ is a non-zero solution of (2.24) and (2.3). More generally, N-fold iteration gives the Wronskian solution

$$v[N] = v + (\log W(\theta_1, \theta_2, \ldots, \theta_N))_x,$$

where $\theta_k, k = 1, 2, \ldots, N$ are the non-zero independent solutions of (2.24) and (2.3).

The binary Darboux transformation gives a new solution of the dpKP equation

$$\widehat{v} = v + (\log \omega(\theta, \rho))_x,$$

where $\omega(\theta, \rho)$ is defined by (2.15) and (2.16), and θ, ρ are non-zero solutions of (2.24), (2.3) and (2.25), (2.4), respectively. Its N-fold iteration gives the Grammian solution

$$v[N] = v + (\log|\boldsymbol{\Omega}(\boldsymbol{\Theta}, \boldsymbol{P})|)_x,$$

where the $N \times N$ matrix $\boldsymbol{\Omega}(\boldsymbol{\Theta}, \boldsymbol{P}) = (\omega(\theta_i, \rho_j))_{i,j=1,\ldots,N}$, $\omega(\theta_i, \rho_j)$ is defined by (2.15) and (2.16), and θ_i, ρ_j are non-zero solutions of (2.24), (2.3) and (2.25), (2.4), respectively.

3. Reductions to the dKdV and dpKdV equations

3.1. The dKP equation to the dKdV equation

The discrete KdV (dKdV) equation is a 2-reduction of the dKP equation (1.3*a*). In the reduction, it is necessary that one takes $a_2 + a_3 = 0$, since the solutions satisfy the reduction condition $\tau_{23} = \tau$, cf [3, 10]. The reduction condition for the eigenfunction of the linear system (2.1) and (2.3) is $\phi_{23} = (1 - \lambda^2)\phi$, where the form of the coefficient is chosen for its correspondence with the discrete one-dimensional Schrödinger equation [26, 27]. To realise this reduction, we make the ansatz

$$\tau(n_1, n_2, n_3) = \tau(n_1, n_2 - n_3)$$
(3.1)

$$\phi(n_1, n_2, n_3) = (1 - \lambda^2)^{n_3}\phi(n_1, n_2 - n_3).$$
(3.2)

Now taking the reduction conditions, the dKP equation (1.3*a*) can be written as either

$$(a_2 - a_1)\frac{\tau_{12}\tau}{\tau_1\tau_2} + (a_2 + a_1)\frac{\tau_{1\bar{2}}\tau}{\tau_1\tau_{\bar{2}}} = 2a_2\frac{\tau}{\tau_{\bar{2}}}\frac{\tau}{\tau_2}, \tag{3.3a}$$

or

$$(a_2 - a_1)\frac{\tau_1\tau_{\bar{2}}}{\tau_{1\bar{2}}\tau} + (a_2 + a_1)\frac{\tau_1\tau_2}{\tau_{12}\tau} = 2a_2\frac{\tau_1}{\tau_{1\bar{2}}}\frac{\tau_1}{\tau_{12}}. \tag{3.3b}$$

We then express the above two equations in terms of nonlinear variable

$$u(n_1, n_2) := \frac{\tau_1\tau_3}{\tau_{13}\tau} = \frac{\tau_1\tau_{\bar{2}}}{\tau_{1\bar{2}}\tau}. \tag{3.4}$$

By eliminating the tau-function parts on the right-hand sides in both (3.3*a*) and (3.3*b*), we obtain

$$\frac{1}{u_1} - \frac{1}{u_2} = \frac{a_1 - a_2}{a_1 + a_2}(u_{12} - u), \tag{3.5}$$

which is the discrete KdV equation, first found by Hirota [17].

