Hermitian dynamics in a class of pseudo-Hermitian networks

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(Received 24 July 2011; published 19 October 2011)

We investigate a pseudo-Hermitian lattice system, which consists of a set of isomorphic pseudo-Hermitian clusters coupled together in a Hermitian manner. We show that such non-Hermitian systems can act as Hermitian systems. This is made possible by considering the dynamics of a state involving an identical eigenmode of each isomorphic cluster. It still holds when multiple eigenmodes are involved and additional restrictions on the state are imposed. This Hermitian dynamics is demonstrated for the case of an exactly solvable $\mathcal{PT}$-symmetric ladder system.

DOI: 10.1103/PhysRevA.84.042116 PACS number(s): 11.30.Er, 03.65.−w, 03.75.−b

I. INTRODUCTION

Hermitian quantum mechanics is a well-developed framework because a Hermitian Hamiltonian leads to real spectrum and unitary time evolution, which preserves the probability normalization. However, a decade ago it was observed that many non-Hermitian Hamiltonians possess real spectra [1,2] and a pseudo-Hermitian Hamiltonian connects with its equivalent Hermitian Hamiltonian via a similarity transformation [3,4]. Thus, quantum theory based on non-Hermitian Hamiltonians was established [4–13]. This framework also indicates the preservation of probability normalization if a positive-definite inner product is employed to replace the Dirac inner product. Nevertheless, the interpretation and experimental measurement of such an inner product are not clear, although the Dirac probability (Dirac inner product) can be measured in a universal manner. The Dirac probability is of central importance to most practical physical problems.

Parity and time-reversal symmetric ($\mathcal{PT}$-symmetric) systems have attracted much attention because of recent experimental investigations in $\mathcal{PT}$-symmetric optical systems, observation of passive $\mathcal{PT}$-symmetry breaking in passive optical double-well structures [14], and observation of spontaneous symmetry breaking together with power oscillations in optical coupled systems [15]. Optics has advantages in detecting evolution of wave functions and seems to be the most readily applicable [16,17]. In the past two decades, general issues of quantum effects in quantum systems have been successfully investigated in the framework of quantum optics based on the fact that paraxial propagation of light in optical guided structures is governed by a Schrödinger-like equation [16]. The intensity observed in optical experiments corresponds to the Dirac probability of an electrical field envelope. The Dirac probability might not be preserved for generic systems even when the non-Hermitian Hamiltonian is time independent. Nevertheless, the violation of the conservation of Dirac probability in non-Hermitian system does not contradict the Copenhagen interpretation. The implications of a pseudo-Hermitian system are still under consideration; peculiar features have been observed such as double refraction and power oscillations [18,19]. Realization of a $\mathcal{PT}$-symmetric structure in the realm of optics has been suggested [20], while nonreciprocal Bloch oscillation with no classical correspondence was also shown in a $\mathcal{PT}$-symmetric complex crystal [21].

We propose a class of non-Hermitian lattice systems in this work: The system is composed of a set of isomorphic pseudo-Hermitian clusters, which connect with each other in a Hermitian way. We show that in such non-Hermitian systems, Hermitian-like dynamics can be observed, such as Dirac-probability-preserving time evolution. This is made possible by considering the dynamics of a state involving the superposition of an identical eigenmode of each isomorphic cluster in the general case. In the case of additional orthogonal modes, it still holds when multiple eigenmodes are involved. This Hermitian dynamics, as well as the quasi-Hermitian behavior, are specifically demonstrated for the case of an exactly solvable pseudo-Hermitian system.

This paper is organized as follows. In Sec. II, we present our model and its basic properties. Section III consists of an exactly solvable example to illustrate our main idea. Section IV is the summary and discussion.

II. HAMILTONIAN AND BASIC PROPERTIES

A general tight-binding network is constructed topologically by the sites and the various connections between them. As a simplified model, it captures the essential features of many discrete systems. It is a nice testing ground for the study of the non-Hermitian quantum mechanics because of its analytical and numerical tractability. Much effort has been devoted in recent years to the pseudo-Hermitian lattice system [22–34]. The Hamiltonian of the concerned tight-binding network reads as follows:

$$H = \sum_{\alpha=1}^{N} H_{\alpha} + \sum_{\alpha<\beta} H_{\alpha\beta}, \tag{1}$$

$$H_{\alpha} = \lambda_{\alpha} \sum_{i,j=1}^{N_d} J_{ij} a_{\alpha,i}^{\dagger} a_{\alpha,j}, \tag{2}$$

$$H_{\alpha\beta} = \kappa_{\alpha\beta} \sum_{l=1}^{N_d} a_{\alpha,l}^{\dagger} a_{\beta,l} + \text{H.c.}, \tag{3}$$

which consists of N isomorphic clusters H_{α}, with each cluster having a dimension N_d. The label α denotes the αth subgraph of N clusters, and $a_{\alpha,i}^{\dagger}$ ($a_{\alpha,i}$) is the boson or fermion creation

