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New travelling wave solutions for a nonlinearly dispersive wave equation of Camassa–Holm equation type

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ABSTRACT

In this paper, the integral bifurcation method is used to study a nonlinearly dispersive wave equation of Camassa–Holm equation type. Loop soliton solution and periodic loop soliton solution, solitary wave solution and solitary cusp wave solution, smooth periodic wave solution and non-smooth periodic wave solution of this equation are obtained, their dynamic characters are discussed. Some solutions have an interesting phenomenon that one solution admits multi-waves when parameters vary.

1. Introduction

In this paper, we consider the following nonlinearly dispersive wave equation of Camassa–Holm equation type (see [1,2])

$$u_t - u_{xxt} + \omega u_x + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx}), \tag{1}$$

where ω is real number, γ is a physical parameter. This equation has recently been investigated with regard to singularity formation by Yin [2]. He also presented the existence of smooth solitary wave for certain values of the parameters ω and γ. For $\gamma = 1$, Eq. (1) is well known as the Camassa–Holm equation which is first found by Fokas and Fuchssteiner [3], and later rediscovered as a water wave model by Camassa et al. [4,5].

When $\omega = 0$, Eq. (1) is reduced to a model equation for mechanical vibrations in a compressible elastic rod, as being derived by Dai and Huo [6,7], where the range of the parameter γ is roughly from -29.5 to 3.4. In 2000, Constantin and Strauss [8] proved that the solitary waves of Eq. (1) were orbitally stable for $\omega = 0, \gamma \neq 1$. In 2004, Liu and Chen considered the Eq. (1) for $\omega = 0$, they employed both bifurcation method and numerical simulation to investigate bounded travelling waves in a general compressible hyper-elastic rod (see [9]). In 2006, Lenells [10] also studied the Eq. (1) when $\omega = 0$, and discussed the weak travelling wave solutions, but the author did not give the expression of exact travelling wave solutions. For $\omega = \gamma = 0$, the equation is well known as the BBM equation [11], which is a model for surface waves in a channel. In 2004, the dynamic stability of solitary wave solutions of (1) was considered by Henrik Kalisch (see [1] and the references cited therein). Meanwhile, Li et al. refined Johnson's implementation of Constantin's method for obtaining a multiple-soliton solution of the Eq. (1) while $\gamma = 1$ (see [12]). However, the loop soliton solution and periodic loop soliton solution of Eq. (1) have not been investigated. In this paper, by using the integral bifurcation method [13–15], we will try to obtain the loop soliton solution and periodic loop soliton solution, solitary wave solution and solitary cusp wave solution, smooth periodic wave solution and non-smooth periodic wave solution of (1), and also discuss their dynamic characters and relations.

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2. Loop soliton solution, solitary cusp wave solution, solitary wave solutions and periodic wave solutions for Eq. (1)

Let $u(x,t) = \phi(\xi) = \phi(x - ct)$, where constant c is wave velocity. Under this transformation, Eq. (1) can be reduced to the following ODE,

$$-c\phi_\xi + c\phi_{\xi\xi\xi} + \omega\phi_\xi + 3\phi\phi_\xi = \gamma(2\phi_\xi\phi_{\xi\xi} + \phi\phi_{\xi\xi\xi}). \tag{2}$$

Integrating (2) once, we have

$$(\omega - c)\phi + c\phi_{\xi\xi} + \frac{3}{2}\phi^2 - \gamma\left[\phi\phi_{\xi\xi} + \frac{1}{2}(\phi_\xi)^2\right] = g, \tag{3}$$

where g is an integral constant. Clearly, (3) is equivalent to the following two-dimensional system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-g + (\omega - c)\phi + \frac{3}{2}\phi^2 - \frac{1}{2}\gamma y^2}{\gamma\phi - c}. \tag{4}$$

We notice that the function $dy/d\xi$ is not continuous when $\phi = c/\gamma$ in Eq. (4). So, we make a transformation

$$d\xi = (\gamma\phi - c)d\tau. \tag{5}$$

Under this transformation, Eq. (5) can be reduced to

$$\frac{d\phi}{d\tau} = (\gamma\phi - c)y, \quad \frac{dy}{d\tau} = -g + (\omega - c)\phi + \frac{3}{2}\phi^2 - \frac{1}{2}\gamma y^2, \tag{6}$$

where τ is a parametric variable. From Eq. (4), it is easy to know that y has no definition when $\phi = \phi_s = \frac{c}{\gamma}$. In other words, on such straight line $\phi = \phi_s$, $\frac{d\phi}{d\xi}$ is nonexistent. It is for this reason that some of the solutions of Eq. (1) have the phenomenon of non-smooth or singular characters such as blow up phenomenon, possessing vertical tangent line etc.