From the linear system (2.1) and using the 2-reduction, we get the linear system of the dKdV equation

$$a_1\phi_1 - a_2\phi_2 = (a_1 - a_2)u_2\phi, \tag{3.6a}$$

$$a_1\phi_1 + a_2(1 - \lambda^2)\phi_{\bar{2}} = (a_1 + a_2)\frac{1}{u}\phi. \tag{3.6b}$$

Note here that, by eliminating the ϕ_1 in these two equations, we have

$$a_2\phi_2 + a_2(1 - \lambda^2)\phi_{\bar{2}} = \left((a_2 - a_1)u_2 + (a_2 + a_1)\frac{1}{u}\right)\phi,$$

which is a discrete one-dimensional Schrödinger equation [26, 27].

The dKdV equation (3.5) is also invariant with respect to the reflections $n_i \rightarrow -n_i$, for $i = 1, 2$. So applying the reflections to the system (3.6) gives a new linear system of the dKdV equation

$$a_1\psi_{\bar{1}} - a_2\psi_{\bar{2}} = (a_1 - a_2)u_{\bar{2}}\psi, \tag{3.7a}$$

$$a_1\psi_{\bar{1}} + a_2(1 - \lambda^2)\psi_2 = (a_1 + a_2)\frac{1}{u}\psi. \tag{3.7b}$$

This system could also be obtained from the linear system (2.2), by using the 2-reduction $\tau_{\bar{2}\bar{3}} = \tau$, and $\psi_{\bar{2}\bar{3}} = (1 - \lambda^2)\psi$ with $a_2 + a_3 = 0$.

For the linear differential–difference equations (2.3) and (2.4), the 2-reduction does not affect the n_1- or n_2- parts, but the n_3-parts become

$$\phi_x = -a_2(1 - \lambda^2)\phi_{\bar{2}} + \left(\left(\frac{\tau_x}{\tau}\right)_{\bar{2}} - \frac{\tau_x}{\tau} + a_2\right)\phi, \tag{3.8}$$

$$\psi_x = a_2(1 - \lambda^2)\psi_2 + \left(\left(\frac{\tau_x}{\tau}\right)_2 - \frac{\tau_x}{\tau} - a_2\right)\psi. \tag{3.9}$$

The fundamental Darboux transformation for the dKdV equation is

Proposition 3.1. *Suppose θ is a non-zero solution of the linear system (3.6) and (3.8) for some u and τ, together with $u = \tau_1\tau_{\bar{2}}/\tau_{1\bar{2}}\tau$, then the transformations*

$$\phi \rightarrow \widetilde{\phi} = a_i(\phi_i - \theta_i\theta^{-1}\phi) = \phi_x - \theta_x\theta^{-1}\phi, \quad i = 1, 2, \tag{3.10a}$$

and

$$\tau \rightarrow \widetilde{\tau} = \theta\tau, \quad u \rightarrow \widetilde{u} = u\frac{\theta_1\theta_{\bar{2}}}{\theta_{1\bar{2}}\theta}, \tag{3.10b}$$

leave (3.6) and (3.8) invariant.

Proof. One way to complete the proof is a straightforward computation. A second way is using the idea that the Darboux transformation of the dKP equation in the proposition 2.1 also works for the dKdV equation after taking the 2-reduction. From the linear system (3.6), ϕ and θ are its solutions, so we have

$$a_i(\phi_i - \theta_i\theta^{-1}\phi) = -a_2(1 - \lambda^2)(\phi_{\bar{2}} - \theta_{\bar{2}}\theta^{-1}\phi), \tag{3.11}$$

for $i = 1, 2$. On the other hand, the 2-reduction gives

$$a_3(\phi_3 - \theta_3\theta^{-1}\phi) = -a_2(1 - \lambda^2)(\phi_{\bar{2}} - \theta_{\bar{2}}\theta^{-1}\phi). \tag{3.12}$$

For the potential u, we have

$$\widetilde{u} = \frac{\widetilde{\tau}_1\widetilde{\tau}_{\bar{2}}}{\widetilde{\tau}_{1\bar{2}}\widetilde{\tau}} = u\frac{\theta_1\theta_{\bar{2}}}{\theta_{1\bar{2}}\theta}.$$

So together with the transformation in (2.6), it means that after taking 2-reduction, the transformations (3.10) leave the linear system (3.6) and (3.8) invariant. □

Similarly, by using the reflections $n_i \to -n_i$ and $x \to -x$, or the 2-reduction of the Darboux transformation of the dKP equation in proposition 2.3, we get the Darboux transformation of the linear system (3.7) and (3.9).

