Improvement of Synchronizability of Scale-Free Networks *

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We investigate the factors that affect synchronizability of coupled oscillators on scale-free networks. Using the memory Tabu search (MTS) algorithm, we improve the eigen-ratio Q of a coupling matrix by edge intercrossing. The numerical results show that the synchronization-improved scale-free networks should have distinctive both small average distance and larger clustering coefficient, which are consistent with some real-world networks. Moreover, the synchronizability-improved networks demonstrate the disassortative coefficient.

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Over the past few years, the analysis of complex systems from the viewpoint of networks has become an important interdisciplinary research topic.[1−11] Recent empirical studies have demonstrated that many real-world networks cannot be treated as either regular or random networks. The most important two common statistical characteristics are called the small-world effect[1] and scale-free property.[2] Small-world networks (SW) can be generated from regular networks by randomly rewiring connections, which results in a small average distance. Scale-free networks display a power-law degree distribution $p(k) \sim k^{-\gamma}$, where the exponent γ observed in real-world networks is often between 2 and 3.

Synchronization in coupled dynamical systems has been studied for many years based on nonlinear dynamics. In particular, synchronization in networks of coupled chaotic systems has received a great deal of attention over the past two decades.[12−15] However, most of these works have been concentrated on networks with regular topological structures such as chains, grids, lattices, as well as fully connected graphs.[16] It has been shown that physical and dynamical processes taking place on networks, such as cascading failures,[17] epidemic spreading,[18] and network synchronization,[5,19−24] are strongly influenced by the underlying structures of the networks. Recently, an increasing number of studies have been devoted to investigating synchronization phenomena in complex networks with small-world and scale-free topologies.[25−28]

One goal of studying network synchronization is to understand how the network topology affects synchronizability. Network synchronizability can be measured well by the eigen-ratio Q of the largest eigenvalue and the smallest nonzero eigenvalue.[22,29−31] Therefore, we focus on the relationship between the network structure and its eigenvalues. Since there are several topological characters of scale-free networks, we intend to find out what is the most important factor to determine the synchronizability of a system. The characteristics of some real-world networks can be found in Ref. [6].

Here we investigate the synchronizability of a continuously dynamical network model with a scale-free structural property. Some detailed comparisons between various networks are presented such that network synchronizability will be stronger with smaller heterogeneity, which can be measured by the variance of a degree distribution or betweenness distribution.[22,32,33] However, no strict and clear conclusions can be achieved because previous studies are of varying both average distances and degree variances. Average distance D has also been extensively studied. Some works indicate that the average distance D is one of the key factors in network synchronizability.[34,35] However, no consistent conclusions have been achieved so far.[22,36−38]

Some researchers considered that the randomicity is the more intrinsic factors leading to better synchronizability,[39] which means that the intrinsic reason making small-world and scale-free networks having better synchronizability than regular ones is their random structures.

Recently, several researchers examined the effect of clustering coefficient on synchronization by using the Kuramoto model[40] or a master stability function.[41,42] Other researchers focus on the role

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played by maximal betweenness $B_{\max}$, they found that network synchronizability with smaller $B_{\max}$ will be better.[21,32] Zhao *et al.*[33] enhanced synchronizability by structural perturbations. They found that maximal betweenness plays a major role in network synchronization.[34] Zhou *et al.* [35] studied the effect of average distance D on synchronizability by a crossed double cycle. Zhou *et al.*[43,44] investigated the synchronizability in weighted complex networks and showed that the synchronizability of random networks with a large minimum degree can be determined by two leading parameters: the mean degree and the heterogeneity of the distribution of intensity of a node, where the intensity of a node is defined as the total strength of input connections. However, many structural characteristics have been used to describe a network, such as degree distribution $p(k)$, average distance D, clustering coefficient C, and betweenness B. In fact, the dynamical behaviour of a network, such as synchronizability, can be affected by these characteristics simultaneously. Therefore, the effect of these structural factors on the synchronizability should be investigated separately. Furthermore, an optimal algorithm has been used to find the improved topology for network dynamics.[45,46] However, they have not discussed the synchronizability of scale-free networks in detail.

