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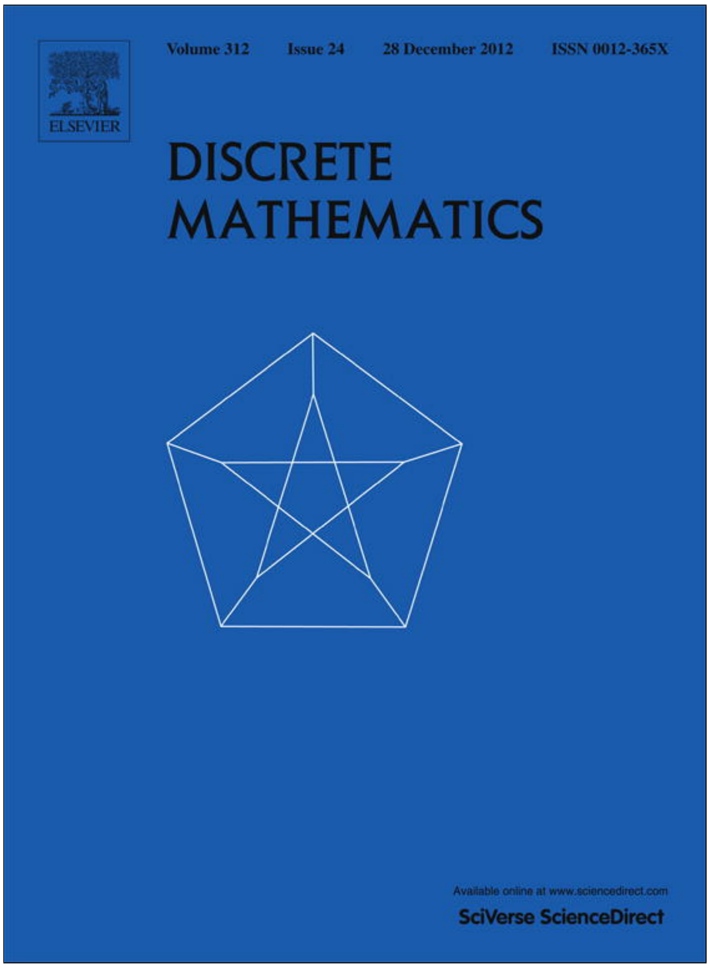

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Acyclic edge coloring of sparse graphs

Yingqian Wang *, Ping Sheng

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

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ABSTRACT

Let Δ denote the maximum degree of a graph. Fiamčík first, Alon, Sudakov and Zaks later conjectured that every graph is acyclically edge $(\Delta+2)$-colorable. In this paper, we prove this conjecture for graphs with maximum average degree less than 4. As a corollary, triangle-free planar graphs are acyclically edge $(\Delta+2)$-colorable.

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1. Introduction

Graphs considered in this paper are finite, simple and undirected. Used but undefined terminology and notation can be found in [5].

A *proper edge coloring* of a graph $G=(V,E)$ is a mapping from the edge set E of G to an available color set such that any two adjacent edges receive distinct colors. In a proper edge coloring of G, a cycle is *bichromatic* if only two colors appear on the cycle. An *acyclic edge coloring* of a graph G is a proper edge coloring that results in no bichromatic cycle. If an acyclic edge coloring of G uses at most k colors, then it is called an *acyclic edge k-coloring*. A graph G is *acyclically edge k-colorable* if it admits an acyclic edge k-coloring. The *acyclic chromatic index* of a graph G, denoted by $\chi'_a(G)$, is the minimum number k such that G is acyclically edge k-colorable. Note that if G is acyclically edge k-colorable, then E can be decomposed into k subsets so that the union of any two of the k subsets is a forest.

The concept of acyclic edge coloring was first introduced by Fiamčík [8]. A conjecture proposed first by Fiamčík [8] and then again by Alon et al. [1] states that, for every graph G, $\chi'_a(G)\le\Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. Even for planar graphs, this conjecture remains open with large gap. For the main achievements on acyclic edge coloring of graphs in the literature, we would like to refer the reader to the introductions of [2–4,7,9,10].

A graph is *planar* if it can be embedded into the plane so that its edges meet only at their ends. The acyclic edge colorability for planar graphs has been extensively studied. Let G be a planar graph. Fiedorowicz et al. [9] proved that $\chi'_a(G)\le 2\Delta(G)+29$. Hou et al. [13] proved that $\chi'_a(G)\le\max\{2\Delta(G)-2,\Delta(G)+22\}$. Recently Basavaraju et al. [4] have showed that every planar graph G is acyclically edge $(\Delta(G)+12)$-colorable.

A *k-cycle* is a cycle of length k. A *triangle* is a 3-cycle. Let G be a triangle-free planar graph. Fiedorowicz et al. [9] proved that $\chi'_a(G)\le\Delta(G)+6$. This bound was improved to $\Delta(G)+5$ in [7]. Hou et al. [10] showed that $\chi'_a(G)\le\Delta(G)+3$ if $\Delta(G)\ge 8$ (the condition $\Delta(G)\ge 8$ was dropped in [3]).

For other interesting results on acyclic edge coloring of sparse (not necessarily planar) graphs, we refer the reader to [2,6,11,12,14,15]. Here we would like to emphasize one result in [2]: $\chi'_a(G)\le\Delta+2=6$ if G satisfies $\Delta(G)=4$ and $2|E(G)|\le 4|V(G)|-1$.

* Corresponding author.
E-mail address: yqwang@zjnu.cn (Y. Wang).

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In this paper, we improve some results stated above by proving the following result.

Theorem 1. *If* $\mathrm{mad}(G) := \max\{\frac{2|E(H)|}{|V(H)|} | H \subseteq G\} < 4$, *then* $\chi'_a(G) \leq \Delta(G) + 2$.

Combining a well known fact that, for a planar graph G with girth g (the length of the shortest cycles in G), $\mathrm{mad}(G) < \frac{2g}{g-2}$, we immediately have the following.

Corollary 1. *Every triangle-free planar graph is acyclically edge* $(\Delta + 2)$*-colorable.*

The rest of this section is mainly devoted to some terminology and notation used later. For a vertex v in a graph G, the degree of v in G, denoted by $d_G(v)$ or simply $d(v)$, is the number of edges incident with v in G. Call v a *k-vertex*, or a *k^+-vertex*, or a *k^--vertex* if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. A *k-neighbor* of v is a k-vertex that is adjacent to v.

Let $e = xy$ be an edge of G with two ends x and y. We call e a *$(d(x), d(y))$-edge*. $G - e$ or $G - xy$ is the graph obtained from G by deleting the edge $e = xy$.

A *partial acyclic edge coloring* of G is an acyclic edge coloring of a proper subgraph of G. Let ϕ be a partial acyclic edge coloring of G using colors from the color set $[k] = \{1, 2, \ldots, k\}$. An *$(\alpha, \beta)$-maximal bichromatic path* under ϕ is a non-extendable path consisting of edges that are colored with colors α and β alternatingly. An *(α, β, u, v)-maximal bichromatic path* is an (α, β)-maximal bichromatic path that starts at vertex u and ends at vertex v and the first edge on the path is colored α. Note that under a partial acyclic edge coloring ϕ, if there exists an *(α, β, u, v)-maximal bichromatic path*, then no edge colored β is incident with u, namely, the color β is *missing* at u. Meanwhile exactly one of α and β is missing at v. The following fact is obvious by the definition of a partial acyclic edge coloring.

Fact 1. *Under a partial acyclic edge coloring ϕ of a graph G, given a pair of colors α and β, there is at most one (α, β)-maximal bichromatic path containing a particular vertex v.*

An edge colored α is called an *α-edge*. For an edge $uv \in E(G)$, an (α, β, u, v)-maximal bichromatic path not passing through uv (uv may be colored or uncolored) is called an *(α, β, uv)-critical path* if it starts and ends via an α-edge.

For any vertex $u \in V(G)$, we define $\phi(u) = \{\phi(uz) | z \in N_G(u)\}$. For an edge $uv \in E(G)$, we define $S_{uv} = \phi(v) \setminus \{\phi(uv)\}$. Note that S_{uv} need not to be the same as S_{vu}.

A color θ is *valid* for an uncolored edge $e = xy$ under a partial acyclic edge coloring of G if none of the adjacent edges of e is colored θ and assigning the color θ to $e = xy$ results in no bichromatic cycle; *invalid* otherwise.

2. Lemmas

Suppose Theorem 1 is false. Let G be a counterexample to Theorem 1 with the fewest edges. It is obvious that G is connected and has no vertex of degree 1. This section is devoted to investigating some structural properties of G. Since the first two lemmas below are known results, we directly cite them without proof.

Lemma 1 ([12]). *G has no $(3^-, 3^-)$-edge.*

Lemma 2 ([11]). *Let x be a d-vertex and y a 2-vertex adjacent to x, where $d \geq 4$. Then x is adjacent to at most $(d-3)$ 3^--vertices.*

By Lemma 1, every neighbor of a 3-vertex in G is a 4^+-vertex. Call a vertex in G *special* if it is a 3-vertex with at least one 4-neighbor.

Lemma 3. *A special vertex has only one 4-neighbor.*

Proof. Suppose to the contrary that x is a special vertex in G that has at least two 4-neighbors, say w and y. Let z be the neighbor of x other than w and y, also y_1, y_2, y_3 and w_1, w_2, w_3 the three neighbors of y and w other than x, respectively. Let $G' = G - xy$. By the choice of G, G' admits an acyclic edge coloring ϕ using colors from $[\Delta + 2]$. Let $F(xy) = \phi(x) \cup \phi(y)$. Without loss of generality, we may assume that $\phi(yy_i) = i$, $i = 1, 2, 3$. There are three cases under consideration.

1. $|\phi(x) \cap \phi(y)| = 0$.

Since $|\phi(x) \cup \phi(y)| \leq 2 + (\Delta - 1) = \Delta + 1 < \Delta + 2$, there exists a color in $[\Delta + 2] \setminus (\phi(x) \cup \phi(y))$ that is valid for xy under ϕ.

2. $|\phi(x) \cap \phi(y)| = 2$.

Clearly $|F(xy)| = |\phi(y)| = |\{1, 2, 3\}| = 3$. For convenience, we rename the neighbors of x other than y by x_i, $i = 1, 2$. Without loss of generality, we may assume that $\phi(xx_i) = i$, $i = 1, 2$. If there is a color in $[\Delta + 2] \setminus F(xy)$ that is valid for xy, then we are done. Otherwise, for every $\theta \in [\Delta + 2] \setminus F(xy)$, there is a $(1, \theta, xy)$- or a $(2, \theta, xy)$-critical path under ϕ. There are two subcases under consideration.

2.1. $(S_{xx_1} \cup S_{xx_2}) \cap F(xy) = \emptyset$.

Clearly, $S_{xx_1}, S_{xx_2} \subseteq [\Delta + 2] \setminus F(xy) = \{4, 5, \ldots, \Delta + 2\}$.

We are going to prove that $S_{yy_i} = \{4, 5, \ldots, \Delta + 2\}$, $i = 1, 2, 3$. If there is not only a $(1, 4, xy)$-critical path but also a $(2, 4, xy)$-critical path under ϕ, then we exchange the colors on xx_1 and xx_2, obtaining a new acyclic edge coloring of G',

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under which 4 is valid for xy. So there is either a $(1, 4, xy)$-critical path or a $(2, 4, xy)$-critical path under ϕ. Similarly, for every $\theta \in \{5, 6, \dots, \Delta + 2\}$, there is either a $(1, \theta, xy)$-critical path or a $(2, \theta, xy)$-critical path under ϕ. Without loss of generality, we may assume that for every $4 \le i \le k$, there is a $(1, i, xy)$-critical path, and for every $k + 1 \le j \le \Delta + 2$, there is a $(2, j, xy)$-critical path under ϕ. Now we exchange the colors on xx_1 and xx_2, obtaining a new acyclic edge coloring ϕ' of G' that differs from ϕ only on xx_1 and xx_2. If there is a color in $\{4, 5, \dots, \Delta + 2\}$ that is valid for xy under ϕ', then we are done. Otherwise for every $4 \le i \le k$, there is a $(2, i, xy)$-critical path, and for every $k + 1 \le j \le \Delta + 2$, there is a $(1, j, xy)$-critical path under ϕ'. It follows that $S_{yy_1} = \{4, 5, \dots, \Delta + 2\}$ and $S_{yy_2} = \{4, 5, \dots, \Delta + 2\}$. By recoloring xx_1 with 3, as above, we know that $\{4, 5, \dots, k\} \subset S_{yy_3}$. Similarly, by recoloring xx_2 with 3, we know that $\{k + 1, k + 2, \dots, \Delta + 2\} \subset S_{yy_3}$. To conclude, $S_{yy_3} = \{4, 5, \dots, \Delta + 2\}$.

Now by exchanging the colors on yy_1 and yy_3 from ϕ, we obtain a new acyclic edge coloring of G', under which 4 to k is valid for xy.

2.2. $(S_{xx_1} \cup S_{xx_2}) \cap F(xy) \neq \emptyset$.