Proposition 3.2. *Suppose ρ is a non-zero solution of the linear system (3.7) and (3.9) for some u and τ, together with $u = \tau_{\bar{1}}\tau_2/\tau_{\bar{1}2}\tau$, then the transformations*

$$\psi \to \widetilde{\psi} = a_i(\psi_{\bar{i}} - \rho_{\bar{i}}\rho^{-1}\psi) = \psi_x - \rho_x\rho^{-1}\psi, \quad i = 1, 2, \tag{3.13a}$$

and

$$\tau \to \widetilde{\tau} = \rho\tau, \quad u \to \widetilde{u} = u\frac{\rho_{\bar{1}}\rho_2}{\rho_{\bar{1}2}\rho}, \tag{3.13b}$$

leave (3.7) and (3.9) invariant.

Similarly, in the light of the reductions, the binary Darboux transformations of the dKdV equation can also be obtained directly from the one in proposition 2.5. Thus the results of the N-applications of the Darboux and binary Darboux transformations for the dKdV equation can be gotten from the ones in propositions 2.2, 2.4 and 2.6. For this reason, we will not go to talk about them in detail here.

3.2. The dpKP equation to the dpKdV equation

Here again we use the 2-reduction, $v_{23} = v$ and $a_2 + a_3 = 0$. Now by introducing the potential $v(n_1, n_2; x) := \tau_x/\tau$ into (2.22), we get

$$(a_1 - a_2)u_2 = v_2 - v_1 + a_1 - a_2, \tag{3.14a}$$

$$(a_1 + a_2)\frac{1}{u} = v_{\bar{2}} - v_1 + a_1 + a_2. \tag{3.14b}$$

By eliminating the potential u from (3.14), we obtain

$$(v_2 - v_1 + a_1 - a_2)(v - v_{12} + a_1 + a_2) = a_1^2 - a_2^2. \tag{3.15}$$

This is the dpKdV equation [18], and we see that its solution can be written as $v = (\log \tau)_x$. The dpKdV equation could also be obtained by the permutability property of the Bäcklund transformations of the continuous potential KdV equation, cf [18]. Taking the potential transformation $v \to v + a_1n_1 + a_2n_2 + \gamma$, where γ is an arbitrary constant, (3.15) becomes

$$(v_2 - v_1)(v - v_{12}) = a_1^2 - a_2^2, \tag{3.16}$$

which is called the H1 equation [19].

By taking the 2-reduction, the linear system of the dpKP equation (2.24) and (2.25) give the Lax pairs of the dpKdV equation (3.15),

$$a_1\phi_1 - a_2\phi_2 = (v_2 - v_1 + a_1 - a_2)\phi, \tag{3.17a}$$

$$a_1\phi_1 + a_2(1 - \lambda^2)\phi_{\bar{2}} = (v_{\bar{2}} - v_1 + a_1 + a_2)\phi, \tag{3.17b}$$

and with reflections $n_i \to -n_i$, this also gives

$$a_1\psi_{\bar{1}} - a_2\psi_{\bar{2}} = (v_{\bar{2}} - v_{\bar{1}} + a_1 - a_2)\psi, \tag{3.18a}$$

$$a_1\psi_{\bar{1}} + a_2(1 - \lambda^2)\psi_2 = (v_2 - v_{\bar{1}} + a_1 + a_2)\psi. \tag{3.18b}$$

Their corresponding differential–difference linear systems are (2.3) and (2.4) with $\tau_x/\tau = v$, for $i = 1, 2$, respectively. Note here that the Lax pair (3.17) is not the same as but can be transformed into the one derived from the multidimensional consistency property, cf [28–30], by using appropriate parameters transformations, such as $r = a_2\lambda$.