Since some real-world networks have both power-law distribution and small-world effect, and inspired by the idea that a structure determine functions, while functions affect its structure, in this Letter we investigate the synchronizability of a class of continuous-time dynamical networks with scale-free topology. Based on the synchronization criterion, we improve the ratio Q of the eigenvalues of a coupling matrix by edge-intercrossing procedures, which provides a way for observing the correlation between synchronizability and those characteristics by keeping the degree distribution unchanged. The numerical results might explain why some real-world networks have the distinctive larger clustering coefficient and small average distances.

We start by considering a network of N linearly coupled identical oscillators. The equation of motion reads

$$\dot{x}^i = \boldsymbol{F}(x^i) + \sigma \sum_{j=1}^{N} G_{ij} \boldsymbol{H}(x^j), \quad i = 1, \ldots, N, \quad (1)$$

where $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ governs the local dynamics of the vector field x^i in each node, $\boldsymbol{H}(\boldsymbol{x})$ is a linear vectorial function, σ is the coupling strength, and G is a coupling matrix.

Stability of the synchronous state $x^i(t) = x^s(t)$, $i = 1, \ldots, N$ can be accounted for by diagonalizing the linear stability equation, yielding N blocks in the form $\dot{\zeta}_i = \boldsymbol{JF}(\boldsymbol{x}_s)\zeta_i - \sigma\lambda_i \boldsymbol{H}(\zeta_i)$, where $\boldsymbol{J}$ is the Jacobian operator. Replacing $\sigma\lambda_i$ by ν in the equation, the master stability function (MSF)[29] fully accounts for the linear stability of the synchronization manifold. For a large class of oscillatory systems, the MSF is negative in a finite parameter interval $I_{st} \equiv (\nu_1 \le \nu \le \nu_2)$.[29]. When the whole set of eigenvalues (multiplied by σ) enters the interval I_{st}, the stability condition is satisfied. This is accomplished when $\sigma\lambda_2 > \nu_2$ and $\sigma\lambda_N < \nu_2$ simultaneously. As ν_2 and ν_1 depend on the specific choice of $\boldsymbol{F}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$, the key quantity for assessing the synchronization of a network is the eigen-ratio

$$Q = \lambda_N/\lambda_2, \quad (2)$$

which only depends on the topology of the network. The smaller the value of λ_N/λ_2 is, the more packed the eigenvalues of G are, leading to an enhanced σ interval for which stability is obtained.[20] In this study, for universality, we do not address any particular dynamical system, but concentrate on how the network topology affects eigen-ratio Q.

The heuristic algorithm (MTS)[47] is presented as follows:

Step 1. Generate an initial matrix G_0 of an extensional BA network[48,49] with N nodes and E edges. Set the improved network's coupling matrix $G_k^* = G_0$ and the improved network's taibu table $G_k = G_0$, and the time step $k = 0$. Compute the ratio Q of G_k^*.

Step 2. If a prescribed termination condition is satisfied, stop (Since the MTS algorithm is heuristic, it can only find the approximate optimal solution. Thus, the termination condition of Step 2 should confirm by the experimentation solution); otherwise, intercrossing a pairs of edges chosen randomly based on the network remains connected, denote by $\boldsymbol{G}$.

Step 3. If $Q_G \le Q_{G_k^*}$, $G_{k+1}^* = G$, $G_{k+1} = G$, else if $Q_G \le Q_{G_k}$, $G_{k+1} = G$, else if G does not satisfy the taibu condition $|Q_{G_k} - Q_G|/R_G > \delta$ (where δ is a random number between 0.5 and 0.75), then $G_{k+1} = G$, else $G_{k+1} = G_k$. Go to step 2.