Without loss of generality, let $S_{xx_1} \cap F(xy) \neq \emptyset$. It follows that at least one color in $\{4, 5, \dots, \Delta + 2\}$ is missing in S_{xx_1}. Without loss of generality, let $4 \notin \phi(x_1)$. We may assume that there is a $(2, 4, xy)$-critical path under ϕ since otherwise 4 is valid for xy. Hence $4 \in \phi(x_2)$ as well as $\phi(y_2)$. By Fact 1, there is no $(2, 4, xx_1)$-critical path under ϕ. By recoloring xx_1 with 4, we obtain a new acyclic edge coloring ϕ' of G' that differs ϕ only on xx_1. If there is a color $\theta \in \{5, 6, \dots, \Delta + 2\}$ that is valid for xy under ϕ', then we are done. Otherwise for every color $\theta \in \{5, 6, \dots, \Delta + 2\}$, there is a $(2, \theta, xy)$-critical path under ϕ as well as under ϕ'. Hence $S_{xx_2} = \{4, 5, \dots, \Delta + 2\} = S_{yy_2}$. If there is also a $(1, 5, xy)$-critical path under ϕ, then we recolor xx_1, xx_2 with 4, 1, respectively, obtaining a new acyclic edge coloring of G', under which 5 is valid for xy. Hence there is no $(1, 5, xy)$-critical path under ϕ. Similarly, there is no $(1, \theta, xy)$-critical path for $\theta \in \{6, 7, \dots, \Delta + 2\}$ under ϕ. To conclude, for every $\theta \in \{4, 5, \dots, \Delta + 2\}$, there is no $(1, \theta, xy)$-critical path under ϕ. By recoloring xx_2 with 3, we obtain a new acyclic edge coloring ϕ'' of G' that differs from ϕ only on xx_2. Argument as above yields that, for every $\theta \in \{4, 5, \dots, \Delta + 2\}$, there is a $(3, \theta, xy)$-critical path under ϕ''. This implies that $S_{yy_3} = \{4, 5, \dots, \Delta + 2\}$. Now by exchanging the colors on yy_2 and yy_3 from ϕ, we obtain a new acyclic edge coloring of G', under which every color in $\{4, 5, \dots, \Delta + 2\}$ is valid for xy.

3. $|\phi(x) \cap \phi(y)| = 1$.

Without loss of generality, we may assume that $\phi(x) \cap \phi(y) = \{1\}$, and 4 the remaining color in $\phi(x)$. Clearly $F(xy) = \phi(x) \cup \phi(y) = \{1, 2, 3, 4\}$. If there is a color in $[\Delta + 2] \setminus F(xy) = \{5, 6, \dots, \Delta + 2\}$ that is valid for xy under ϕ, then we are done. Otherwise,

$$\text{for every } \theta \in \{5, 6, \dots, \Delta + 2\}, \text{ there is a } (1, \theta, xy)\text{-critical path under } \phi. \qquad (*)$$

3.1. $\phi(xz) = 1$ and $\phi(xw) = 4$.

By $(*)$, $\{5, 6, \dots, \Delta + 2\} \subseteq S_{xz}$. We first claim that $\{1, 2, 3\} \subseteq \phi(w)$. If $1 \notin \phi(w)$, then by recoloring xw with 1, xz with $\alpha \in \{2, 3\} \setminus \phi(z)$ from ϕ, we obtain a new acyclic edge coloring ϕ' of G', and hence return to case 2. So $1 \in \phi(w)$. If $2 \notin \phi(w)$, then we recolor xw with 2 from ϕ, obtaining a new proper edge coloring ϕ_1 of G'. If ϕ_1 is acyclic, then we return to case 2. Otherwise, there is a $(1, 2)$-bichromatic cycle under ϕ_1. Thus $2 \in \phi(z)$. It follows that $\phi(z) = \{1, 2, 5, 6, \dots, \Delta + 2\}$. Clearly $1, 4 \in \phi(w)$. Now if $3 \notin \phi(w)$, then by recoloring xw with 3 from ϕ, we obtain a new acyclic edge coloring of G', and return to case 2. To conclude, if $2 \notin \phi(w)$, then $3 \in \phi(w)$. Now, at least one of 5 and 6 is not in $\phi(w)$, say $5 \notin \phi(w)$. By $(*)$ and Fact 1, there exists no $(1, 5, xw)$-critical path under ϕ. Now by recoloring xw with 5, we obtain a new acyclic edge coloring of G' from ϕ, under which 4 is valid for xy. This proves $2 \in \phi(w)$. Similarly $3 \in \phi(w)$. Our claim is proved.

Now $\phi(w) = \{1, 2, 3, 4\}$. Note that there is no $(1, 5, xw)$-critical path. By recoloring xw with 5, we obtain a new acyclic edge coloring ϕ'' of G' that differs from ϕ only on xw. If 4 is valid for xy under ϕ'', then we are done. Otherwise there is a $(1, 4, xy)$-critical path under ϕ as well as under ϕ''. So $S_{xz} = S_{yy_1} = \{4, 5, \dots, \Delta + 2\}$. Combining with $(*)$, we can conclude that

$$\text{for every } \theta \in \{4, 5, \dots, \Delta + 2\}, \text{ there is a } (1, \theta, xy)\text{-critical path under } \phi. \qquad (**)$$

Let us first show that there exist $(4, 2, xz)$- and $(4, 3, xz)$-critical paths under ϕ. Suppose not, say no $(4, 2, xz)$-critical path under ϕ. By recoloring xz with 2, we obtain a new acyclic edge coloring ϕ' of G' that differs from ϕ only on xz, as above, for every $\theta \in \{4, 5, \dots, \Delta + 2\}$, there is a $(2, \theta, xy)$-critical path under ϕ'. It follows that for every $\theta \in \{4, 5, \dots, \Delta + 2\}$, there is a $(2, \theta, y, z)$-maximal bichromatic path under ϕ; hence $S_{yy_2} = S_{xz} = \{4, 5, \dots, \Delta + 2\}$. Now by exchanging the colors on yy_1 and yy_2 from ϕ, we obtain a new acyclic edge coloring of G', under which every color in $\{5, 6, \dots, \Delta + 2\}$ is valid for xy.

Let us next show that for every color $\eta \in \{2, 3\}$ and every color $\xi \in \{5, 6, \dots, \Delta+2\}$, there exists an (η, ξ, w, z)-maximal bichromatic path under ϕ. Suppose not, say no $(2, 5, w, z)$-maximal bichromatic path under ϕ. By recoloring xz, xw with 2, 5 respectively, we obtain a new acyclic edge coloring ϕ'' of G' that differs from ϕ only on xz and xw. By Fact 1, there is no $(2, 4, y, z)$-maximal bichromatic path under ϕ since there is already $(4, 2, xz)$-critical paths under ϕ. It follows that 4 is valid for xy under ϕ''.

Now, without loss of generality, we may assume that $\phi(ww_i) = i$, $i = 1, 2, 3$. So $\{4, 5, \dots, \Delta + 2\} \subseteq S_{ww_i}$, $i = 2, 3$. Since $d(w_i) \le \Delta$, $|S_{ww_i}| \le \Delta - 1$, $i = 2, 3$. It follows that $S_{ww_i} = \{4, 5, \dots, \Delta + 2\}$, $i = 2, 3$. Now, we first exchange the colors on ww_2 and ww_3, then recolor xz, xw with 2, 5, respectively, obtaining a new acyclic edge coloring φ of G' from ϕ. By Fact 1, there is no $(2, 4, y, z)$-maximal bichromatic path under φ as well as under ϕ since there is already $(4, 2, xz)$-critical paths under ϕ. It follows that 4 is valid for xy under φ.

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3.2. $\phi(xw) = 1$ and $\phi(xz) = 4$.

By $(*)$, $\{5, 6, \ldots, \Delta + 2\} \subseteq S_{xw}$. This implies that $(\Delta + 2) - 5 + 1 \le |S_{xw}| = 3$, i.e., $\Delta \le 5$. Clearly $\Delta \ge 4$.

If $\Delta = 4$, then $d(z) \le 4$. Without loss of generality, we may assume that $d(z) = 4$. This returns to case 3.1.

Suppose $\Delta = 5$. Observe that $S_{xw} = \{5, 6, 7\}$. If $5 \notin S_{xz}$, then by $(*)$ and Fact 1, there is no $(1, 5, xz)$-critical path under ϕ since $z \ne y$. Now by recoloring xz with 5 from ϕ, we obtain a new acyclic edge coloring of G', under which 4 is valid for xy. So $5 \in S_{xz}$. Similarly, $6, 7 \in S_{xz}$. Since $\Delta = 5$, at least one of 2 and 3, say 2, is not in S_{xz}. Now, by recoloring xz with 2 from ϕ, we obtain a new acyclic edge coloring of G', under which 4 is valid for xy. $\square$

Lemma 4. *A 5-vertex x in G has at most three 3-neighbors.*

Proof. Suppose to the contrary that x is a 5-vertex in G that has at least four 3-neighbors, say y, x_1, x_2, x_3. Let z be the remaining neighbor of x, and y_1, y_2 the two neighbors of y other than x. Let $G' = G - xy$. By the choice of G, G' has an acyclic edge coloring ϕ using colors from $[\Delta+2]$. We may assume that $\phi(xx_i) = i$, $i = 1, 2, 3$ and $\phi(xz) = 4$. Let $F(xy) = \phi(x) \cup \phi(y)$.

If $\phi(x) \cap \phi(y) = \emptyset$, then we can find a color in $[\Delta + 2] \setminus (\phi(x) \cup \phi(y))$ that is valid for xy under ϕ, since $|\phi(x) \cup \phi(y)| \le \Delta - 1 + 2 = \Delta + 1 < \Delta + 2$. Suppose $|\phi(x) \cap \phi(y)| \ge 1$. There are two cases under consideration.

1. $|\phi(x) \cap \phi(y)| = 1$.

Let α be the unique color in $\phi(x) \cap \phi(y)$. If we can find a color in $[\Delta + 2] \setminus F(xy)$ that is valid for xy under ϕ, then we get an acyclic edge coloring of G using $\Delta + 2$ colors, a contradiction. Hence for each color $\theta \in [\Delta + 2] \setminus F(xy)$, there exists an (α, θ, xy)-critical path under ϕ. Clearly, $\alpha \in \{1, 2, 3, 4\}$. We discuss by distinguishing two subcases as follows.

(a) $\alpha \in \{1, 2, 3\}$.

Without loss of generality, we may assume that $\phi(yy_1) = \alpha = 1$ and $\phi(yy_2) = 5$. Now,

• *for each color $\theta \in [\Delta + 2] \setminus F(xy) = \{6, 7, \ldots, \Delta + 2\}$, there exists a $(1, \theta, xy)$-critical path under ϕ.*

By •, $S_{yy_1}, S_{xx_1} \supseteq \{6, 7, \ldots, \Delta + 2\}$. Hence $2 = |S_{xx_1}| \ge (\Delta + 2) - 6 + 1 = \Delta - 3$, i.e., $\Delta \le 5$. Since $d(x) = 5$, $\Delta = 5$. Thus $S_{xx_1} = \{6, 7\}$ and $\{6, 7\} \subseteq S_{yy_1}$.

Now by recoloring xx_1 with 5 from ϕ, we obtain a new acyclic edge coloring ϕ' of G', under which neither 6 nor 7 is valid for xy, since otherwise G would be acyclically edge $(\Delta + 2)$-colorable. This implies that there exists a $(5, 6, y, x_1)$- as well as a $(5, 7, y, x_1)$-maximal bichromatic path under ϕ. Hence $6, 7 \in S_{yy_2}$.

If there is a color $i \in \{2, 3, 4\} \setminus (S_{yy_1} \cup S_{yy_2})$, then by recoloring yy_2 with color i from ϕ, we obtain a new acyclic edge coloring of G', under which 5 would be valid for xy. So $\{2, 3, 4\} \subseteq S_{yy_1} \cup S_{yy_2}$.

If $1 \notin S_{yy_2}$, then by recoloring yy_2 with a color in $\{2, 3, 4\} \setminus S_{yy_2}$, we would obtain a new acyclic edge coloring of G', under which 5 is valid for xy. So $1 \in S_{yy_2}$.

In summary, $|S_{yy_1}| + |S_{yy_2}| \ge 2 + 2 + 4 = 8$. On the other hand, $\Delta = 5$ implies $|S_{yy_1}| + |S_{yy_2}| \le 4 + 4 = 8$. So $|S_{yy_1}| + |S_{yy_2}| = 8$. So $5 \notin \phi(y_1)$.

Now by recoloring xx_1 with 5, and yy_1 with a color in $\{2, 3, 4\} \setminus \phi(y_1)$, we would obtain a new acyclic edge coloring of G' from ϕ, under which 1 is valid for xy.

(b) $\alpha = 4$.

We may assume that $\phi(yy_1) = 4$ and $\phi(yy_2) = 5$. Clearly, for each color $\theta \in [\Delta + 2] \setminus F(xy) = \{6, 7, \ldots, \Delta + 2\}$, there exists a $(4, \theta, xy)$-critical path under ϕ. Hence $\{6, 7, \ldots, \Delta + 2\} \subseteq S_{xz} \cap S_{yy_1}$.

If $5 \notin S_{yy_1}$, then by recoloring yy_1 with a color in $\{1, 2, 3\} \setminus S_{yy_1}$, we obtain a new acyclic edge coloring from ϕ, and return to (a). So $5 \in S_{yy_1}$.

If $4 \notin S_{yy_2}$, then by recoloring yy_1, yy_2 with a color $i \in \{1, 2, 3\} \setminus S_{yy_1}$, 4, respectively, we obtain a new acyclic edge coloring ϕ' of G'. For each color $\theta \in \{6, 7, \ldots, \Delta + 2\}$, since there is already a $(4, \theta, xy)$-critical path under ϕ, there is a $(4, \theta, x, y_1)$-maximal bichromatic path under ϕ'. It follows that, for each color $\theta \in \{6, 7, \ldots, \Delta + 2\}$, there is no $(4, \theta, xy)$-critical path passing through y_2 under ϕ' by Fact 1. Note that $5 \notin S_{yy_2}$ and $|S_{xx_i}| = 2$. Now a color in $\{5, 6, 7, \ldots, \Delta + 2\} \setminus S_{xx_i}$ would be valid for xy under ϕ'. This proves $4 \in S_{yy_2}$.

If there is a color $\beta \in \{1, 2, 3\} \setminus (S_{yy_1} \cup S_{yy_2})$, then by recoloring yy_1 with β, we obtain a new acyclic edge coloring of G', and we return to (a). So $\{1, 2, 3\} \subset (S_{yy_1} \cup S_{yy_2})$.