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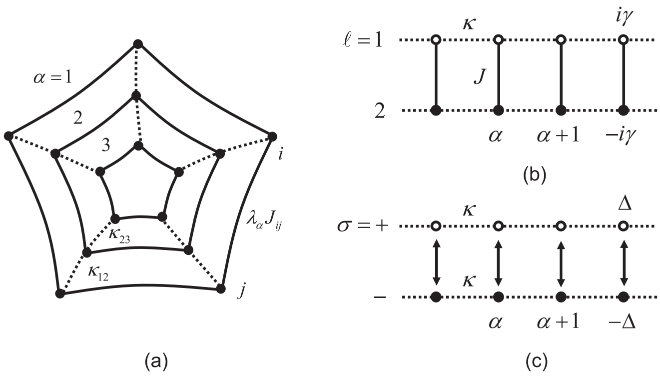

FIG. 1. Schematic illustration of the concerned networks. (a) A lattice consists of three five-site isomorphic clusters, where the different sizes indicate the factor λ_α. The dotted lines denote the similarity-mapping Hermitian structure couplings across the clusters. (b) A concrete example is a two-legged ladder. Each rung is a non-Hermitian cluster. (c) Equivalent two-band model Eq. (24). Here the double-headed arrow denotes the quasicanonical commutation relations between the eigenmodes $\sigma = \pm$ for the same cluster.

(annihilation) operator at the ith site in the αth cluster. The cluster H_α is defined by the distribution of the hopping integrals $\{\lambda_\alpha J_{ij}\}$ where λ_α is real. The set of clusters are isomorphic because they have the same eigenfunctions and spectral structures. Note that terms $\sum_{\alpha<\beta} H_{\alpha\beta}$ are self-adjoint since $H_{\alpha\beta} = H_{\alpha\beta}^\dagger$, which describes the Hermitian connection between clusters. Such couplings are *similarity mapping*, which is crucial for the conclusion of this paper. The total Hamiltonian H is not Hermitian when the matrix J_{ij} is not Hermitian. Figure 1(a) shows a schematic example.

In this paper, we consider the case of H_α being pseudo-Hermitian; that is, H_α is non-Hermitian but has an entirely real spectrum. H is also pseudo-Hermitian in the case of real λ_α, possessing the common exceptional point as H_α. In general, a pseudo-Hermitian Hamiltonian does not guarantee the preservation of Dirac probability. The Dirac norm of an evolved wave packet ceases to preserve Dirac probability as long as it touches the region of on-site imaginary potentials [35]. In the following, we show that because the pseudo-Hermitian clusters are combined together in a Hermitian way, quantum states exist that preserve Dirac probability, even if their profiles cover the imaginary potentials.

We start with the eigen problem of the Hamiltonian H_α. In single-particle invariant subspace, following the well-established pseudo-Hermitian quantum mechanics [11–13], we always have

$$H_\alpha \bar{a}_{\alpha,\sigma}|\mathrm{vac}\rangle = \lambda_\alpha \epsilon_\sigma \bar{a}_{\alpha,\sigma}|\mathrm{vac}\rangle \tag{4}$$

and

$$H_\alpha^\dagger a_{\alpha,\sigma}^\dagger|\mathrm{vac}\rangle = \lambda_\alpha \epsilon_\sigma a_{\alpha,\sigma}^\dagger|\mathrm{vac}\rangle, \tag{5}$$

where $\alpha \in [1,N]$ and $\sigma \in [1,N_d]$, and the operators $\bar{a}_{\alpha,\sigma}$ and $a_{\alpha,\sigma}$ have the form

$$\bar{a}_{\alpha,\sigma} = \sum_l f_{l\sigma} a_{\alpha,l}^\dagger,\ a_{\alpha,\sigma} = \sum_l g_{l\sigma}^* a_{\alpha,l}, \tag{6}$$

where

$$\sum_\sigma g_{l\sigma}^* f_{l'\sigma} = \delta_{ll'}, \quad \sum_l g_{l\sigma}^* f_{l\sigma'} = \delta_{\sigma\sigma'}. \tag{7}$$