After many numerical experiments, we find that the termination can be set to 3000 time steps. From Fig. 1, we can see that Q reaches a stability value after 3000 steps.

The numerical results are experimented on the extensional BA model[48,49] for different network scales. When the network size $N = 100$, the fluctuations are greatly emerged because of the too small network size, therefore we start from a network of sizes $N = 200, 300, 400$ and 500 and the average degree $\langle k \rangle = 6$ and then perform the optimization processes. At each time step, we record the structural properties, such as D, C, r and average node betweenness, as the objective function Q is reduced. Figure 1 demonstrates the values of the eigen-ratio Q vs the evolving

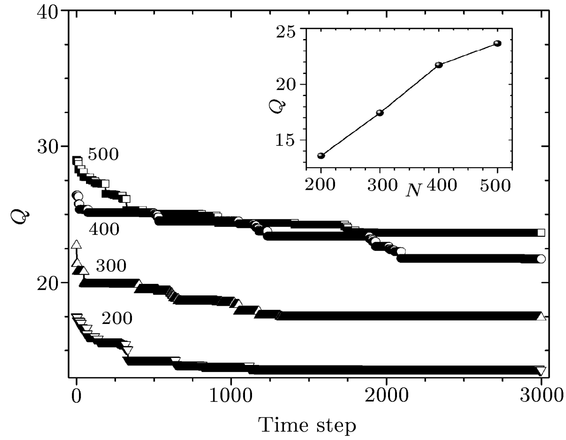

Fig. 1. Eigen-ratio Q of an extensional BA model with $\langle k \rangle = 6$ for different network size $N = 200, 300, 400$ and 500. The horizonal coordinate denotes the steps of the value Q being improved. All the data are obtained by 10 independent runs. Inset: the stability value vs the network size N.

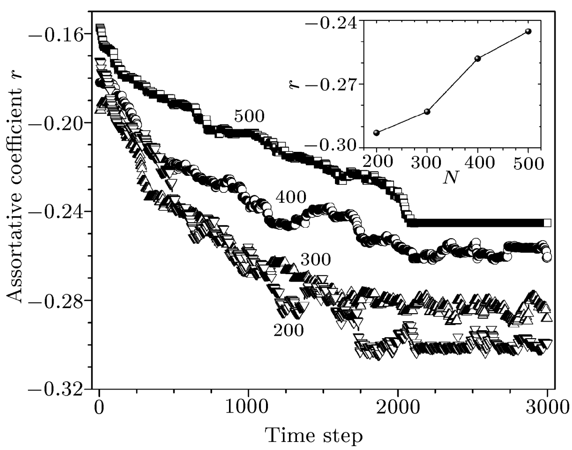

Fig. 2. Assortative coefficient r vs optimal algorithm time step. All the data are obtained by 10 independent runs.

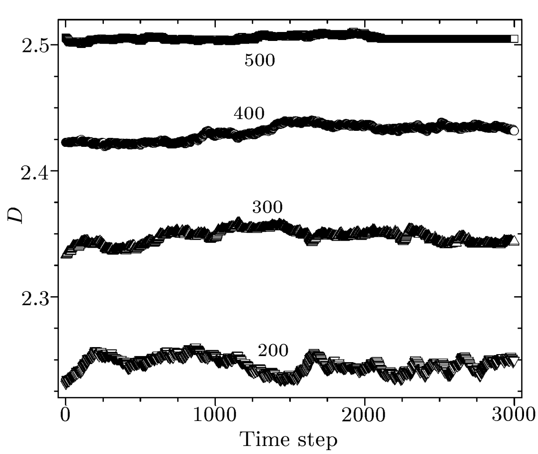

Fig. 3. Average distance D vs optimal algorithm time steps. All the data are obtained by 10 independent runs.

steps. The inset shows the stability value vs the network size N, which indicates that network synchronizability increases with N.