If there is a color γ in $\{6, 7, \ldots, \Delta + 2\}$ that is not in S_{yy_2}, then by recoloring yy_2 with γ, we would obtain a new acyclic edge coloring of G' (since there is already a $(4, \gamma, xy)$-critical path under ϕ, by Fact 1, there is no $(4, \gamma, x, y_2)$-maximal bichromatic path under ϕ), under which 5 is valid for xy. Hence $\{6, 7, \ldots, \Delta + 2\} \subseteq S_{yy_2}$.

To conclude, we have shown that $\{4, 5, 6, \ldots, \Delta + 2\} \subseteq \phi(y_2)$, $\{4, 6, \ldots, \Delta + 2\} \subseteq \phi(y_1)$, and $\{1, 2, 3\} \subset (S_{yy_1} \cup S_{yy_2})$. These imply $5 \notin S_{yy_1}$.

Now, by recoloring yy_1 with 5, yy_2 with a color in $\{1, 2, 3\} \setminus S_{yy_2}$, we obtain a new acyclic edge coloring of G' from ϕ, and return to (a).

2. $|\phi(x) \cap \phi(y)| = 2$.

In this case, $\phi(x) \cup \phi(y) = \{1, 2, 3, 4\}$; hence $[\Delta + 2] \setminus F(xy) = \{5, 6, \ldots, \Delta + 2\}$. Since $\Delta \ge 5$, $\{5, 6, 7\} \subseteq \{5, 6, \ldots, \Delta + 2\}$. There are two subcases under consideration.

(i) $4 \notin \phi(y)$.

By the symmetry of 1, 2 and 3, we may assume that $\phi(yy_1) = 1$ and $\phi(yy_2) = 2$. If we can find a color in $[\Delta+2] \setminus F(xy) = \{5, 6, \ldots, \Delta+2\}$ that is valid for xy under ϕ, then we are done. Otherwise, for every $\theta \in \{5, 6, \ldots, \Delta+2\}$, there is a $(1, \theta, xy)$- or a $(2, \theta, xy)$-critical path under ϕ. It follows that $\{5, 6, \ldots, \Delta+2\} \subseteq S_{xx_1} \cup S_{xx_2}$. Since $d(x_1) = d(x_2) = 3$, $|S_{xx_1}| = |S_{xx_2}| = 2$.

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We may assume that $S_{xx_1} = \{5, 6\}$ and $7 \in S_{xx_2}$. Now, by recoloring xx_1 with 7 from ϕ, we obtain a new acyclic edge coloring of G', under which a color in $\{5, 6\} \setminus S_{xx_2}$ would be valid for xy.

(ii) $4 \in \phi(y)$.

By the symmetry of 1, 2 and 3, we may assume that $\phi(yy_1) = 1$ and $\phi(yy_2) = 4$. If we can find a color in $[\Delta+2] \setminus F(xy) = \{5, 6, \ldots, \Delta+2\}$ that is valid for xy under ϕ, then we are done. Otherwise, for every $\theta \in \{5, 6, \ldots, \Delta+2\}$, there is a $(1, \theta, xy)$-or a $(4, \theta, xy)$-critical path under ϕ.

If there is a color $\beta \in \{5, 6, \ldots, \Delta+2\} \setminus S_{yy_1}$, then there is a $(4, \beta, xy)$-critical path under ϕ. It follows that there is no $(4, \beta, yy_1)$-critical path under ϕ by Fact 1. Now, by recoloring yy_1 with β, we obtain a new acyclic edge coloring of G', and return to case 1. So $\{5, 6, \ldots, \Delta+2\} \subseteq S_{yy_1}$. Similarly, $\{5, 6, \ldots, \Delta+2\} \subseteq S_{yy_2}$.

If $|\{2, 3\} \setminus (S_{yy_1} \cup S_{yy_2})| \geq 1$, then by recoloring yy_2 with a color in $\{2, 3\} \setminus (S_{yy_1} \cup S_{yy_2})$, we obtain a new acyclic edge coloring of G', and return to (i). So $\{2, 3\} \subset (S_{yy_1} \cup S_{yy_2})$.

Combining conclusions above with $|\{5, 6, \ldots, \Delta+2\}| = \Delta - 2$ and $|S_{yy_1}|, |S_{yy_2}| \leq \Delta - 1$, we get $1 \notin S_{yy_2}$. Now by recoloring yy_2 with a color in $\{2, 3\} \setminus S_{yy_2}$, we obtain a new acyclic edge coloring of G', and return to (i). Lemma 4 is proved. □

Lemma 5. *If a 5-vertex x in G has a special neighbor y, then it has at most two 3-neighbors.*

Proof. Suppose to the contrary that a 5-vertex x in G has at least three 3-neighbors, say y, x_1 and x_2 with y special. Let z_1, z_2 be the remaining two neighbors of x, and y_1, y_2 the two neighbors of y other than x. Let $G' = G - xy$. By the choice of G, G' has an acyclic edge coloring ϕ using colors from $[\Delta+2]$. We may assume that $\phi(xx_i) = i$, $i = 1, 2$, $\phi(xz_1) = 3$ and $\phi(xz_2) = 4$. Let $F(xy) = \phi(x) \cup \phi(y)$.

If $\phi(x) \cap \phi(y) = \emptyset$, then we can find a color in $[\Delta+2] \setminus (\phi(x) \cup \phi(y))$ that is valid for xy under ϕ since $|\phi(x) \cup \phi(y)| \leq \Delta - 1 + 2 = \Delta + 1 < \Delta + 2$. Suppose $|\phi(x) \cap \phi(y)| \geq 1$. There are two cases under consideration.

1. $|\phi(x) \cap \phi(y)| = 1$.

Let α be the unique color in $\phi(x) \cap \phi(y)$. If we can find a color in $[\Delta+2] \setminus F(xy)$ that is valid for xy under ϕ, then we get an acyclic edge coloring of G using $\Delta+2$ colors, a contradiction. Hence for each color $\theta \in [\Delta+2] \setminus F(xy)$, there exists an (α, θ, xy)-critical path under ϕ. Note that $\alpha \in \{1, 2, 3, 4\}$. We discuss by distinguishing two subcases as follows.

(a) $\alpha \in \{1, 2\}$.

We may assume that $\phi(yy_1) = \alpha = 1$ and $\phi(yy_2) = 5$. Thus, for each color $\theta \in [\Delta+2] \setminus F(xy) = \{6, 7, \ldots, \Delta+2\}$, there exists a $(1, \theta, xy)$-critical path under ϕ. It follows that $S_{yy_1}, S_{xx_1} \supseteq \{6, 7, \ldots, \Delta+2\}$. Since $d(x_1) = 3$, $\Delta = 5$. Hence $S_{xx_1} = \{6, 7\}$ and $\{6, 7\} \subseteq S_{yy_1}$. Now by recoloring xx_1 with 5 from ϕ, we obtain a new acyclic edge coloring ϕ' of G', under which neither 6 nor 7 is valid for xy, since otherwise G would be acyclically edge $(\Delta+2)$-colorable. This implies that there exists a $(5, 6, y, x_1)$- as well as a $(5, 7, y, x_1)$-maximal bichromatic path under ϕ. Hence $6, 7 \in S_{yy_2}$. If there is a color $i \in \{2, 3, 4\} \setminus (S_{yy_1} \cup S_{yy_2})$, then by recoloring yy_2 with color i from ϕ, we obtain a new acyclic edge coloring of G', under which 5 would be valid for xy as $5 \notin S_{yy_2}$. So $\{2, 3, 4\} \subseteq (S_{yy_1} \cup S_{yy_2})$. It follows that $|S_{yy_1}| + |S_{yy_2}| \geq 2 + 2 + 3 = 7$. Notice that $\Delta = 5$ and y being special implies $|S_{yy_1}| + |S_{yy_2}| \leq 4 + 3 = 7$. So $|S_{yy_1}| + |S_{yy_2}| = 7$. Therefore, $1 \notin S_{yy_2}$. Now by recoloring yy_2 with a color in $\{2, 3, 4\} \setminus S_{yy_2}$, we would obtain a new acyclic edge coloring of G', under which 5 is valid for xy.

(b) $\alpha \in \{3, 4\}$.

We may assume that $\phi(yy_1) = 4$ and $\phi(yy_2) = 5$. Note that for each color $\theta \in [\Delta+2] \setminus F(xy) = \{6, 7, \ldots, \Delta+2\}$, there exists a $(4, \theta, xy)$-critical path under ϕ. Hence $S_{xz_2}, S_{yy_1} \supseteq \{6, 7, \ldots, \Delta+2\}$. We are going to show that $[\Delta+2] \subseteq (S_{yy_1} \cup S_{yy_2})$. This will be done by the following four claims.

- $5 \in S_{yy_1}$.
  Suppose $5 \notin S_{yy_1}$. Then we first have $1, 2 \in S_{yy_1}$, since otherwise, $\{1, 2\} \setminus S_{yy_1} \neq \emptyset$, by recoloring yy_1 with a color in $\{1, 2\} \setminus S_{yy_1}$ from ϕ, we obtain a new acyclic edge coloring of G', and return to (a). Now $|S_{yy_1}| \geq |\{1, 2, 6, 7, \ldots, \Delta+2\}| = \Delta+2-5+2 = \Delta-1$ implies $d(y_1) = \Delta \geq 5$. Since y is special, $d(y_2) = 4$. If $\{1, 2\} \setminus S_{yy_2} \neq \emptyset$, then by recoloring yy_1 with 5, yy_2 with a color in $\{1, 2\} \setminus S_{yy_2}$ from ϕ, we obtain a new acyclic edge coloring of G', and return to (a). Hence $\{1, 2\} \subset S_{yy_2}$. If $4 \notin S_{yy_2}$, then by exchanging the colors on yy_1 and yy_2 from ϕ, we obtain a new acyclic edge coloring of G', under which 6 is valid for xy (by Fact 1). Hence $4 \in S_{yy_2}$. Since y_2 is a 4-neighbor of y, $S_{yy_2} = \{1, 2, 4\}$. Thus $3 \notin (S_{yy_1} \cup S_{yy_2})$. Now by recoloring yy_2 with 3 from ϕ, we obtain a new acyclic edge coloring of G', under which 5 is valid for xy. This proves $5 \in S_{yy_1}$.
- $4 \in S_{yy_2}$.
  Suppose $4 \notin S_{yy_2}$. Since now $\{5, 6, \ldots, \Delta+2\} \subseteq S_{yy_1}$, at least one of 1 and 2, say 1, is not in S_{yy_1}. By recoloring yy_1 with 1, yy_2 with 4 from ϕ, we obtain a new acyclic edge coloring ϕ' of G'. For each color $\theta \in \{6, 7, \ldots, \Delta+2\}$, since there is already a $(4, \theta, xy)$-critical path under ϕ, there is a $(4, \theta, x, y_1)$-maximal bichromatic path under ϕ'. By Fact 1, there is no $(4, \theta, xy)$-critical path passing through y_2 under ϕ' for each color $\theta \in \{6, 7, \ldots, \Delta+2\}$. By $5 \notin S_{yy_2}$ and $d(x_1) = 3$, There exists a color in $\{5, 6, 7, \ldots, \Delta+2\} \setminus S_{xx_1}$ that is valid for xy under ϕ'. This proves $4 \in S_{yy_2}$.
- $1, 2 \in (S_{yy_1} \cup S_{yy_2})$.
  If there is a color $\beta \in \{1, 2\} \setminus S_{yy_1} \cup S_{yy_2}$, then by recoloring yy_1 with β, we obtain a new acyclic edge coloring of G', and return to (a).
- $3 \in (S_{yy_1} \cup S_{yy_2})$.

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If $3 \notin (S_{yy_1} \cup S_{yy_2})$, then by recoloring yy_1 with 3, we obtain a new acyclic edge coloring ϕ' of G' that differs from ϕ only on yy_1, under which there would be a $(3, \theta, xy)$-critical path for each $\theta \in \{6, 7, \ldots, \Delta + 2\}$. It follows that there is no $(3, \theta, x, y_2)$-maximal bichromatic path under ϕ for each $\theta \in \{6, 7, \ldots, \Delta + 2\}$ by Fact 1. Now by recoloring yy_2 with 3, yy_1 with a color $i \in \{1, 2\} \setminus S_{yy_1}$ from ϕ, we obtain a new acyclic edge coloring of G', under which a color in $\{5, 6, \ldots, \Delta + 2\} \setminus S_{xx_i}$ would be valid for xy.

Recall that y has a 4-neighbor. $|S_{yy_1}| + |S_{yy_2}| \le (\Delta - 1) + 3 = \Delta + 2$. According to $[\Delta + 2] \subseteq (S_{yy_1} \cup S_{yy_2})$, $[\Delta + 2] = S_{yy_1} \cup S_{yy_2}$. It follows that $|S_{yy_1} \cap S_{yy_2}| = |S_{yy_1} \cup S_{yy_2}| - (|S_{yy_1}| + |S_{yy_2}|) = 0$, i.e., $S_{yy_1} \cap S_{yy_2} = \emptyset$. So $\{5, 6, \ldots, \Delta + 2\} \subseteq S_{yy_1}$ and $S_{yy_2} \subseteq \{1, 2, 3, 4\}$. According to which of y_1 and y_2 is the 4-neighbor of y, there are two subcases under consideration.

(b1) $d(y_1) = 4$, and $d(y_2) \ge 5$.

Since $\{5, 6, \ldots, \Delta + 2\} \subseteq S_{yy_1}$ and $|S_{yy_1}| = 3$, $|\{5, 6, \ldots, \Delta + 2\}| = \Delta + 2 - 4 = \Delta - 2 \le 3$, i.e., $\Delta \le 5$. Since $d(x) = 5$, $\Delta \ge 5$. So $\Delta = 5$. Hence $S_{yy_1} = \{5, 6, 7\}$ and $S_{yy_2} = \{1, 2, 3, 4\}$. Now by recoloring yy_2 with 6 from ϕ, we obtain a new acyclic edge coloring ϕ' of G'. (Since there is already a $(4, 6, xy)$-critical path under ϕ, by Fact 1, there is no $(4, 6, yy_2)$-critical path under ϕ.) Since G is not acyclically edge $(\Delta + 2)$-colorable, there exists a $(4, 5, xy)$-critical path under ϕ as well as under ϕ'.