Systems (4) and (6) have the same first integral as follows:

$$y^2 = \frac{-2g\phi + (\omega - c)\phi^2 + \phi^3 + h}{\gamma\phi - c}, \tag{7}$$

where h is an integral constant. Eq. (7) can be rewritten as

$$y = \pm\frac{\sqrt{\gamma\phi^4 + [(\omega - c)\gamma - c]\phi^3 + [-2g\gamma - (\omega - c)c]\phi^2 + (h\gamma + 2cg)\phi - hc}}{\gamma\phi - c}. \tag{8}$$

Substituting Eq. (8) into the first expression in Eq. (6), we have

$$\left(\frac{d\phi}{d\tau}\right)^2 = \gamma\phi^4 + [(\omega - c)\gamma - c]\phi^3 + [-2g\gamma - (\omega - c)c]\phi^2 + (h\gamma + 2cg)\phi - hc. \tag{9}$$

Under some parametric conditions, Eq. (9) can be reduced to two well known auxiliary equations which were given in Refs. [16–19]. Using these references' results, we can obtain loop soliton solutions and periodic loop soliton solutions of Eq. (1). First, we discuss periodic solutions of Eq. (1), see the next discussion.

2.1. When $c = \frac{\omega\gamma}{\gamma+1}$ and $h = -\frac{2g\omega}{\gamma+1}$, Eq. (9) can be reduced to

$$\left(\frac{d\phi}{d\tau}\right)^2 = P\phi^4 + Q\phi^2 + R, \tag{10}$$

where $P = \gamma, Q = -\frac{\gamma(2g\gamma^2+4g\gamma+2g+\omega^2)}{(\gamma+1)^2}$, $R = \frac{2g\gamma\omega^2}{(\gamma+1)^2}$.

2.1.1. If $0 < \gamma < 1$ and $m = \sqrt{1-\gamma}$, $\omega = -\gamma - 1$, $g = \frac{\gamma-1}{2\gamma}$ (or $m = \sqrt{1-\gamma}$, $\omega = +\gamma + 1$, $g = \frac{\gamma-1}{2\gamma}$), then $P = 1 - m^2, Q = 2m^2 - 1, R = -m^2$. In this case, Eq. (10) has a exact solution

$$\phi = \mathrm{nc}(\tau, m). \tag{11}$$

Substituting Eq. (11) into Eq. (5) and integrating it, we obtain a periodic solution of parametric type of Eq. (1) as follows:

$$\begin{cases} \phi = \mathrm{nc}(\tau, m), \\ \xi = -c\tau + \frac{\gamma}{\sqrt{1-m^2}}\ln\left|\frac{\sqrt{1-m^2}\,\mathrm{sn}(\tau,m) + \mathrm{dn}(\tau,m)}{\mathrm{cn}(\tau,m)}\right|. \end{cases} \tag{12}$$

Similarly, under other conditions, we also obtain many periodic solutions of parametric type of Eq. (1) as follows:

2.1.2. If $0 < \gamma < 1$ and $m = \sqrt{\gamma}$, $\omega = -\gamma - 1$, $g = \frac{1}{2\gamma}$ (or $m_1 = \sqrt{\gamma}$, $\omega = \gamma + 1$, $g = \frac{1}{2\gamma}$ or $m_1 = \sqrt{\gamma}$, $\omega = \pm\sqrt{\frac{1}{\gamma}}(\gamma + 1)$, $g = \frac{1}{2}$), then $P = m^2, Q = -(1 + m^2), R = 1$. In this case, Eq. (1) has a periodic solution,

$$\phi = \mathrm{sn}(\tau, m), \quad \xi = -c\tau + \frac{\gamma}{m}\ln|\mathrm{dn}(\tau, m) - m\,\mathrm{cn}(\tau, m)|. \tag{13}$$

or

$$\phi = \mathrm{cd}(\tau, m), \quad \xi = -c\tau + \frac{\gamma}{m} \ln\left|\frac{1 + m\mathrm{sn}(\tau, m)}{\mathrm{dn}(\tau, m)}\right|. \tag{14}$$