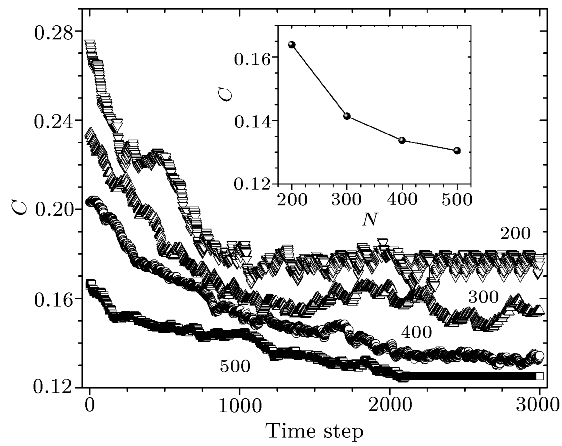

Fig. 4. Average clustering coefficient C vs optimal algorithm time steps. All the data are obtained by 10 independent runs.

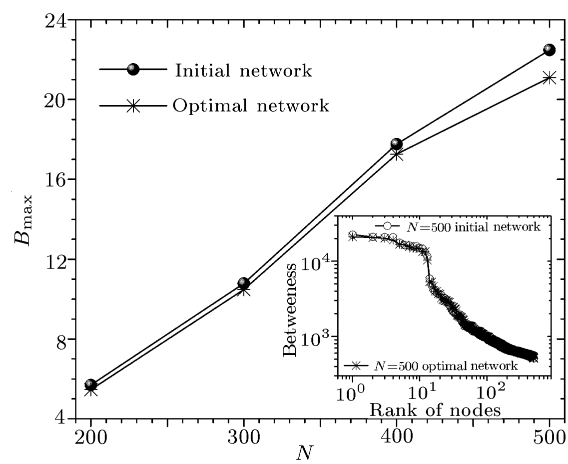

Fig. 5. Maximal node betweenness $B_{\max}$ vs network size N. Inset: the Zipf plot of node betweenness with size $N = 500$. All the data are obtained by 10 independent runs.

As shown in Fig. 2, the assortative coefficient r decreases to a stability negative value as Q decreases, and the stability value increases with N, the inset shows that the assortative coefficient r increases with N. Because the algorithm has found the improved solution of Q, the tail line of $N = 500$ is straight.

Figure 3 shows that the average distance D keeps approximately stable as Q decreases, which indicates that D may not be the key factor in the synchronizability of scale-free networks.

Figure 4 shows that the clustering coefficient C decreases to a stability value as Q decreases, and the inset shows the stable value decreases as N increases, which is consistent with the conclusion presented in Refs. [40–42].

Figure 5 demonstrates the maximal node betweenness $B_{\max}$ of the initial network and improved network. It is shown that when $B_{\max}$ of the improved network is slightly smaller than that of the initial network. The inset shows the node betweenness Zipf plot of the initial and improved network with $N = 500$, which is almost superposition. The statistical properties of the improved networks for different size N show similar trends.

In summary, using the MTS optimal algorithm, we have optimized network synchronizability by changing the connection pattern between different pairs of nodes while keeping the degree distribution. Starting from scale-free networks, we have studied the dependence between the structural characteristics of scale-free networks and synchronizability. The numerical results suggest that a scale-free network with shorter path length D, lower degree of clustering C, and nodes connecting according to disassortive pattern can be easily synchronized.

Because the MTS algorithm needs much of the computer system source, the network size of our numerical results can only be limited on smaller than $N = 500$. Further work should improve the algorithm to enlarge the network size. Although network synchronizability is determined by the network structural characteristics simultaneously, we also wonder which characteristic is the key factor governing network synchronizability.

By maximizing and minimizing Q respectively using the optimal algorithm, the characteristic which changes dramatically can be seen as the factor. The further work would focus on this issue.

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