We are going to prove $3 \in (S_{xx_1} \cap S_{xx_2})$. Suppose $3 \notin S_{xx_1}$. We first claim $2 \in S_{xx_1}$. If $2 \notin S_{xx_1}$, then by recoloring xx_1 with a color in $\{5, 6, 7\} \setminus S_{xx_1}$, we obtain a new edge coloring ϕ' of G' from ϕ. ϕ' is also acyclic since there is already a $(4, \theta, xy)$-critical path under ϕ' as well as under ϕ, by Fact 1, there is no $(4, \theta, xx_1)$-critical path under ϕ' for each $\theta \in \{5, 6, 7\}$. Now, 1 would be valid for xy under ϕ' since $1 \notin S_{xx_1}$ and $1 \notin S_{yy_1}$. This proves $2 \in S_{xx_1}$. Now, if $\{5, 6, 7\} \setminus (S_{xx_1} \cup S_{xx_2}) \ne \emptyset$, then we could recolor xx_1 with a color in $\{5, 6, 7\} \setminus (S_{xx_1} \cup S_{xx_2})$, and then 1 is valid for xy; otherwise, $S_{xx_1} \cup S_{xx_2} = \{2, 5, 6, 7\}$, we could recolor xx_1 with a color in S_{xx_2}, and then 1 is valid for xy. This proves $3 \in S_{xx_1}$. Similarly, $3 \in S_{xx_2}$. So $3 \in (S_{xx_1} \cap S_{xx_2})$.

Now recolor xx_1 with a color $\vartheta \in \{5, 6, 7\} \setminus (S_{xx_1} \cup S_{xx_2})$ from ϕ. If we obtain a new acyclic edge coloring φ of G', then 1 is valid for xy under φ; hence we are done. Otherwise there is a $(3, \vartheta)$-bichromatic cycle under φ, hence there is a $(3, \vartheta, xx_1)$-critical path under ϕ as well as under φ. By Fact 1, there is no $(3, \vartheta, xx_2)$-critical path under ϕ (since $x_1 \ne x_2$). Hence by recoloring xx_2 with ϑ from ϕ, we obtain a new acyclic edge coloring of G', under which 2 would be valid for xy.

(b2) $d(y_2) = 4$ and $5 \le d(y_1) \le \Delta$.

By recoloring yy_2 with 6 from ϕ, we obtain a new acyclic edge coloring ϕ' of G', under which there is a $(4, 5, xy)$-critical path, since otherwise, by coloring xy with 5, we would obtain an acyclic edge $(\Delta + 2)$-coloring of G from ϕ'. Thus there is a $(4, 5, xy)$-critical path under ϕ.

If neither 1 nor 2 is in S_{yy_1}, then arguing as in (b1) could yield an acyclic edge $(\Delta + 2)$-coloring of G. So $|\{1, 2\} \cap S_{yy_1}| \ge 1$. Since $\{5, 6, \ldots, \Delta + 2\} \subseteq S_{yy_1}$, $|\{1, 2\} \cap S_{yy_1}| = 1$. We may assume that $1 \in S_{yy_1}$ and $S_{yy_2} = \{2, 3, 4\}$ (since $d(y_2) = 4$ and $S_{yy_1} \cap S_{yy_2} = \emptyset$).

In what follows we shall show that we can finally recolor yy_1 with 2 from some acyclic edge $(\Delta + 2)$-coloring of G', obtaining a new acyclic edge $(\Delta + 2)$-coloring of G', and return to subcase (a), hence completing the proof of case 1. This will be completed based on the following four claims.

• *For each $\eta \in \{5, 6, \ldots, \Delta + 2\}$, there is an $(\eta, 2, y_1, y_2)$-maximal bichromatic path under ϕ.*

Suppose, for some $\eta_0 \in \{5, 6, \ldots, \Delta + 2\}$, there is no $(\eta_0, 2, y_1, y_2)$-critical path under ϕ. By recoloring yy_1 with 2, yy_2 with η_0 (if $\eta_0 \ne 5$) from ϕ, we obtain a new acyclic edge coloring of G', and return to (a).

• *There is a $(1, 2, y_1, y_2)$-maximal bichromatic path under ϕ.*

If not, then by recoloring yy_1, yy_2 with 2, 1, respectively, we obtain a new acyclic edge coloring of G', under which there would be a color in $\{5, 6, \ldots, \Delta + 2\} \setminus S_{xx_2}$ that is valid for xy (since $S_{yy_2} = \{2, 3, 4\}$ and $d(x_2) = 3$).

• *There is a $(5, 3, yy_1)$-critical path under ϕ.*

If not, by recoloring yy_1 with 3 from ϕ, we obtain a new acyclic edge coloring ϕ' of G', under which there would be a $(3, \theta, xy)$-critical path for each $\theta \in \{6, 7, \ldots, \Delta + 2\}$. By Fact 1, for each $\theta \in \{6, 7, \ldots, \Delta + 2\}$, there is no $(3, \theta, yy_2)$-critical path under ϕ'. Now by recoloring yy_2 with 6 from ϕ', we obtain a new acyclic edge coloring φ of G', under which, there would be a $(3, 5, xy)$-critical path. By Fact 1, for each $\theta \in \{5, 6, \ldots, \Delta + 2\}$, there is no $(3, \theta, xx_2)$-critical path under ϕ as well as under φ. If $1 \notin S_{xx_2}$, by recoloring yy_1 with 3, xx_2 with a color $\gamma \in \{5, 6, \ldots, \Delta + 2\} \setminus S_{xx_2}$ from ϕ, we obtain a new acyclic edge coloring of G' (since there is already a $(4, \gamma, xy)$-critical path, by Fact 1, there is no $(4, \gamma, xx_2)$-critical path under ϕ), under which 2 is valid for xy ($2 \notin S_{yy_1}$ and $2 \notin S_{xx_2}$). Hence $1 \in S_{xx_2}$. If $\{5, 6, \ldots, \Delta + 2\} \setminus (S_{xx_1} \cup S_{xx_2}) \ne \emptyset$, then we recolor xx_2 with a color in $\{5, 6, \ldots, \Delta + 2\} \setminus (S_{xx_1} \cup S_{xx_2})$ from ϕ, obtaining a new acyclic edge $(\Delta + 2)$-coloring of G', under which, arguing as just above, 2 is valid for xy. So $\{5, 6, \ldots, \Delta + 2\} \subset S_{xx_1} \cup S_{xx_2}$. Note that $|\{5, 6, \ldots, \Delta + 2\}| \ge 3$. Since $1 \in S_{xx_2}$ and $d(x_1) = d(x_2) = 3$, $S_{xx_1} \subset \{5, 6, \ldots, \Delta + 2\}$. Now by recoloring xx_2 with a color in $S_{xx_1} \setminus S_{xx_2}$, say γ, from ϕ, we obtain a new edge coloring φ of G'. If φ is acyclic, then 2 is valid for xy under φ. Otherwise, there is a cycle alternately colored by γ and 1 under φ. Namely, there is a $(1, \gamma, xx_2)$-critical path under ϕ. In this case, we recolor xx_1 with 2 from φ, obtaining an acyclic edge coloring φ' of G'. If 1 is valid for xy under φ', then we are done. Otherwise, there is a $(1, 4, xy)$-critical path under ϕ as well as φ'. By recoloring yy_1 with 3 from ϕ, we obtain a new acyclic edge coloring ϕ'. A similar argument (or by symmetry of z_1 and z_2) yields that there is a $(1, 3, xy)$-critical path under ϕ'. It follows that $S_{xz_1} = S_{xz_2} = \{1, 5, 6, \ldots, \Delta + 2\}$. Now we first exchange the colors on xz_1 and xz_2 from ϕ, then recolor xx_2 with γ, xx_1 with 2, yielding an acyclic edge coloring of G', under which 1 is valid for xy.

• *For each $\xi \in \{1, 6, 7, \ldots, \Delta + 2\}$ there is a $(\xi, 3, y_1, y_2)$-maximal bichromatic path under ϕ.*

If there is no $(\xi, 3, y_1, y_2)$-maximal bichromatic path under ϕ for some color $\xi \in \{1, 6, 7, \ldots, \Delta + 2\}$, then by recoloring yy_1 with 3, yy_2 with ξ, we obtain a new acyclic edge coloring ϕ' of G' that differs from ϕ only on yy_1 and yy_2. Since there is

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already a $(5, 3, yy_1)$-critical path under ϕ, there is a $(3, 5, yy_2)$-critical path under ϕ'. It follows that there is no $(3, 5, xy)$-critical path under ϕ' by Fact 1. Note that $5 \notin S_{yy_2}$. Therefore, 5 is valid for xy under ϕ'.

Now we can complete the proof of case 1. Recall that $S_{yy_2} = \{2, 3, 4\}$. We may assume that v_1, v_2 and v_3 are the three neighbors of y_2 other than y with $\phi(y_2v_1) = 2$ and $\phi(y_2v_2) = 3$. According to the four claims above, $S_{y_2v_1} = \{1, 5, 6, \ldots, \Delta + 2\} = S_{y_2v_2}$, and for each color $\alpha \in \{1, 5, 6, \ldots, \Delta + 2\}$, $\beta \in \{2, 3\}$, there is an $(\alpha, \beta, y_1, y_2)$-maximal bichromatic path under ϕ. Now by exchanging the colors on y_2v_1 and y_2v_2 from ϕ, we destroy the old $(5, 2, yy_1)$-critical path in ϕ, and by Fact 1, we do not create any new $(5, 2, yy_1)$-critical path in the new acyclic edge coloring ψ of G'. Hence by recoloring yy_1 with 2 from ψ, we obtain a new acyclic edge coloring of G', and return to (a).

2. $|\phi(x) \cap \phi(y)| = 2$.

Let $\phi(x) \cap \phi(y) = \{\alpha, \beta\}$. If we can find a color in $[\Delta + 2] \setminus F(xy)$ that is valid for xy under ϕ, then we get an acyclic edge coloring of G using $\Delta + 2$ colors, a contradiction. Hence for each color $\theta \in [\Delta + 2] \setminus F(xy) = \{5, 6, \ldots, \Delta + 2\}$ there exists an (α, θ, xy)-critical path or a (β, θ, xy)-critical path under ϕ. Note that $\{\alpha, \beta\} \subset F(xy) = \{1, 2, 3, 4\}$. We may assume that $\phi(yy_1) = \alpha$ and $\phi(yy_2) = \beta$.

If ξ is a color in $\{5, 6, \ldots, \Delta + 2\} \setminus S_{yy_1}$, then there is a (β, ξ, xy)-critical path. By Fact 1, there is no (β, ξ, yy_1)-critical path under ϕ. Recoloring yy_1 with ξ from ϕ, we obtain a new acyclic edge coloring of G', and return to case 1. So $\{5, 6, \ldots, \Delta + 2\} \subseteq S_{yy_1}$. Similarly, $\{5, 6, \ldots, \Delta + 2\} \subseteq S_{yy_2}$. By symmetry, we discuss by distinguishing three subcases as follows.

(i) $\{\alpha, \beta\} = \{1, 2\}$.

We may assume that $\phi(yy_i) = i$, $i = 1, 2$. In this case, for each color $\theta \in \{5, 6, \ldots, \Delta+2\}$, there exists a $(1, \theta, xy)$-critical path or a $(2, \theta, xy)$-critical path under ϕ. Hence $\{5, 6, \ldots, \Delta+2\} \subseteq (S_{xx_1} \cup S_{xx_2})$. Since $\Delta \geq 5$, $\{5, 6, 7\} \subseteq \{5, 6, \ldots, \Delta+2\}$. Note that $d(x_1) = d(x_2) = 3$. We may assume that $S_{xx_1} = \{5, 6\}$ and $7 \in S_{xx_2}$. Now by recoloring xx_1 with 7 from ϕ, we obtain a new acyclic edge coloring of G', and return to case 1.

(ii) $\alpha \in \{1, 2\}$ and $\beta \in \{3, 4\}$.

We may assume that $\phi(yy_1) = 1$ and $\phi(yy_2) = 4$. If $2 \in S_{yy_2}$, then $\phi(y_2) = \{5, 6, \ldots, \Delta + 2\} \cup \{2, 4\}$. So $d(y_2) = \Delta$. Hence $d(y_1) = 4$. So $S_{yy_1} = \{5, 6, 7\}$. By recoloring yy_1 with 2, yy_2 with 1 from ϕ, we obtain a new acyclic edge coloring of G', and return to subcase (i). So $2 \notin S_{(yy_2)}$. If there is no $(1, 2, yy_2)$-critical path under ϕ, then by recoloring yy_2 with 2 from ϕ, we obtain a new acyclic edge coloring of G', and return to subcase (i). Otherwise, there is a $(1, 2, yy_2)$-critical path under ϕ. So, $2 \in S_{yy_1}$ and $1 \in S_{yy_2}$. It follows that $d(y_i) \geq 5$, $i = 1, 2$, contradicting that y is special.

(iii) $\{\alpha, \beta\} = \{3, 4\}$.

We may assume that $d(y_1) = 4$, $\phi(yy_1) = 3$ and $\phi(yy_2) = 4$. Recall that $\{5, 6, \ldots, \Delta + 2\} \subseteq S_{yy_1}$. This implies $\Delta = 5$ and $\phi(y_1) = \{3, 5, 6, 7\}$. Now we can recolor yy_1 with 1 from ϕ, obtaining a new acyclic edge coloring of G', and returning to subcase (ii).

The proof of Lemma 5 is completed. $\square$

Lemma 6. *A 7-vertex x with a special neighbor y has at most six 3^--neighbors (including y).*

Proof. Suppose to the contrary that x has seven 3^--neighbors. Without loss of generality, we may assume that all the seven neighbors of x are 3-vertices. Let $x_1, x_2, \ldots, x_6$ be the six neighbors of x other than y, and y_1, y_2 the two neighbors of y other than x. Let $G' = G - xy$. By the choice of G, G' has an acyclic edge coloring ϕ using colors from the color set $[\Delta + 2]$. Without loss of generality, we may assume that $\phi(xx_i) = i$ for $i = 1, 2, \ldots, 6$. Let $F(xy) = \phi(x) \cup \phi(y)$.