2.1.3. If $-1 < \gamma < 0$ and $m = \sqrt{-\gamma}$, $\omega = -\gamma - 1$, $g = \frac{\gamma+1}{2\gamma}$ (or $m_2 = \sqrt{-\gamma}$, $\omega = \gamma + 1$, $g = \frac{\gamma+1}{2\gamma}$ or $m_2 = \sqrt{-\gamma}$, $\omega = \pm\sqrt{\frac{\gamma+1}{\gamma}}(\gamma + 1)$, $g = \frac{1}{2}$), then $P = -m^2, Q = 2m^2 - 1, R = 1 - m^2$. In this case, Eq. (1) has a periodic solution,

$$\phi = \mathrm{cn}(\tau, m), \quad \xi = -c\tau + \frac{\gamma}{m} \arccos(\mathrm{dn}(\tau, m)). \tag{15}$$

2.1.4. If $-1 < \gamma < 0$ and $m = \sqrt{\gamma + 1}$, $\omega = -\gamma - 1$, $g = -\frac{1}{2\gamma}$ (or $m_3 = \sqrt{\gamma + 1}$, $\omega = \gamma + 1$, $g = -\frac{1}{2\gamma}$), then $P = m^2 - 1, Q = 2 - m^2, R = -1$. In this case, Eq. (1) has a periodic solution,

$$\begin{cases} \phi = \mathrm{nd}(\tau, m), \\ \xi = -c\tau + \frac{\gamma}{\sqrt{1-m^2}} \arctan \frac{\sqrt{1-m^2}\mathrm{sn}(\tau,m) - \mathrm{cn}(\tau,m)}{\sqrt{1-m^2}\mathrm{sn}(\tau,m) + \mathrm{cn}(\tau,m)}. \end{cases} \tag{16}$$

Among the above periodic solutions, solutions (13) is a peculiar solutions, because the profiles of this solution have four kinds of wave-form including two kinds of periodic loop waves, periodic cusp wave and smooth periodic wave when the parameters vary. As an example, when ω vary, the profiles of (13) are shown in Fig. 1a–d while $g = 1, \gamma = 0.9, \tau \in [-33.2, 28.2]$:

Second, we discuss soliton-like solutions of Eq. (1), see the next.

2.2. When $h = 0$, $g = 0$, Eq. (9) can be reduced to

$$\left(\frac{d\phi}{d\tau}\right)^2 = A\phi^2 + B\phi^3 + \gamma\phi^4, \tag{17}$$

where $A = c(c - \omega)$, $B = ((\omega - c)\gamma - c)$. Denote $\Delta = B^2 - 4\gamma A = (\omega\gamma - c\gamma + c)^2$. As in Ref. [20], using the results of Refs. [18,19], we obtain kinds of exact solutions of parametric type of Eq. (1).