As we showed for the dpKP equation, the Darboux transformation gives the new solution of the dpKdV equation in differential form,

$$\widetilde{v} = v + (\log\theta)_x,$$

where θ is the non-zero solution of (3.17) and (2.3). Its N-fold iteration gives the solution in Wronskian form

$$v[N] = v + (\log W(\theta_1, \theta_2, \ldots, \theta_N))_x, \tag{3.19}$$

where θ_k, for $k = 1, 2, \ldots, N$, are the non-zero solutions of (3.17) and (2.3).

The soliton solutions of the dpKdV equation (3.15) can be obtained from (3.19). From the linear system (3.17) and (2.3), by choosing the seed solution $v = 0$, the eigenfunctions of the τ-function in Wronskian determinant are

$$\theta_k(n_1, n_2) = e^{a_2\lambda_k x}(1 + \lambda_k\beta)^{n_1}(1 + \lambda_k)^{n_2} + e^{-a_2\lambda_k x}(1 - \lambda_k\beta)^{n_1}(1 - \lambda_k)^{n_2}, \quad \beta = \frac{a_2}{a_1},$$

for $k = 1, 2, \ldots, N$. This result coincides with the one given by Hietarinta and Zhang [29]. In their paper, they employed $f = |\widehat{N-1}|$ and $g = |\widehat{N-2}, N|$, which were given as Casoratian determinants with respect to an auxiliary discrete variable. This auxiliary variable is compatible with the original independent variables, n_1 and n_2, but is external to the system. We will denote this auxiliary variable to be n_4 here. Then for an arbitrary constant c, $v = \frac{g}{f} + c$ satisfies the dpKdV equation (3.15). f and g in Casoratian determinants in compact form are

$$f = |\widehat{N-1}| = \left|\phi, T_{n_4}(\phi), T_{n_4}^2(\phi), \ldots, T_{n_4}^{N-1}(\phi)\right|,$$

and

$$g = |\widehat{N-2}, N| = \left|\phi, T_{n_4}(\phi), T_{n_4}^2(\phi), \ldots, T_{n_4}^{N-2}(\phi), T_{n_4}^N(\phi)\right|,$$

where $\phi = \phi(n_1, n_2, n_4, x)$ and $T_{n_4}^j(\phi) = \phi(n_1, n_2, n_4 + j, x)$, $j = 0, 1, \ldots, N$. Similarly, τ and τ_x in Wronskian determinants are

$$\tau = |\widehat{N-1}| = \left|\phi, \partial_x\phi, \partial_x^2\phi, \ldots, \partial_x^{N-1}\phi\right|,$$

and

$$\tau_x = |\widehat{N-2}, N| = \left|\phi, \partial_x\phi, \partial_x^2\phi, \ldots, \partial_x^{N-2}\phi, \partial_x^N\phi\right|.$$

So for the linear systems (3.17) and (2.3) with $\tau_x/\tau = v$, we could also introduce a virtual discrete variable n_4, say, which is another compatible discrete flow, with parameter a_4. If we choose the seed solution of the dpKdV equation $v = 0$, then the differential–difference linear system (2.3), with respect to the discrete variable n_4, gives the entries of the above Wronskian and Carsoratian determinants satisfying the relations

$$\phi_x = a_4(T_{n_4} - 1)(\phi),$$

and
$$\partial_x^N \phi = a_4^N (T_{n_4} - 1)^N (\phi).$$
So we have
$$\tau = a_4^{\frac{N(N-1)}{2}} f, \quad \tau_x = a_4^{\frac{N(N-1)}{2}+1} (g - Nf).$$
By taking $a_4 = 1$, the soliton solution of the dpKdV equation is
$$v = (\log \tau)_x = \frac{\tau_x}{\tau} = \frac{g}{f} - N.$$
So if the soliton solution of the dpKdV equation is expressed in Wronskian determinant, by using a compatible continuous variable, then it needs to employ τ and its derivative τ_x. If it is expressed in Casoratian determinant, through a virtual discrete variable n_4, then it needs to employ f and g. But there is no direct relation between f and g.