If $\phi(x) \cap \phi(y) = \emptyset$, then $|\phi(x) \cup \phi(y)| \leq (\Delta - 1) + 2 = \Delta + 1 < \Delta + 2$; hence there exists at least one color $\alpha \in [\Delta + 2] \setminus F(xy)$ that is valid for xy under ϕ. So we may assume that $\phi(x) \cap \phi(y) \neq \emptyset$. There are two cases under consideration.

(1) $|\phi(x) \cap \phi(y)| = 1$.

Without loss of generality, we may assume that $\phi(yy_1) = 1$ and $\phi(yy_2) = 7$. If we can find a color $\alpha \in [\Delta + 2] \setminus F(xy)$ that is valid for xy under ϕ, then we get an acyclic edge coloring of G using $\Delta + 2$ colors, a contradiction. Hence for each color $\theta \in [\Delta + 2] \setminus F(xy)$, there exists a $(1, \theta, xy)$-critical path under ϕ. It follows that $[\Delta + 2] \setminus F(xy) = \{8, 9, \ldots, \Delta + 2\} \subseteq S_{xx_1}$ as well as S_{yy_1}. Since $d(x_1) = 3$, $\Delta = 7$. It follows that $S_{xx_1} = \{8, 9\}$ and $\{8, 9\} \subseteq S_{yy_1}$. Now by recoloring xx_1 with 7, we obtain a new acyclic edge coloring ϕ' of G', under which neither 8 nor 9 is valid for xy since otherwise G would be acyclically edge $(\Delta + 2)$-colorable. This implies that there are $(7, 8, y, x_1)$- and $(7, 9, y, x_1)$-maximal bichromatic paths under ϕ as well as under ϕ'. Hence $8, 9 \in S_{yy_2}$. If there is a color $i \in \{2, 3, 4, 5, 6\} \setminus (S_{yy_1} \cup S_{yy_2})$, then by recoloring yy_2 with color i, we obtain a new acyclic edge coloring of G' from ϕ, under which, 7 would be valid for xy. So $\{2, 3, 4, 5, 6\} \subseteq S_{yy_1} \cup S_{yy_2}$. It follows that $|S_{yy_1}| + |S_{yy_2}| \geq 2 + 2 + 5 = 9$. Note that y is a special vertex, i.e., y has a 4-neighbor that is y_1 or y_2. Hence $|S_{yy_1}| + |S_{yy_2}| \leq 3 + 6 = 9$ (recall that $\Delta = 7$). It follows that $|S_{yy_1}| + |S_{yy_2}| = 9$. So $1 \notin S_{yy_2}$. Now by recoloring yy_2 with a color in $\{2, 3, 4, 5, 6\} \setminus \phi(y_2)$, we obtain a new acyclic edge coloring of G', under which, 7 is valid for xy, a contradiction.

(2) $|\phi(x) \cap \phi(y)| = 2$.

Without loss of generality, we may assume that $\phi(yy_1) = 1$ and $\phi(yy_2) = 2$. If we can find a color in $\{7, 8, \ldots, \Delta+2\}$ that is valid for xy under ϕ, then we are done. Otherwise, for every $\theta \in \{7, 8, \ldots, \Delta+2\}$, there is a $(1, \theta, xy)$- or $(2, \theta, xy)$-critical path under ϕ. It follows that $\{7, 8, \ldots, \Delta + 2\} \subseteq S_{xx_1} \cup S_{xx_2}$. Note that $d(x_1) = d(x_2) = 3$ implies $|S_{xx_1}| = |S_{xx_2}| = 2$. Since $\Delta \geq 7$, $\{7, 8, 9\} \subseteq S_{xx_1} \cup S_{xx_2}$. Without loss of generality, we may assume that $S_{xx_1} = \{7, 8\}$ and $9 \in S_{xx_2}$. So $|\{7, 8\} \setminus S_{xx_2}| \geq 1$ (note that $|S_{xx_2}| = 2$). Now by recoloring xx_1 with 9, we obtain a new acyclic edge coloring of G' from ϕ, under which, the color in $\{7, 8\} \setminus S_{xx_2}$ would be valid for xy. This completes the proof of Lemma 5. $\square$

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Lemma 7. *A 6-vertex x with a special neighbor y has at most four 3^--neighbors (including y).*

Proof. Suppose to the contrary that x has at least five 3^--neighbors, say, x_1, x_2, x_3, x_4, y. Let z be the remaining neighbor of x. Without loss of generality, we may assume that all the 3^--neighbors of x are 3-vertices. Let $G' = G - xy$. By the choice of G, G' has an acyclic edge coloring ϕ using colors from $[\Delta + 2]$. Without loss of generality, we may assume that $\phi(xx_i) = i$ for $i = 1, 2, 3, 4$ and $\phi(xz) = 5$. Let $F(xy) = \phi(x) \cup \phi(y)$.

If $\phi(x) \cap \phi(y) = \emptyset$, then $|\phi(x) \cup \phi(y)| \le (\Delta - 1) + 2 = \Delta + 1 < \Delta + 2$. Hence there exists at least one color $\alpha \in [\Delta + 2] \setminus F(xy)$ that is valid for xy under ϕ. So we may assume that $\phi(x) \cap \phi(y) \ne \emptyset$. There are two cases under consideration.

(1) $|\phi(x) \cap \phi(y)| = 1$.

Let $\phi(x) \cap \phi(y) = \{\alpha\}$. Clearly, $\alpha \in \{1, 2, 3, 4, 5\}$. We discuss by distinguishing two subcases as follows.

(a) $\alpha \in \{1, 2, 3, 4\}$.

Without loss of generality, we may assume that $\phi(yy_1) = 1$ and $\phi(yy_2) = 6$. If we can find a color $\alpha \in [\Delta + 2] \setminus F(xy)$ that is valid for xy under ϕ, then we get an acyclic edge coloring of G using $\Delta + 2$ colors, a contradiction. Hence for each color $\theta \in [\Delta + 2] \setminus F(xy)$, there exists a $(1, \theta, xy)$-critical path under ϕ. It follows that $[\Delta + 2] \setminus F(xy) = \{7, 8, \ldots, \Delta + 2\} \subseteq S_{xx_1}$ as well as S_{yy_1}. Since $d(x_1) = 3$, $\Delta = 6$. Thus $S_{xx_1} = \{7, 8\}$ and $\{7, 8\} \subseteq S_{yy_1}$. Now by recoloring xx_1 with 6, we obtain a new acyclic edge coloring ϕ' of G', under which neither 7 nor 8 is valid for xy, since otherwise G would be acyclically edge $(\Delta + 2)$-colorable. This implies that there are $(6, 7, y, x_1)$- and $(6, 8, y, x_1)$-maximal bichromatic paths under ϕ as well as under ϕ'. Hence $7, 8 \in S_{yy_2}$. If there is a color $i \in \{2, 3, 4, 5\} \setminus (S_{yy_1} \cup S_{yy_2})$, then by recoloring yy_2 with color i, we obtain a new acyclic edge coloring of G', under which 6 would be valid for xy. So $\{2, 3, 4, 5\} \subseteq S_{yy_1} \cup S_{yy_2}$. It follows that $|S_{yy_1}| + |S_{yy_2}| \ge 2 + 2 + 4 = 8$. Note that y is special, i.e., one of y_1 and y_2 is a 4-vertex. Hence $|S_{yy_1}| + |S_{yy_2}| \le 3 + 5 = 8$ (recall that $\Delta = 6$). It follows that $|S_{yy_1}| + |S_{yy_2}| = 8$. So $1 \notin S_{yy_2}$. Now by recoloring yy_2 with a color in $\{2, 3, 4, 5\} \setminus \phi(y_2)$, we obtain a new acyclic edge coloring of G', under which, 6 would be valid for xy.

(b) $\alpha = 5$.

Without loss of generality, we may assume that $\phi(yy_1) = 5$ and $\phi(yy_2) = 6$.

If we can find a color $\alpha \in [\Delta + 2] \setminus F(xy)$ that is valid for xy under ϕ, then we get an acyclic edge coloring of G using $\Delta + 2$ colors, a contradiction. Hence for each color $\theta \in [\Delta + 2] \setminus F(xy)$, there exists a $(5, \theta, xy)$-critical path under ϕ. It follows that $[\Delta + 2] \setminus F(xy) = \{7, 8, \ldots, \Delta + 2\} \subseteq S_{yy_1}$ as well as S_{xz}. Since $d(y_1) \le \Delta$, $|S_{yy_1}| \le \Delta - 1$. Note that $|\{7, 8, \ldots, \Delta + 2\}| = \Delta + 2 - 7 + 1 = \Delta - 4$. It follows that $|S_{yy_1} \setminus \{7, 8, \ldots, \Delta + 2\}| \le (\Delta - 1) - (\Delta - 4) = 3$. Hence $|\{1, 2, 3, 4\} \setminus S_{yy_1}| \ge 1$. Without loss of generality, we may assume that $1 \notin S_{yy_1}$.

If $6 \notin S_{yy_1}$, then by recoloring yy_1 with color 1, we obtain a new acyclic edge coloring of G' from ϕ, and return to (a). So $6 \in S_{yy_1}$.

We are going to prove $5 \in S_{yy_2}$. If not, by recoloring yy_1, yy_2 with 1, 5, respectively, we obtain a new acyclic edge coloring ϕ' of G' that differs from ϕ only on yy_1 and yy_2. Note that $d(x_1) = 3$, $\{6, 7, 8\} \setminus S_{xx_1} \ne \emptyset$. Let $\alpha \in \{6, 7, 8\} \setminus S_{xx_1}$. If $\alpha = 6$, then 6 is valid for xy under ϕ' (since 6 belongs to neither S_{xx_1} nor S_{yy_2}), a contradiction. So $\alpha \in 7, 8$. Without loss of generality, we may assume that $\alpha = 7$. Since G is not acyclically edge $(\Delta + 2)$-colorable, coloring xy with 7 under ϕ' produces a bichromatic cycle C. Clearly C is either a $(1, 7)$- or a $(5, 7)$-bichromatic cycle. If C is a $(1, 7)$-bichromatic cycle, then $7 \in S_{xx_1}$, a contradiction. So C is a $(5, 7)$-bichromatic cycle. It follows that there is a $(5, 7, xy)$-critical path under ϕ', namely, there is a $(5, 7, x, y_2)$-maximal bichromatic path under ϕ. By Fact 1, there is no $(5, 7, x, y_1)$-maximal bichromatic path under ϕ. However, there is already a $(5, 7, x, y_1)$-maximal bichromatic path under ϕ. This contradiction shows that $5 \in S_{yy_2}$.

If there is a color $\alpha \in \{1, 2, 3, 4\} \setminus (S_{yy_1} \cup S_{yy_2})$, then by recoloring yy_1 with α, we obtain a new acyclic edge coloring of G' from ϕ, and return to (a). So $\{1, 2, 3, 4\} \subset S_{yy_1} \cup S_{yy_2}$. To conclude, $[\Delta + 2] \subseteq S_{yy_1} \cup S_{yy_2}$. On the other hand, $S_{yy_1} \cup S_{yy_2} \subseteq [\Delta + 2]$. So $S_{yy_1} \cup S_{yy_2} = [\Delta + 2]$. Since y is special, $|S_{yy_1}| + |S_{yy_2}| \le \Delta - 1 + 3 = \Delta + 2$. Hence $|S_{yy_1} \cap S_{yy_2}| = |S_{yy_1}| + |S_{yy_2}| - |S_{yy_1} \cup S_{yy_2}| \le (\Delta + 2) - (\Delta + 2) = 0$. Therefore $S_{yy_1} \cap S_{yy_2} = \emptyset$. Since $\{6, 7, 8, \ldots, \Delta + 2\} \subseteq S_{yy_1}, S_{yy_2} \subseteq \{1, 2, 3, 4, 5\}$. According to which of y_1 and y_2 is the 4-neighbor of y, there are two cases under consideration.

(b1) $d(y_1) = 4$.

Clearly, $|S_{yy_1}| = 3$. Since $\{6, 7, 8, \ldots, \Delta + 2\} \subseteq S_{yy_1}$, $|\{6, 7, 8, \ldots, \Delta + 2\}| \le 3$, i.e., $3 \ge \Delta + 2 - 6 + 1 = \Delta - 3$. Hence $\Delta \le 6$. Since $d(x) = 6$, $\Delta \ge 6$. So $\Delta = 6$. It follows that $S_{yy_1} = \{6, 7, 8\}$. Recall that $[\Delta + 2] \subseteq S_{yy_1} \cup S_{yy_2}$, i.e., $|S_{yy_2}| \ge (\Delta + 2) - |S_{yy_1}|$. Hence $d(y_2) = |S_{yy_2}| + 1 \ge (\Delta + 2) - |S_{yy_1}| + 1 = \Delta$. So $d(y_2) = \Delta = 6$, hence $|S_{yy_2}| = 5$. Note that $S_{yy_2} \subseteq \{1, 2, 3, 4, 5\}$. Hence $S_{yy_2} = \{1, 2, 3, 4, 5\}$. Let u_1, u_2, u_3 be the three neighbors of y_1 other than y. Without loss of generality, we may assume that $\phi(y_1u_1) = 7$ and $\phi(y_1u_2) = 8$. Now we claim that for every color $\eta \in \{1, 2, 3, 4\}$, $\xi \in \{7, 8\}$, there is a (η, ξ, y_2, y_1)-maximal bichromatic path under ϕ. Suppose not, without loss of generality, we may assume that there is no $(1, 7, y_2, y_1)$-maximal bichromatic path under ϕ. Then by recoloring yy_1, yy_2 with 1, 7, respectively, we obtain a new acyclic edge coloring of G' from ϕ, and return to (a). It follows that $\{1, 2, 3, 4\} \subset S_{y_1u_i}$, $i = 1, 2$. Note that there are $(5, 7, xy)$-, $(5, 8, xy)$-critical paths under ϕ. Hence $5 \in S_{y_1u_i}$, $i = 1, 2$. It follows that $\{1, 2, 3, 4, 5\} \subseteq S_{y_1u_i}$, $i = 1, 2$. Since $\Delta = 6$, $|S_{y_1u_i}| \le 5$, $i = 1, 2$. Therefore, $S_{y_1u_i} = \{1, 2, 3, 4, 5\}$, $i = 1, 2$. Now, we first exchange the colors on y_1u_1 and y_1u_2, then recolor yy_1, yy_2 with 1, 7, respectively, obtaining a new acyclic edge coloring of G', and returning to (a).