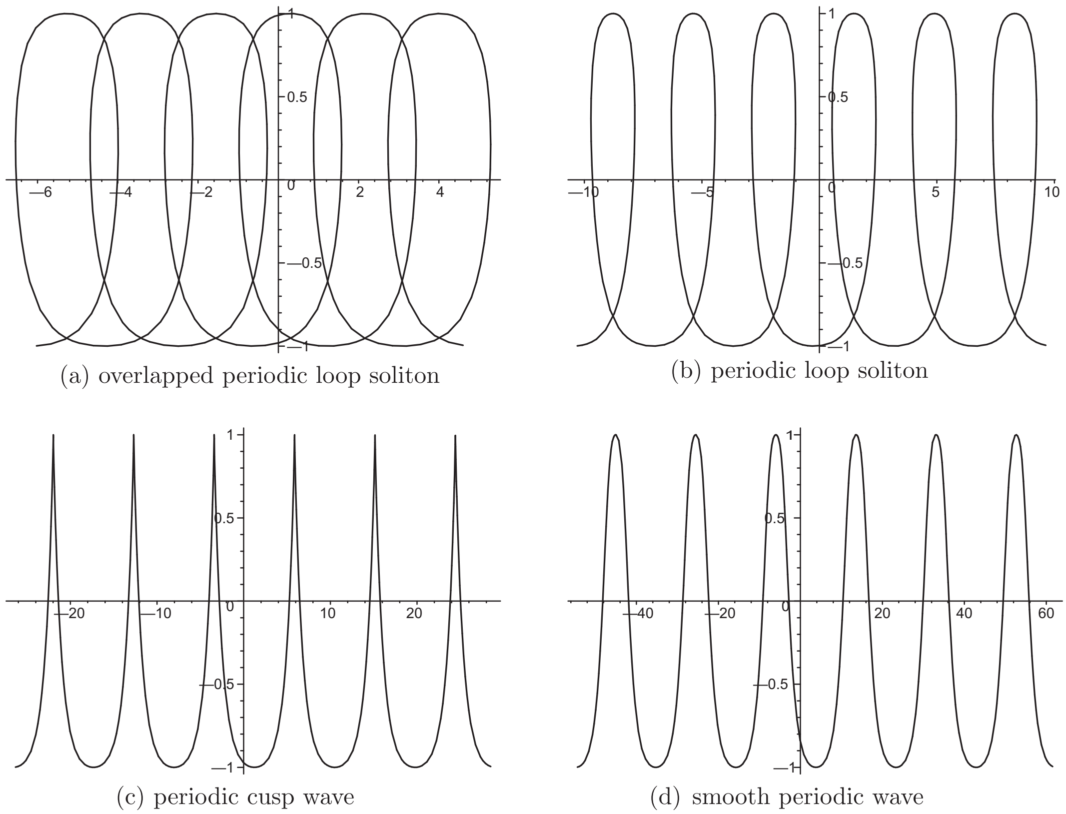


Fig. 1. (a) $\omega = 0.38$; (b) $\omega = 0.7$; (c) $\omega = 1.9$; and (d) $\omega = 4$.

2.2.1. When $A>0$, $\gamma>0$ or $\gamma<0$, we obtain six kinds of soliton-like solutions as follows:

$$\begin{cases} \phi = \dfrac{-AB\,\mathrm{sech}^2\left(\frac{\sqrt{A}}{2}\tau\right)}{B^2-\gamma A\left(1+\epsilon\tanh\left(\frac{\sqrt{A}}{2}\tau\right)\right)^2}, \quad (A>0,\gamma>0) \\ \xi = -c\tau - 2\epsilon\sqrt{\gamma}\tanh^{-1}\left[\frac{\sqrt{\gamma A}}{B}\left(1+\epsilon\tanh\left(\frac{\sqrt{A}}{2}\tau\right)\right)\right], \end{cases} \tag{18}$$

$$\begin{cases} \phi = \dfrac{AB\,\mathrm{csch}^2\left(\frac{\sqrt{A}}{2}\tau\right)}{B^2-\gamma A\left(1+\epsilon\coth\left(\frac{\sqrt{A}}{2}\tau\right)\right)^2}, \quad (A>0,\gamma>0) \\ \xi = -c\tau - 2\epsilon\sqrt{\gamma}\tanh^{-1}\left[\frac{\sqrt{\gamma A}}{B}\left(1+\epsilon\coth\left(\frac{\sqrt{A}}{2}\tau\right)\right)\right], \end{cases} \tag{19}$$

$$\begin{cases} \phi = \dfrac{2A\,\mathrm{sech}\left(\sqrt{A}\tau\right)}{\epsilon\sqrt{\Delta}-B\,\mathrm{sech}\left(\sqrt{A}\tau\right)}, \quad (A>0,\gamma<0) \\ \xi = -c\tau - 2\epsilon\sqrt{-\gamma}\arctan\left(\frac{2\sqrt{-\gamma A}}{\sqrt{\Delta}-B}\tanh\left(\frac{1}{2}\sqrt{A}\tau\right)\right), \end{cases} \tag{20}$$