The binary Darboux transformation gives the exact solution of the dpKdV equation,
$$\widehat{v} = v + (\log \omega(\theta, \rho))_x,$$
where $\omega(\theta, \rho)$ is defined by (2.15) and (2.16), and θ, ρ are non-zero solutions of (3.17), (2.3) and (3.18), (2.4), respectively, with $\tau_x/\tau = v$. Its N-fold iteration gives the exact solution in Grammian form,
$$v[N] = v + (\log |\mathbf{\Omega}(\mathbf{\Theta}, \mathbf{P})|)_x,$$
where the $N \times N$ matrix $\mathbf{\Omega}(\mathbf{\Theta}, \mathbf{P}) = (\omega(\theta_i, \rho_j))_{i,j=1,\dots,N}$, $\omega(\theta_i, \rho_j)$ is defined by (2.15) and (2.16), and θ_i, ρ_j are non-zero solutions of (3.17), (2.3) and (3.18), (2.4), respectively, with $\tau_x/\tau = v$.

4. Conclusions

In this paper, we have revisited the Darboux and binary transformations of the Hirota–Miwa equation given in [5, 6] not only in difference form in a more general case, including the lattice parameters a_i, but also in differential form [22, 23]. These allow one to obtain solutions of the Hirota–Miwa equation in both Casoratian and Wronskian forms, and as discrete and continuous Grammians as well. It is straightforward to obtain corresponding results for reductions of the Hirota–Miwa equation. In this paper we do this for the dKdV and dpKdV equations. The Lax pair obtained by reductions are not the same as the ones given by the multidimensional consistency property, and allow the application of the classical Darboux transformations [22, 23]. We find that the Hirota–Miwa equation can be written in 'nonlinear' form in two distinct ways: as the discrete KP equation in terms of the variable $u^{ij} = \tau_{ij}\tau/\tau_i\tau_j$ and, by using a compatible continuous variable x, as the discrete potential KP (dpKP) equation in variable $u^{ij} = (v_j - v_i + (a_i - a_j))/(a_i - a_j)$, where $v = (\log \tau)_x$. This leads us to see clearly the relationship between the dKP and dpKP equations, which is similar to the relationship $u = v_x$ in the continuous case. Thus we understand better the form of the Casoratian solutions to the H1 equation in [29], and it will be helpful in future to deal with other members in the ABS list, such as the H2 equation.