(b2) $d(y_2) = 4$.

In this case, $d(y_1) = \Delta$. Clearly $|S_{yy_2}| = 3$. By the symmetry of 1, 2, 3 and 4, we may assume that $S_{yy_2} = \{3, 4, 5\}$ and $S_{yy_1} = \{1, 2, 6, 7, \ldots, \Delta + 2\}$. Let v_1, v_2, v_3 be the three neighbors of y_2 other than y. Without loss of generality, we may

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assume that $\phi(y_2v_1) = 3$ and $\phi(y_2v_2) = 4$. We first claim that for every color $\eta_1 \in \{1, 2\}$, $\xi \in \{3, 4\}$, there is a (η_1, ξ, y_1, y_2)-maximal bichromatic path under ϕ. Suppose not, without loss of generality, we may assume that there is no $(1, 3, y_1, y_2)$-maximal bichromatic path under ϕ. Then by recoloring yy_1, yy_2 with 3, 1, respectively, we obtain a new acyclic edge coloring of G', under which, a color in $\{6, 7, 8\} \setminus S_{xx_3}$ is valid for xy (note that $|S_{xx_3}| = 2$, $S_{yy_2} = \{3, 4, 5\}$). This proves the claim. Hence $1, 2 \in S_{y_2v_i}$, $i = 1, 2$. We next claim that for every color $\eta_2 \in \{6, 7, \dots, \Delta+2\}$, $\xi \in \{3, 4\}$, there is a (η_2, ξ, y_1, y_2)-maximal bichromatic path under ϕ. Suppose not, without loss of generality, we may assume that there is no $(6, 3, y_1, y_2)$-maximal bichromatic path under ϕ. Then by recoloring yy_1, yy_2 with 3, 6, respectively, we obtain a new acyclic edge coloring of G', and return to (a). It follows that $\{6, 7, \dots, \Delta + 2\} \subseteq S_{y_2v_i}$, $i = 1, 2$. To conclude, $\{1, 2, 6, 7, \dots, \Delta + 2\} \subseteq S_{y_2v_i}$, $i = 1, 2$. Since $|\{1, 2, 6, 7, \dots, \Delta + 2\}| = \Delta + 2 - 6 + 1 + 2 = \Delta - 1$ and $|S_{y_2v_i}| \le \Delta - 1$, $S_{y_2v_i} = \{1, 2, 6, 7, \dots, \Delta + 2\}$, $i = 1, 2$. Now, we first exchange the colors on y_2v_1 and y_2v_2, then recolor yy_1 with 3, obtaining a new acyclic edge coloring of G' from ϕ, and returning to (a).

(2) $|\phi(x) \cap \phi(y)| = 2$.

In this case, $F(xy) = \phi(x) \cup \phi(y) = \{1, 2, 3, 4, 5\}$; hence $[\Delta + 2] \setminus F(xy) = \{6, 7, \dots, \Delta + 2\}$. Since $\Delta \ge 6$, $\{6, 7, 8\} \subseteq \{6, 7, \dots, \Delta + 2\}$. There are two subcases under consideration.

(i) $5 \notin \phi(y)$.

By the symmetry of 1, 2, 3 and 4, we may assume that $\phi(yy_1) = 1$ and $\phi(yy_2) = 2$. If we can find a color in $[\Delta+2]\setminus F(xy) = \{6, 7, \dots, \Delta+2\}$ that is valid for xy under ϕ, then we are done. Otherwise, for every $\theta \in \{6, 7, \dots, \Delta+2\}$, there is $(1, \theta, xy)$- or $(2, \theta, xy)$-critical paths under ϕ. It follows that $\{6, 7, \dots, \Delta+2\} \subseteq S_{xx_1} \cup S_{xx_2}$. Since $d(x_1) = d(x_2) = 3$, $|S_{xx_1}| = |S_{xx_2}| = 2$. Without loss of generality, we may assume that $S_{xx_1} = \{6, 7\}$ and $8 \in S_{xx_2}$. Now, by recoloring xx_1 with 8, we obtain a new acyclic edge coloring of G', under which, a color in $\{6, 7\} \setminus S_{xx_2}$ would be valid for xy.

(ii) $5 \in \phi(y)$.

By the symmetry of 1, 2, 3 and 4, we may assume that $\phi(yy_1) = 1$ and $\phi(yy_2) = 5$. If we can find a color in $[\Delta+2]\setminus F(xy) = \{6, 7, \dots, \Delta+2\}$ that is valid for xy under ϕ, then we are done. Otherwise, for every $\theta \in \{6, 7, \dots, \Delta+2\}$, there is a $(1, \theta, xy)$- or a $(5, \theta, xy)$-critical path under ϕ. If there is a color $\beta \in \{6, 7, \dots, \Delta+2\}\setminus S_{yy_1}$, then there is a $(5, \beta, xy)$-critical path under ϕ. It follows that there is no $(5, \beta, yy_1)$-critical path under ϕ by Fact 1. Now, by recoloring yy_1 with β, we obtain a new acyclic edge coloring of G', and return to (1). So $\{6, 7, \dots, \Delta + 2\} \subseteq S_{yy_1}$. Similarly, $\{6, 7, \dots, \Delta + 2\} \subseteq S_{yy_2}$. It follows that $\Delta - 3 = |\{6, 7, \dots, \Delta + 2\}| \le \min\{|S_{yy_1}|, |S_{yy_2}|\} = 3$, i.e., $\Delta \le 6$; hence $\Delta = 6$, and one of S_{yy_1} and S_{yy_2} is equal to $\{6, 7, 8\}$, and at least one of 2, 3 and 4 is not contained in $S_{yy_1} \cup S_{yy_2}$ (otherwise $\Delta > 6$). Now by recoloring yy_2 with a color in $\{2, 3, 4\} \setminus (S_{yy_1} \cup S_{yy_2})$, we obtain a new acyclic edge coloring of G', and return to (i). $\square$

Lemma 8. *G has no* $(2, 5^-)$*-edge.*

Proof. Suppose to the contrary that xy is a $(2, 5^-)$-edge with $d(x) = 2$. Let z be the neighbor of x other than y. Without loss of generality, we may assume that $d(y) = 5$ and y_i, $i = 1, 2, 3, 4$, the four neighbors of y other than x. Let $G' = G - xy$. By the minimality of G, G' has an acyclic edge coloring ϕ using colors from the color set $[\Delta + 2]$. We may assume that $\phi(yy_i) = i$, $i = 1, 2, 3, 4$. If $\phi(xz) \notin \{1, 2, 3, 4\}$, then we can color xy with a color from $[\Delta + 2] \setminus \{1, 2, 3, 4, \phi(xz)\}$ as $|[\Delta + 2] \setminus \{1, 2, 3, 4, \phi(xz)\}| = (\Delta + 2) - 5 = \Delta - 3 \ge d(y) - 3 = 5 - 3 = 2$, giving an acyclic edge coloring of G using $\Delta + 2$ colors, a contradiction. Thus, $\phi(xz) \in \{1, 2, 3, 4\}$.

Without loss of generality, we may assume that $\phi(xz) = 1$ and, for $i = 1, 2, 3, 4$, $d(y_i) = d(z) = \Delta$. If $([\Delta + 2] \setminus \{1, 2, 3, 4\}) \setminus S_{yy_1} \ne \emptyset$, then we can color xy with a color $\theta \in ([\Delta + 2] \setminus \{1, 2, 3, 4\}) \setminus S_{yy_1}$, giving an acyclic edge coloring of G using colors from $[\Delta + 2]$, a contradiction. So $([\Delta + 2] \setminus \{1, 2, 3, 4\}) \setminus S_{yy_1} = \emptyset$. Namely, $\{5, 6, \dots, \Delta + 2\} \subset S_{yy_1}$. Since G is not acyclic edge $(\Delta + 2)$-colorable, for every $i \in \{5, 6, \dots, \Delta + 2\}$, there is a $(1, i, xy)$-critical path passing through xz under ϕ. It follows that $\{5, 6, \dots, \Delta + 2\} \subset S_{xz}$. Without loss of generality, we may assume that $S_{xz} = \{4, 5, 6, \dots, \Delta + 2\}$. If $\{5, 6, \dots, \Delta + 2\} \setminus \phi(y_2) \ne \emptyset$, then we can recolor xz with 2 and color xy with a color in $\{5, 6, \dots, \Delta + 2\} \setminus \phi(y_2)$, giving an acyclic edge coloring of G using colors from $[\Delta + 2]$, a contradiction. So $\{5, 6, \dots, \Delta + 2\} \setminus \phi(y_2) = \emptyset$, i.e., $\{5, 6, \dots, \Delta + 2\} \subset S_{yy_2}$. Similarly, $\{5, 6, \dots, \Delta + 2\} \subset S_{yy_3}$.

Let us prove that $\{5, 6, \dots, \Delta + 2\} \subset S_{yy_4}$, too. If not, say $5 \notin S_{yy_4}$, then we can recolor yy_4 with 5, getting a new proper edge coloring ϕ' of G' that differs from ϕ only on yy_4. We are going to show that ϕ' is also acyclic, i.e., there is no bichromatic cycle under ϕ'. Suppose to the contrary that C is a bichromatic cycle under ϕ'. As ϕ is acyclic, C must be colored with 5 and a color $\alpha \in \{1, 2, 3\}$. If $\alpha = 1$, then there is a $(1, 5, yy_4)$-critical path under ϕ. By Fact 1, there is no $(1, 5, yx)$-critical path under ϕ (note that $x \ne y_4$). However, there has already been a $(1, 5, yx)$-critical path under ϕ, a contradiction. Suppose $\alpha = 2$. Clearly there is a $(2, 5, yy_4)$-critical path under ϕ. This implies that there is no $(2, 5, y, z)$-maximal chromatic path by Fact 1. (Note that $z \ne y_4$ since otherwise $2 \in \phi(y_4) = \phi(z)$ contradicting $2 \notin \phi(z)$.) Then we can recolor xz with 2 and then color xy with 5, modifying ϕ into an acyclic edge coloring of G using colors from $[\Delta + 2]$, a contradiction. Similar argument tells us that $\alpha \ne 3$. Hence ϕ' is acyclic indeed. Now, if 4 is valid for edge xy under ϕ', then we are done. Otherwise, there is a $(1, 4, xy)$-critical path under ϕ as well as under ϕ'. It follows that $4 \in \phi(y_1)$. By recoloring xz with 2 (resp. 3), we know that $4 \in \phi(y_2)$ (resp. $4 \in \phi(y_3)$). To conclude, $S_{yy_i} = S_{xz} = \{4, 5, \dots, \Delta + 2\}$ for $i = 1, 2, 3$ under ϕ. Now, by exchanging the colors on yy_1 and yy_2, we obtain a new proper edge coloring ϕ_1 of G' from ϕ. If ϕ_1 is acyclic, then 5 is valid for xy. Otherwise, there is either a $(4, 1)$-bichromatic cycle or a $(4, 2)$-bichromatic cycle under ϕ_1. It follows that there is either a $(4, 1, yy_2)$-critical path P_{12} or a $(4, 2, yy_1)$-critical path P_{21} under ϕ. Similar argument can yield that there is either a $(4, 1, yy_3)$-critical path P_{13} or a $(4, 3, yy_1)$-critical path P_{31} under ϕ. Also note that, by recoloring xz with 2, arguing as

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above, there is either a $(4, 2, yy_3)$-critical path P_{23} or a $(4, 3, yy_2)$-critical path P_{32} under ϕ. Without loss of generality, we may assume that P_{12} exists under ϕ. By Fact 1, P_{13} does not exist, hence P_{31} exists, by the same reason, P_{32} does not exist, P_{23} exists, hence P_{21} does not exists by Fact 1 under ϕ. Now, by recoloring yy_1, yy_2, yy_3 with 2, 3, 1, respectively, we obtain a new acyclic edge coloring of G', under which, 5 is valid for xy. This proves $\{5, 6, \ldots, \Delta + 2\} \subset S_{yy_4}$.

Now let α_i be the unique uncertain color in $\phi(y_i)$, $i \in \{1, 2, 3, 4\}$. By the property of ϕ, these uncertain colors cannot be identical since they all are in $\{1, 2, 3, 4\}$. We shall complete the proof by distinguishing three cases as follows.

(1) *Exactly three of the four uncertain colors are identical.*

Let the three uncertain identical colors be equal to α, and β the remaining color of the four uncertain colors.

We first consider the case $\alpha_4 = \beta$. Clearly $\alpha = 4$; hence $\beta \in \{1, 2, 3\}$. If $\beta = 1$, then we first recolor xz with 2, then exchange the colors on yy_2 and yy_3, obtaining a new acyclic edge coloring of G', under which, 5 is valid for xy. Suppose $\beta \in \{2, 3\}$. Without loss of generality, we may assume that $\beta = 2$. In this case, we can exchange the colors on yy_1 and yy_3, obtaining a new acyclic edge coloring of G', under which, 5 is valid for xy.