$$\begin{cases} \phi = \dfrac{-A\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{A}\tau\right)}{B+2\epsilon\sqrt{\gamma A}\tanh\left(\frac{1}{2}\sqrt{A}\tau\right)}, \quad (A>0,\gamma>0) \\ \xi = -c\tau - \sqrt{\gamma}\ln\left|B+2\epsilon\sqrt{\gamma A}\tanh\left(\frac{1}{2}\sqrt{A}\tau\right)\right|, \end{cases} \tag{21}$$

$$\begin{cases} \phi = \dfrac{A\,\mathrm{csch}^2\left(\frac{1}{2}\sqrt{A}\tau\right)}{B+2\epsilon\sqrt{\gamma A}\coth\left(\frac{1}{2}\sqrt{A}\tau\right)}, \quad (A>0,\gamma>0) \\ \xi = -c\tau - \epsilon\sqrt{\gamma}\ln\left|B+2\epsilon\sqrt{\gamma A}\coth\left(\frac{1}{2}\sqrt{A}\tau\right)\right|, \end{cases} \tag{22}$$

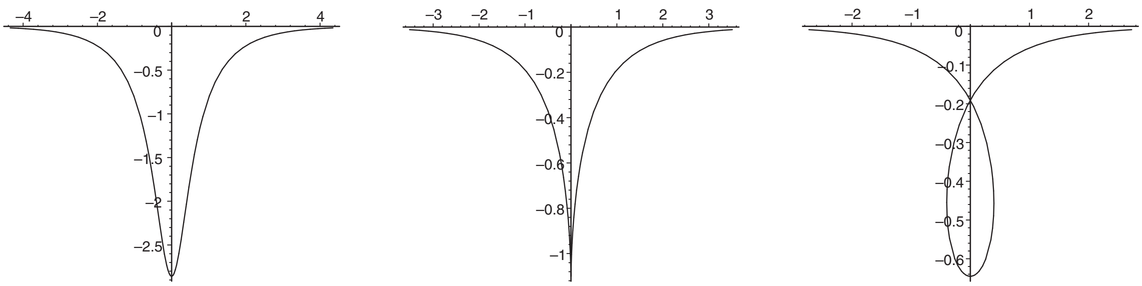
$$\begin{cases} \phi = \dfrac{4A\exp\left(\epsilon\sqrt{A}\tau\right)}{\left[\exp(\epsilon\sqrt{A}\tau)-B\right]^2-4\gamma A}, \quad (A>0,\gamma>0) \\ \xi = -c\tau - 2\sqrt{\gamma}\tanh^{-1}\frac{\exp\left(\epsilon\sqrt{A}\tau\right)-B}{2\sqrt{\gamma A}}, \end{cases} \tag{23}$$

We find that the solution (20) has three kinds of wave-form including smooth solitary wave, non-smooth solitary cusp wave and loop soliton when the parameter γ vary, their profiles are shown in Fig. 2a–c when $\epsilon=-1, c=1, \omega=-1, \tau\in[-5,5]$. In solution (20), the value $\gamma=\frac{c}{\omega-c}$ is one point of bifurcation. When $\gamma>\frac{c}{\omega-c}$, the wave-form of (20) is a smooth solitary wave. When $\gamma=\frac{c}{\omega-c}$, the wave-form of (20) is a non-smooth solitary cusp wave. When $\gamma<\frac{c}{\omega-c}$, the wave-form of (20) is a smooth loop soliton. The similar phenomenon can be shown in Ref. [20], this is a very interesting mathematical and physical phenomenon that one solution admits multi-waves (see Figs. 1 and 2).

2.2.2. When $A>0, \Delta=0$, we obtain two kinds of parametric solutions as follows:

$$\begin{cases} \phi = -\frac{A}{B}\left(1+\epsilon\tanh\left(\frac{1}{2}\sqrt{A}\tau\right)\right), \quad (A>0,\Delta=0) \\ \xi = -(c+\frac{A}{B})\tau - \frac{2\epsilon\gamma\sqrt{A}}{B}\ln\left|\cosh\left(\frac{1}{2}\sqrt{A}\tau\right)\right|, \end{cases} \tag{24}$$

$$\begin{cases} \phi = -\frac{A}{B}\left(1+\epsilon\coth\left(\frac{1}{2}\sqrt{A}\tau\right)\right), \quad (A>0,\ \Delta=0) \\ \xi = -(c+\frac{A}{B})\tau - \frac{2\epsilon\gamma\sqrt{A}}{B}\ln\left|\sinh\left(\frac{1}{2}\sqrt{A}\tau\right)\right|. \end{cases} \tag{25}$$



(a) smooth solitary wave (b) non-smooth solitary cusp wave (c) loop soliton

Fig. 2. (a) $\gamma=-0.35$; (b) $\gamma=-0.90909091$; and (c) $\gamma=-1.5$.