References

[1] Hirota R 1981 Discrete analogue of a generalized Toda equation *J. Phys. Soc. Japan* **50** 3785–91
[2] Miwa T 1982 On Hirota's difference equations *Proc. Japan Acad.* A **58** 9–12
[3] Ohta Y, Hirota R, Tsujimoto S and Imai T 1993 Casorati and discrete Gram type determinant representations of solutions to the discrete KP hierarchy *J. Phys. Soc. Japan* **62** 1872–86
[4] Saitoh N and Saito S 1987 General solutions to the Bäcklund transformation of the Hirota's bilinear difference equation *J. Phys. Soc. Japan* **56** 1664–74
[5] Nimmo J J C 1997 Darboux transformations and the discrete KP equation *J. Phys. A: Math. Gen.* **30** 8693–704
[6] Nimmo J J C 2000 Darboux transformations for discrete systems *Chaos Solitons Fractals* **11** 115–20
[7] Doliwa A and Nieszporski J 2009 Darboux transformations for linear operators on two-dimensional regular lattices *J. Phys. A: Math. Gen.* **42** 454001
[8] Willox R, Tokihiro T and Satsuma J 1997 Darboux and binary Darboux transformations for the nonautonomous discrete KP equation *J. Math. Phys.* **38** 6455
[9] Willox R, Tokihiro T and Satsuma J 2000 Nonautonomous discrete integrable systems *Chaos Solitons Fractals* **11** 121–35
[10] Kajiwara K and Ohta Y 2008 Bilinearization and Casorati determinant solution to the Non-autonomous discrete KdV equation *J. Phys. Soc. Japan* **77** 054004
[11] Tsujimoto S 2010 Determinant solutions of the nonautonomuos discrete Toda equation associated with the deautonomized discrete KP hierarchy *J. Syst. Sci. Complex* **23** 153–76
[12] Nimmo J J C 2006 On a non-Abelian Hirota–Miwa equation *J. Phys. A: Math. Gen.* **39** 5053–65
[13] Gilson C R, Nimmo J J C and Ohta Y 2007 Quasideterminant solutions of a non-Abelian Hirota–Miwa equation *J. Phys. A: Math. Theor.* **40** 12607–17
[14] Date E, Jimbo M and Miwa T 1983 Method for generating discrete soliton equations. III *J. Phys. Soc. Japan* **52** 388–93
[15] Ramani A, Grammaticos B and Satsuma J 1992 Integrability of multidimensional discrete systems *Phys. Lett.* A **162** 323–28
[16] Zabrodin A 1997 Hirota's difference equations *Theor. Math. Phys.* **113** 1317–92
[17] Hirota R 1977 Nonlinear partial Difference equations. I. A difference analogue of the Korteweg–de Vries equation *J. Phys. Soc. Japan* **43** 1424–33
[18] Nijhoff F W, Quispel G R and Capel H W 1983 Direct linearization of nonlinear difference–difference equations *Phys. Lett.* A **97** 125–28
[19] Adler V E, Bobenko A I and Suris Y B 2003 Classification of integrable equations on quad-graphs. The consistency approach *Commun. Math. Phys.* **233** 513–43
[20] Nijhoff F W and Walker A J 2001 The discrete and continuous Painlevé VI hierarchy and the Garnier systems *Glasgow Math. J.* A **43** 109–23
[21] Bobenko A I and Suris Yu B 2002 Integrable systems on quad-graphs *Int. Math. Res. Not.* **11** 573–611
[22] Matveev V B 1979 Darboux transformations and the explicit solutions of differential–difference and difference–difference evolution equations *Lett. Math. Phys.* **3** 217–22
[23] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[24] Nijhoff F W, Capel H W, Wiersma G L and Quispel G R W 1984 Bäcklund transformations and three-dimensional lattice equations *Phys. Lett.* A **105** 267–72
[25] Freeman N C and Nimmo J J C 1983 Soliton solutions of the KdV and KP equations: the Wronskian technique *Phys. Lett.* A **95** 1–3
[26] Case K M and Kac M 1973 A discrete version of the inverse scattering problem *J. Math. Phys.* **14** 594–603
[27] Willox R, Nakata Y, Satsuma J, Ramani A and Grammaticos B 2010 Solving the ultradiscrete KdV equation *J. Phys. A: Math. Theor.* **43** 482003
[28] Nijhoff F W, Atkinson J and Hietarinta J 2009 Soliton solutions for ABS lattice equations: I. Cauchy matrix approach *J. Phys. A: Math. Theor.* **42** 404005
[29] Hietarinta J and Zhang D J 2009 Soliton solutions for ABS lattice equations: II. Casoratians and bilinearization *J. Phys. A: Math. Theor.* **42** 4006–35
[30] Butler S and Joshi N 2010 An inverse scattering transform for the lattice potential KdV equation *Inverse Problems* **26** 115012