We next consider the case $\alpha_4 = \alpha$. Note that, if we recolor xz with 2 or 3, then we obtain a new acyclic edge coloring of G' that differs from ϕ only on xz. Thus 1, 2, 3 are symmetric. Without loss of generality, we may assume that $\alpha_1 = \beta$. Clearly $\alpha = 1$; hence $\beta \in \{2, 3, 4\}$. So either $\beta \neq 2$ or $\beta \neq 3$. Without loss of generality, we may assume that $\beta \neq 2$. Now we first recolor xz with 2, then exchange the colors on yy_2 and yy_4 if $\beta = 3$ or exchange the colors on yy_2 and yy_3 if $\beta = 4$, obtaining a new acyclic edge coloring of G', under which, 5 is valid for xy.

(2) *At least one pair of the four uncertain colors are the same.*

Let α be the same color or one of the two same colors among the four uncertain colors.

We first consider the case $\alpha_4 = \alpha$. Clearly $\alpha_4 \in \{1, 2, 3\}$. Since 1, 2, 3 are symmetric, we may assume that $\alpha = 1$ and $\alpha_3 = \alpha = 1$. If $\alpha_1 \notin \{3, 4\}$, then we first recolor xz with 3, and then exchange the colors on yy_3 and yy_4, obtaining a new acyclic edge coloring of G', under which, 5 is valid for xy. Suppose $\alpha_1 = 3(4)$. In this case, we first recolor xz with 2, then exchange the colors on yy_2 and yy_4 if $\alpha_2 = 3$ or exchange the colors on yy_2 and yy_3 if $\alpha_2 = 4$, obtaining a new acyclic edge coloring of G', under which, 5 is valid for xy.

We next consider the case α_4 is distinct from the other three uncertain colors. Clearly $\alpha_4 \in \{1, 2, 3\}$. Since 1, 2 and 3 are symmetric, we may assume that $\alpha_4 = 1$; hence $\alpha \in \{2, 3, 4\}$. If $\alpha = 4$, then by the symmetry of 1, 2 and 3, we may assume that $\alpha_1 = \alpha_2 = 4$, hence $\alpha_3 = 2$, and then we only need exchange the colors on yy_1 and yy_3, obtaining a new acyclic edge coloring of G', under which, 5 is valid for xy. Suppose $\alpha \in \{2, 3\}$. Without loss of generality, we may assume that $\alpha = 2$, hence $\alpha_1 = \alpha_3 = \alpha = 2$. Thus $\alpha_2 \in \{3, 4\}$. If $\alpha_2 = 4$, then we exchange the colors on yy_1 and yy_3; otherwise $\alpha_2 = 3$, then we first recolor xz with 3, and then exchange the colors on yy_3 and yy_4, obtaining a new acyclic edge coloring of G', under which, 5 is valid for xy.

(3) *The four uncertain colors are distinct.*

Clearly $\alpha_4 \in \{1, 2, 3\}$. By the symmetry of 1, 2 and 3, we may assume that $\alpha_4 = 1$. Clearly $\alpha_1 \in 2, 3, 4$. If $\alpha_1 = 4$, then $\alpha_2 = 3$ and $\alpha_3 = 2$. Now, by exchanging the colors on yy_1 and yy_2, we obtain a new acyclic edge coloring of G', under which, 5 is valid for xy. So $\alpha_1 \in \{2, 3\}$. Without loss of generality, we may assume that $\alpha_1 = 2$. In this case, $\alpha_2 = 3$ and $\alpha_3 = 4$. Now, by recoloring yy_1, yy_2, yy_3, yy_4 with 3, 4, 1, 2, respectively, we obtain a new acyclic edge coloring of G', under which, 5 is valid for xy. This completes the proof. □

Lemma 9. *Let $d \geq 6$ be an integer and x a d-vertex in G. If x has a 2-neighbor y and a 3^--neighbor y'. Then x has at most $(d - 6)$ 2-neighbors other than y and y'.*

Proof. Let $v_1, v_2, \ldots, v_{d-1}$ be the neighbors of x other than y, and z the neighbor of y other than x. As before, $G' = G - xy$ and ϕ an acyclic edge coloring of G' using colors from $[\Delta + 2]$. Without loss of generality, we may assume that $\phi(xv_i) = i$ for $i = 1, 2, \ldots, d - 1$. Suppose to the contrary that x has at least $(d - 5)$ 2-neighbors other than y and y'. For convenience, let $v_1, v_2, \ldots, v_{d-5}$ be 2-neighbors of x other than y and y', and $y' = v_{d-4}$. Without loss of generality, we may assume that $d(v_{d-4}) = 3$. Let v_i' be the neighbor of v_i other than x for $i = 1, 2, \ldots, d - 5$, and v_{d-4}', v_{d-4}'' the two neighbors of v_{d-4} other than x. Let $I = \{1, 2, \ldots, d - 4\}$, $J = \{d - 3, d - 2, d - 1\}$ and $K = \{d, d + 1, \ldots, \Delta + 2\}$. According to the value of $\phi(yz)$, there are three cases under consideration.

(1) $\phi(yz) \in K$.

In this case, $|\phi(x) \cup \{\phi(yz)\}| = (d - 1) + 1 = d < \Delta + 2$. So we can find a color in $[\Delta + 2] \setminus (\phi(x) \cup \{\phi(yz)\})$ that is valid for xy under ϕ.

(2) $\phi(yz) \in I$.

Assume that $\phi(yz) = i$. Then $|\phi(v_i)| \leq 3$ (more precisely, $|\phi(v_i)| = 3$ if $i = d - 4$; 2 otherwise). Since $|\phi(x) \cup \phi(v_i)| \leq d - 1 + 2 = d + 1 < \Delta + 2$, we can find a color in $[\Delta + 2] \setminus (\phi(x) \cup \phi(v_i))$ that is valid for xy under ϕ (note that $\phi(yz) = i \in \phi(v_i)$).

(3) $\phi(yz) \in J$.

This is an involving case to deal with. To exclude it, we need the following three claims.

Claim 1. $S_{yz} = I \cup K$.

If there is a color $i \in I \setminus S_{yz}$, by recoloring yz with i, we then obtain a new acyclic edge coloring of G' using colors from $[\Delta + 2]$, and then return to (2). So $I \subset S_{yz}$. If there is a color $k \in K \setminus S_{yz}$, by recoloring yz with k, we then obtain a new

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acyclic edge coloring of G' using colors from $[\Delta+2]$, and then return to (1). So $K \subset S_{yz}$. To conclude, $I \cup K \subseteq S_{yz}$. Since $|S_{yz}| \leq \Delta - 1$ and $|I \cup K| = \Delta - 1$, $S_{yz} = I \cup K$.

Claim 2. *For every $j \in J$, $S_{xv_j} = I \cup K$.*

Clearly, Claim 2 directly follows from the following two assertions: for every $j \in J$, there exists

(a) a (j, k, x, z)-maximal bichromatic path under ϕ for every $k \in K$;
(b) a (j, i, x, z)-maximal bichromatic path under ϕ for every $i \in I$.

Suppose to the contrary that there is no (j_0, k_0, x, z)-maximal bichromatic path under ϕ for some $j_0 \in J$, $k_0 \in K$. Observe that if $\phi(yz) \neq j_0$, then by recoloring yz with j_0, we obtain a new acyclic edge coloring of G' that differs from ϕ only on yz. Hence we may assume that $\phi(yz) = j_0$. Thus there is no (j_0, k_0, xy)-critical path under ϕ. It follows that k_0 is valid for xy under ϕ, a contradiction proving assertion (a).

Suppose to the contrary that there is no (j_0, i_0, x, z)-maximal bichromatic path under ϕ for some $j_0 \in J$, $i_0 \in I$. As above, we may assume that $\phi(yz) = j_0$. So there is no (j_0, i_0, xy)-critical bichromatic path under ϕ. If we can recolor xv_{i_0} with a suitable color obtaining a new acyclic edge coloring of G', say ϕ', which differs from ϕ only on edge xv_{i_0}, then i_0 is valid for xy under ϕ', a contradiction proving assertion (b). Below we show that we can do so indeed.

According to the value of i_0, there are two possibilities.

(i) $i_0 \neq d-4$.

In this case, without loss of generality, we may assume that $i_0 = 1$. If $\phi(v_1v_1') \in K$, then we can recolor xv_1 with a color $k \in K \setminus \{\phi(v_1v_1')\}$, as required. If $\phi(v_1v_1') = i' \in I$, then we can recolor xv_1 with a color in $K \setminus S_{xv_{i'}}$ since $|K| \geq 3$ and $|S_{xv_{i'}}| \leq 2$. Finally suppose $\phi(v_1v_1') = j' \in J$. By assertion (a), there is a (j', k, x, z)-maximal bichromatic path under ϕ for every $k \in K$. By Fact 1, there is no (j', k, x, v_1)-maximal bichromatic path under ϕ for every $k \in K$. It follows that we can recolor xv_1 with any color $k \in K$.

(ii) $i_0 = d-4$.

Let $S_{xv_{d-4}} = \{i_1, i_2\}$. According to the values of i_1 and i_2, there are $2^3 = 8$ subcases under consideration. In each subcase, as above, we can find a color $k \in K$ to recolor xv_{d-4} as required. For example, if $|\{i_1, i_2\} \cap I| = 2$, then we can choose a color $k \in K \setminus \{\phi(v_{i_1}v_{i_1'}), \phi(v_{i_2}v_{i_2'})\}$; if $|\{i_1, i_2\} \cap J| = 2$, then we can choose any color $k \in K$ by assertion (a) as above; if $|\{i_1, i_2\} \cap K| = 2$, then we can choose a color $k \in K \setminus \{i_1, i_2\}$, etc.

We are intent to destroy some (j, k, xy)-critical paths under ϕ such that k is valid for xy under a new acyclic edge coloring of G'. This first requires us to exchange colors on two of edges xv_j, $j \in J$. However it may yield some new (j, i)-bichromatic cycles for some $i \in I$. Intuitively, we should first avoid to produce $(j, d-4)$-bichromatic cycles.

Claim 3. *We can exchange some two of the colors on xv_{d-1}, xv_{d-2} and xv_{d-3} to obtain a proper edge coloring of G', under which there is no $(j, d-4)$-bichromatic cycle for every $j \in J$.*

For convenience, we rewrite $J = \{j_1, j_2, j_3\}$. If $|S_{xv_{d-4}} \cap J| \leq 1$, say $S_{xv_{d-4}} \cap J = \emptyset$, or $S_{xv_{d-4}} \cap J = \{j_1\}$, then we only need to exchange the two colors on xv_{j_2} and xv_{j_3}. Suppose $S_{xv_{d-4}} \subset J$. Without loss of generality, we may assume that $S_{xv_{d-4}} = \{j_1, j_2\}$. Now by exchanging the colors on xv_{j_1} and xv_{j_2}, we obtain a new proper edge coloring ϕ' of G'. If there is neither $(j_1, d-4)$- nor $(j_2, d-4)$-bichromatic cycle under ϕ', then we are done. Otherwise there is a $(j_1, d-4)$-bichromatic cycle, or a $(j_2, d-4)$-bichromatic cycle (or both bichromatic cycles) under ϕ'. Without loss of generality, we may assume that there is a $(j_1, d-4)$-bichromatic cycle under ϕ'. Clearly there is a $(d-4, j_1, xv_{j_2})$-critical path under ϕ as well as under ϕ'. By Fact 1, there is no $(d-4, j_1, xv_{j_3})$-critical path under ϕ. Now by exchanging the colors on xv_{j_1} and xv_{j_3}, we obtain a new proper edge coloring of G', under which there is no $(j, d-4)$-bichromatic cycle for every $j \in J$. This proves Claim 3.

By Claims 1–3, we know that G' admits a proper edge coloring ψ that differs from ϕ only on two of the three edges xv_{d-3}, xv_{d-2} and xv_{d-1}, under which, there is no $((d-4), j)$-bichromatic cycles for every $j \in J$. By symmetry, we may assume that the two edges, on which ψ and ϕ differs, are xv_{d-1} and xv_{d-2}. More precisely, $\psi(xv_{d-1}) = \phi(xv_{d-2}) = d-2$, $\psi(xv_{d-2}) = \phi(xv_{d-1}) = d-1$, $\psi(e) = \phi(e)$, otherwise.

Recall that the motivation of Claim 3 is to destroy the (j, k, xy)-critical path under ϕ for every $k \in K$ when $\phi(xy) = j$. Also observe that under ϕ we can freely recolor xy with any $j \in J$ in G', still giving an acyclic edge coloring of G'. So now we may assume that $\psi(xy) = \phi(xy) = d-1$ and that all the $(d-1, k, xy)$-critical path for $k \in K$ under ϕ are destroyed under ψ. If ψ is acyclic, then every color in K is valid for xy under ψ, we are done. Suppose ψ is not acyclic. We are going to show that we can make ψ into an acyclic edge coloring of G, under which, there is at least one color $k \in K$ that is valid for xy.

Let I_1 (resp. I_2) be the subset of I such that, for every $i_1 \in I_1$ (resp. $i_2 \in I_2$), there is a $(d-1, i_1)$- (resp. $(d-2, i_2)$)-bichromatic cycle under ψ, and $n_i = |I_i|$, $i = 1, 2$. According to the values of n_1 and n_2, there are six cases under consideration.

(0) $n_1 \geq 2$, $n_2 = 0$.

Without loss of generality, we may assume that the bichromatic cycles under ψ are $(i_1, d-1)$-bichromatic cycle for $i_1 \in \{1, 2, \ldots, n_1\}$. By taking a rotation of the colors $1, 2, \ldots, n_1$ on edges $xv_1, xv_2, \ldots, xv_{n_1}$ (i.e., assigning i to xv_{i+1} for $i = 1, 2, \ldots, n_1 - 1$ and n_1 to xv_1), we destroy all the present bichromatic cycles, and produce no new bichromatic cycles (because $d(v_i) = 2$, and $\psi(v_iv_i') = d-1$, $i = 1, 2, \ldots, n_1$), hence make ψ into an acyclic edge coloring of G', under which, every color $k \in K$ is valid for xy.

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(i) $n_1 = 1, n_2 = 0$.