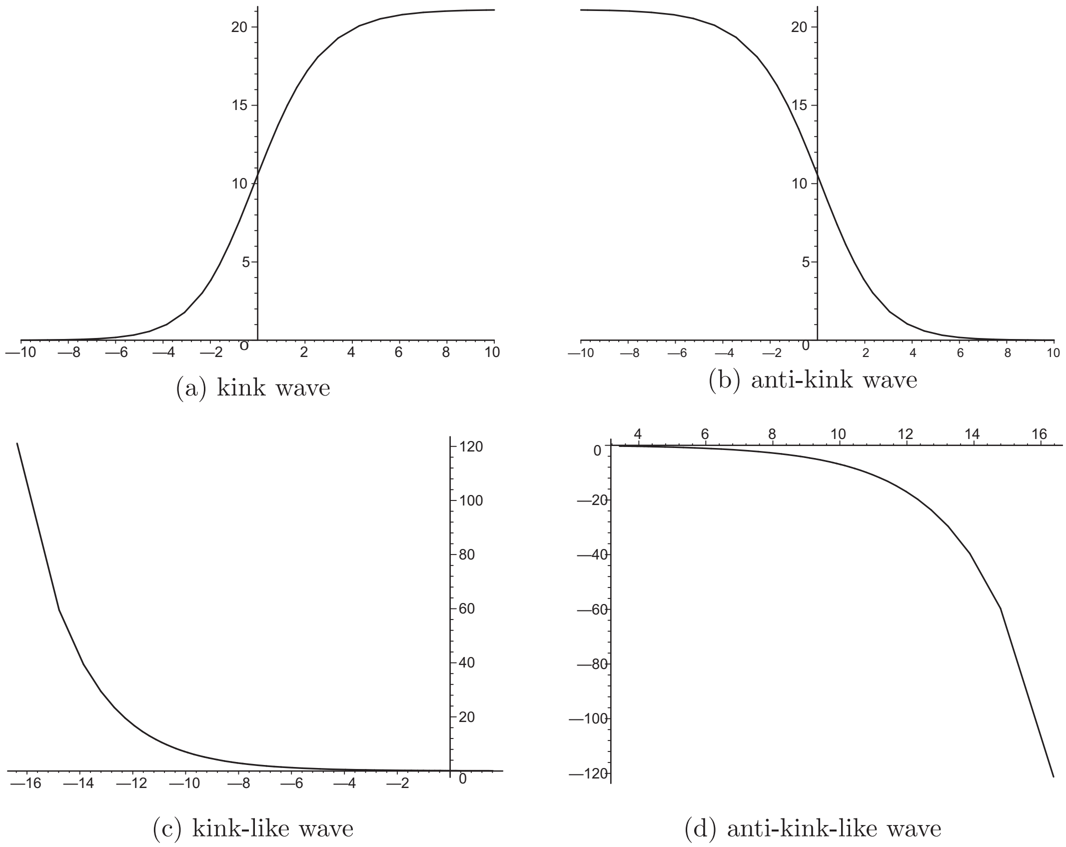


Fig. 3. (a) $c = 1.9,\ \gamma = 0.09, \omega = -19.21, \epsilon = 1, -3 \leqslant \tau \leqslant 1.5$; (b) $c = 1.9,\ \gamma = 0.09\ \ \omega = -19.21, \epsilon = -1,\ -3 \leqslant \tau \leqslant 1.5$; (c) $c = -0.8, \gamma = 5, \omega = -0.64, \epsilon = -1, -2 \leqslant \tau \leqslant 3.5$; and (d) $c = -0.8, \gamma = 5, \omega = -0.64, \epsilon = 1, -2 \leqslant \tau \leqslant 3.5$.

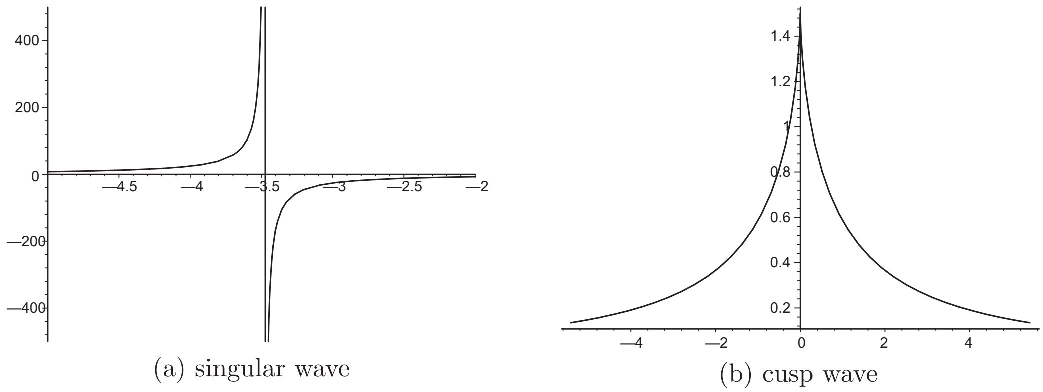


Fig. 4. (a) $c = 1, \omega = 1, \gamma = -1.25, \epsilon = 1, \tau \in [-0.6, 0.6]$, view $= [-5.. -2, -500..500]$; and (b) $c = -3, \omega = -3, \gamma = -2, \tau \in [-3, 3]$.

The solution (24) admits a kink and an anti-kink waves which are shown in Fig. 3a and b for some given parameters. And solution (25) admits a kink-like and an anti-kink-like waves which are shown in Fig. 3c and d for some given parameters.
2.2.3. When $A = 0$ or $B = 0$, we obtain four kinds of soliton-like solutions as follows:

$$\begin{cases} \phi = \sqrt{-\frac{A}{\gamma}}\operatorname{sech}\left(\epsilon\sqrt{A}\tau\right), \quad (A > 0, B = 0, \gamma < 0) \\ \xi = -c\tau - 2\epsilon\sqrt{-\gamma}\arctan\left(\exp\left(\epsilon\sqrt{A}\ \tau\right)\right), \end{cases} \tag{26}$$

$$\begin{cases} \phi = \frac{4B}{B^2\tau^2 - 4\gamma}, \quad (A = 0, \gamma > 0) \\ \xi = -c\tau - 2\sqrt{\gamma}\tanh^{-1}\left(\frac{B}{2\sqrt{\gamma}}\ \tau\right)\Big), \end{cases} \tag{27}$$

$$\begin{cases} \phi = \frac{4A\exp\left(\epsilon\sqrt{A}\,\tau\right)}{\exp\left(2\epsilon\sqrt{A}\,\tau\right)-4\gamma A}, \quad (A>0, B=0, \gamma<0) \\ \xi = -c\tau - 2\epsilon\sqrt{-\gamma}\,\arctan\frac{\exp\left(\epsilon\sqrt{A}\,\tau\right)}{2\sqrt{-\gamma A}}, \end{cases} \tag{28}$$

$$\begin{cases} \phi = \frac{4B}{B^2\tau^2-4\gamma}, \quad (A=0, \gamma<0) \\ \xi = -c\tau - 2\sqrt{-\gamma}\,\arctan\left(\frac{B}{2\sqrt{-\gamma}}\tau\right)\Big), \end{cases} \tag{29}$$

The solution (28) is a singular travelling wave for the given parameters, whose profile is shown in Fig. 4a. The (29) is a cusp wave for the given parameters, whose profile is shown in Fig. 4b.

3. Conclusion

In this work, using the integral bifurcation method, we study a nonlinearly dispersive wave equation of Camassa–Holm equation type. Loop soliton solution and periodic loop soliton solution, solitary wave solution and solitary cusp wave solution, smooth periodic wave solution and non-smooth periodic wave solution of this equation are obtained. The solutions have been verified to be the solutions of Eq. (1). We find that some of these solutions which are obtained in this paper have an interesting phenomenon that one solution admits multi-waves (see Figs. 1 and 2). This is a very interesting mathematical and physical phenomenon.

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