Without loss of generality, we may assume that the unique bichromatic cycle under ψ is a $(1, d-1)$-bichromatic cycle. Observe that under ψ there is a $(d-1, 1, xv_1)$-critical path. By Fact 1, there is no $(d-1, 1, xy)$-critical path under ψ. If we can find a color $k \in K$ such that recoloring xv_1 with k under ψ results in a new acyclic edge coloring ϕ_1 of G', then 1 is valid for xy under ϕ_1, we are done. So we assume that for every $k \in K$, there is a $(d-1, k, xv_1)$-critical path under ψ. It follows that $K \subset \psi(v_1')(=\phi(v_1'))$. Let us prove $I \subset \psi(v_1')$. Clearly $1 \in \psi(v_1')$. Suppose there is a color $i \in I \setminus \{1\}$ that is not in $\psi(v_1')$. We recolor v_1v_1' with i, xv_1 with a color $k \in K \setminus \{\phi(v_i)\}$ under ψ, obtaining an acyclic edge coloring of G', under which 1 is valid for xy. This proves $I \subset \psi(v_1')(=\phi(v_1'))$.

Recall that there is already a $(d-3, 1, x, z)$-maximal bichromatic path under ϕ (as well as under ψ), by Fact 1, there is no $(d-3, 1, x, v_1')$-maximal bichromatic path under ϕ (as well as under ψ) (note that $z \neq v_1'$ since otherwise $\phi(yz) = \phi(v_1v_1') = d-1$ implies $y = v_1$). Now we recolor v_1v_1' with $d-3$ under ψ, obtaining a new acyclic edge coloring of G', under which every $k \in K$ is valid for xy.

(ii) $n_1 = 0, n_2 = 1$.

By recoloring yz with $d-2$, we obtain a new acyclic edge coloring of G' that differs from ϕ only on yz, under which (ii) turns into (i).

(iii) $n_1 \geq 2, n_2 = 1$.

Without loss of generality, we may assume that the bichromatic cycles under ψ are $(i_1, d-1)$-bichromatic cycle for $i_1 \in \{1, 2, \ldots, n_1\}$ and a $(d-5, d-2)$-bichromatic cycle C. By taking a rotation of colors $1, 2, \ldots, n_1$ on edges $xv_1, xv_2, \ldots, xv_{n_1}$, we destroy all the $(d-1, i_1)$-bichromatic cycles, where $i_1 \in I_1$, and obtain a new proper edge coloring φ of G' that differs from ψ only on $xv_1, xv_2, \ldots, xv_{n_1}$, under which there is exactly one bichromatic cycle C, hence return to (ii).

(iv) $n_1 = n_2 = 1$.

Let C_1 and C_2 be the only two bichromatic cycles under ψ. Without loss of generality, we may assume that C_1 is a $(d-1, 1)$-bichromatic cycle and C_2 is a $(d-2, 2)$-bichromatic cycle. In most of the subcases below, we are going to show that we can destroy C_2, and then return to case (i).

If there is a color $k \in K$ that is not in $\psi(v_2')$, then we recolor v_2v_2' with k under ψ. This action clearly destroys C_2 and does not produce any new bichromatic cycle as $d(v_2) = 2$ and $\psi(x) \cap K = \emptyset$, hence return to case (i). So we may assume that $K \subset \psi(v_2')$. Clearly, $2 \in \psi(v_2')$.

Suppose there is a color $i \in I \setminus \{2, d-4\}$ that is not in $\psi(v_2')$. Clearly, by recoloring v_2v_2' with i under ψ, we destroy C_2. If this operation results in no new bichromatic cycle, then we are done. Otherwise, there is a $(2, i)$-bichromatic cycle C_3 under the new proper edge coloring φ obtained by recoloring v_2v_2' with i under ψ. If there is a color $k \in K$ that is not in $\varphi(v_i')$, then we recolor v_iv_i' with k under φ, destroying C_3, and return to (i). So $K \subset \varphi(v_i')$. If $1 \notin \varphi(v_i')$, then we recolor v_iv_i' with 1 under φ, destroying C_3, and return to (i). So $1 \in \varphi(v_i')$. Clearly, $2, i \in \varphi(v_i')$. If there is a color $i^* \in I \setminus \{1, 2, i, d-4\}$ that is not in $\varphi(v_i')$, then we recolor v_iv_i' with i^* under φ, obtaining a new proper edge coloring φ' of G': if there is exactly one bichromatic cycle C_1 under φ', then we return to (i); otherwise, except C_1, there is another (i, i^*)-bichromatic cycle under φ'. But then we can exchange the colors on xv_2 and xv_{i^*} under φ', obtaining a new proper edge coloring of G', under which there is exactly one bichromatic cycle C_1, hence return to (i). Now $(I \setminus \{d-4\}) \cup K \subset \varphi(v_i')$. If $d-3 \notin \varphi(v_i')$, then we recolor v_iv_i' with $d-3$, obtaining a new proper edge coloring of G', under which there is exactly one bichromatic cycle C_1. (Otherwise, there is a $(d-3, i)$-bichromatic cycle, hence there is a $(d-3, i, x, v_i')$-maximal bichromatic path under ϕ. By Fact 1, there is no $(d-3, 2, x, z)$-maximal bichromatic path under ϕ as $z \neq v_i'$. This contradicts assertion (b) in Claim 2), hence return to (i). Thus $(d-3) \in \varphi(v_i')$. If $d-2 \notin \varphi(v_i')$, then we recolor v_iv_i' with $d-2$ under φ, obtaining a new proper edge coloring φ'' of G': if there is exactly one bichromatic cycle C_1 under φ'', then we return to (i); otherwise, except C_1, there is a $(d-2, i)$-bichromatic cycle under φ''. Now by recoloring v_2v_2' with $d-2$ and exchanging the colors on xv_2, xv_i under φ'', we obtain a new proper edge coloring of G', under which there is exactly one bichromatic cycle C_1, hence return to (i). It follows that $d-1 \notin \varphi(v_i')$. Now we recolor v_iv_i' with $d-1$ under φ, obtaining a new proper edge coloring φ''' of G': if there is exactly one bichromatic cycle C_1 under φ''', then we return to (i); otherwise, except C_1, there is another $(d-1, i)$-bichromatic cycle under φ''', hence return to (0). So far, we have proved $(I \setminus \{d-4\}) \cup K \subset \psi(v_2')$.

If $d-1 \notin \psi(v_2')$, then by recoloring v_2v_2' with $d-1$ under ψ, we obtain a new proper edge coloring ψ' of G': if there is exactly one bichromatic cycle C_1 under ψ', then we return to (i); otherwise, except C_1, there is another $(2, d-1)$-bichromatic cycle under ψ', hence return to case (0). So $d-1 \in \psi(v_2')$. It follows that $d-3 \notin \psi(v_2')$. Now recoloring v_2v_2' with $d-3$ under ψ, C_2 is destroyed, and no new bichromatic cycle is produced (otherwise, there is a $(d-3, 2)$-bichromatic cycle, hence there is a $(d-3, 2, x, v_2')$-maximal bichromatic path under ϕ; by Fact 1, there is no $(d-3, 2, x, z)$-maximal bichromatic path under ϕ as $z \neq v_2'$. This contradicts assertion (b) in Claim 2), hence return to case (i).

(v) $n_1 \geq 2, n_2 \geq 2$.

In this case, we only need make two rotations as in case (0).

(vi) $n_1 = 1, n_2 \geq 2$.

By recoloring yz with $d-2$, we obtain a new acyclic edge coloring of G' that differs from ϕ only on yz, which turns (vi) into (iii).

(vii) $n_1 = 0, n_2 \geq 2$.

Now by recoloring yz with $d-2$, we obtain a new acyclic edge coloring of G' that differs from ϕ only on yz, which turns (vii) into (0).

Lemma 9 is proved. □

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3. Discharging

To complete the proof of Theorem 1, we shall derive a contradiction by a discharging procedure proceeded in G.

First, we define the *initial charge function* μ on $V = V(G)$ by letting $\mu(v) = d(v) - 4$ for every $v \in V$. Since $\text{mad}(G) < 4$, the sum of the initial charge is negative. If we can make suitable discharging rules to redistribute charges among vertices so that the *final charge* $\mu'(v)$ of every vertex $v \in V$ is nonnegative, then we get a contradiction completing the proof.

The needed discharging rules are as follows.

R1. Every 2-vertex gets 1 from each of its 6^+-neighbors.
R2. Let v be a 3-vertex. If v is special, then it gets $\frac{1}{2}$ from each of its 5^+-neighbors. Otherwise, v gets $\frac{1}{3}$ from each of its 5^+-neighbors.

We are going to show that $\mu'(v) \geq 0$ for all $v \in V$. Recall that every vertex in G has degree at least 2.

Let v be a 2-vertex. By Lemma 8, the two neighbors of v are 6^+-vertices. By R1, $\mu'(v) \geq -2 + 2 \times 1 = 0$.

Let v be a 3-vertex. By Lemma 1, v has no 3^--neighbors. By Lemma 3, v has at most one 4-neighbor. If v has a 4-neighbor, i.e., v is special, then $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$ by R2. Otherwise, $\mu'(v) \geq -1 + 3 \times \frac{1}{3} = 0$ by R2.

Let v be a 4-vertex. By our rules, v gives and gets nothing. Hence, $\mu'(v) = \mu(v) = 0$.

Let v be a 5^+-vertex. If v has no 3^--neighbor, then v gives and gets nothing, we have $\mu'(v) = \mu(v) \geq 5 - 4 = 1 > 0$. Suppose v has at least one 3^--neighbor.

First assume that v has no 2-neighbor. If $d(v) \geq 8$, then $\mu'(v) \geq \mu(v) - d(v) \times \frac{1}{2} = d(v) - 4 - \frac{d(v)}{2} = \frac{d(v)-8}{2} \geq 0$ by R2. Suppose $d(v) = 7$: if v has at least one special-neighbor, by Lemma 6, $\mu'(v) \geq \mu(v) - 6 \times \frac{1}{2} = 7 - 4 - 3 = 0$; otherwise, $\mu'(v) \geq \mu - 7 \times \frac{1}{3} = 7 - 4 - \frac{7}{3} = \frac{2}{3} > 0$. Suppose $d(v) = 6$: if v has at least one special-neighbor, by Lemma 7, $\mu'(v) \geq \mu(v) - 4 \times \frac{1}{2} = 6 - 4 - 2 = 0$; otherwise, $\mu'(v) \geq \mu - 6 \times \frac{1}{3} = 6 - 4 - 2 = 0$. Finally suppose $d(v) = 5$. By Lemma 4, v has at most three 3-neighbors. If v has at least one special-neighbor, by Lemma 5, $\mu'(v) \geq \mu(v) - 2 \times \frac{1}{2} = 5 - 4 - 1 = 0$; otherwise, $\mu'(v) \geq \mu(v) - 3 \times \frac{1}{3} = 5 - 4 - 1 = 0$.

Next assume that v has at least one 2-neighbor. By Lemma 8, v is a 6^+-vertex. If v has no 3-neighbor, by Lemma 9, v has at most $(d(v) - 4)$ 2-neighbors, hence $\mu'(v) \geq \mu(v) - (d(v) - 4) \times 1 = 0$. If v has exactly one 3-neighbor, by Lemma 9, v has at most $(d(v) - 5)$ 2-neighbors, giving $\mu'(v) \geq \mu(v) - \frac{1}{2} - (d(v) - 5) \times 1 = \frac{1}{2} > 0$. Suppose v has at least two 3-neighbors, by Lemma 2, it has at most $(d(v) - 3) 3^-$-neighbors, we have $\mu'(v) \geq \mu(v) - 2 \times \frac{1}{2} - (d(v) - 3 - 2) \times 1 = d(v) - 4 - 1 - (d(v) - 5) = 0$.

Theorem 1 is completely proved.

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References

[1] N. Alon, B. Sudakov, A. Zaks, Acyclic edge-colorings of graphs, J. Graph Theory 37 (2001) 157–167.
[2] M. Basavaraju, L.S. Chandran, Acyclic edge coloring of graphs with maximum degree 4, J. Graph Theory 61 (2009) 192–209.
[3] M. Basavaraju, L.S. Chandran, Acyclic edge coloring of triangle free planar graphs, 2010. http://arxiv.org/abs/1007.2282v1.
[4] M. Basavaraju, L.S. Chandran, N. Cohen, F. Havet, T. Müller, Acyclic edge-colouring of planar graphs, SIAM J. Discrete Math. 25 (2011) 463–478.
[5] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, Berlin, 2008.
[6] M. Borowiecki, A. Fiedorowicz, Acyclic edge colouring of planar graphs without short cycles, Discrete Math. 310 (2010) 1445–1455.
[7] W. Dong, B. Xu, Some results on acyclic edge coloring of plane graphs, Inform. Process. Lett. 110 (2010) 887–892.
[8] J. Fiamčík, The acyclic chromatic class of a graph, Math. Slovaca 28 (1978) 139–145.
[9] A. Fiedorowicz, M. Hałuszczak, N. Narayanan, About acyclic edge colourings of planar graphs, Inform. Process. Lett. 108 (2008) 412–417.
[10] J. Hou, G. Liu, G. Wang, Improved bounds for acyclic chromatic index of planar graphs, Discrete Appl. Math. 159 (2011) 876–881.
[11] J. Hou, G. Liu, J. Wu, Acyclic edge coloring of planar graphs without small cycles, Graphs Combin. 28 (2012) 215–226.
[12] J. Hou, N. Roussel, J. Wu, Acyclic chromatic index of planar graphs with triangles, Inform. Process. Lett. 111 (2011) 836–840.
[13] J. Hou, J. Wu, G. Liu, B. Liu, Acyclic edge colorings of planar graphs and series–parallel graphs, Sci. China Ser. A 51 (2009) 605–616.
[14] P. Sheng, Y. Wang, On acyclic edge colorings of planar graphs without intersecting triangles, Discrete Math. 311 (2011) 2490–2495.
[15] Y. Wang, P. Sheng, Improved upper bound for acyclic chromatic index of planar graphs without 4-cycles, J. Comb. Optim. http://dx.doi.org/10.1007/s10878-012-9524-5.