Note that $\{f_{l\sigma}\}$, $\{g_{l\sigma}\}$, and $\{\varepsilon_\sigma\}$ are independent of α. Then the operators $\bar{a}_{\alpha,\sigma}$ and $a_{\alpha,\sigma}$ are canonical conjugate pairs, satisfying

$$[a_{\alpha,\sigma},\bar{a}_{\alpha',\sigma'}]_\pm = \delta_{\alpha\alpha'}\delta_{\sigma\sigma'}, \tag{8}$$

$$[a_{\alpha,\sigma},a_{\alpha',\sigma'}]_\pm = [\bar{a}_{\alpha,\sigma},\bar{a}_{\alpha',\sigma'}]_\pm = 0, \tag{9}$$

where $[\cdot,\cdot]_\pm$ denotes the the commutator and anticommutator. Accordingly, the original Hamiltonian can be rewritten as the form

$$H = \sum_{\alpha,\sigma} \lambda_\alpha \epsilon_\sigma \bar{a}_{\alpha,\sigma} a_{\alpha,\sigma} + \sum_{\alpha<\beta,\sigma} (\kappa_{\alpha\beta}\bar{a}_{\alpha,\sigma}a_{\beta,\sigma} + \kappa_{\alpha\beta}^*\bar{a}_{\beta,\sigma}a_{\alpha,\sigma}), \tag{10}$$

which has the following subtle features: (i) The matrix representation of H with respect to the biorthogonal basis $\{\langle \mathrm{vac}|a_{\alpha,\sigma},\bar{a}_{\alpha,\sigma}|\mathrm{vac}\,\rangle\}$ is Hermitian; that is, $\langle \mathrm{vac}|a_{\alpha,\sigma} H \bar{a}_{\alpha',\sigma'}|\mathrm{vac}\rangle = (\langle \mathrm{vac}|a_{\alpha',\sigma'} H \bar{a}_{\alpha,\sigma}|\mathrm{vac}\rangle)^*$. (ii) Although it is a non-Hermitian operator (i.e., $H \neq H^\dagger$), straightforward algebra shows that

$$[a_{\alpha,\sigma},a_{\alpha',\sigma'}^\dagger]_\pm \propto \delta_{\alpha\alpha'}, \quad [a_{\alpha,\sigma},a_{\alpha',\sigma'}]_\pm = 0, \tag{11}$$

which indicates that although nonorthogonality of the eigenstates is an inherent feature of non-Hermitian systems, $a_{\alpha,\sigma}$ and $a_{\alpha',\sigma'}^\dagger$ obey quasicanonical commutation relations as a result of the Hermitian connection structure between clusters. This results in a new type of particle statistics, which is rarely observed in Hermitian systems and thus becomes highly relevant in the presence of non-Hermitian terms.

Consider an arbitrary state in the form

$$|\Phi_\sigma(0)\rangle = \sum_\alpha c_\alpha \bar{a}_{\alpha,\sigma}|\mathrm{vac}\rangle, \tag{12}$$

as the initial state, where $\sum_\alpha |c_\alpha|^2 = 1$ and in which only the eigenmode σ of each cluster is involved. At instant t, we have

$$|\Phi_\sigma(t)\rangle = \sum_\alpha c_\alpha e^{-iHt}\bar{a}_{\alpha,\sigma}|\mathrm{vac}\rangle. \tag{13}$$

In the framework of metric operator theory, H acts as a Hermitian system, obeying unitary time evolution in the positive-definite inner product [4]. However, to date the physical meaning of the positive-definite inner product is unclear, while the Dirac probability can be measured in a universal manner. For example, Dirac probability of wave electric field corresponds to the light intensity in optics and is simple to detect in experiments [16]; therefore, the Dirac norm is of central importance. The aim of this paper is to show that, in contrast to the nonclassical dynamical behavior [18,21], the unitary Dirac probability dynamics can be observed in the pseudo-Hermitian system. By inserting $\sum_{\beta,\sigma'} \bar{a}_{\beta,\sigma'}|\mathrm{vac}\rangle\langle \mathrm{vac}|a_{\beta,\sigma'} = 1$ into Eq. (13), we have

$$\begin{aligned} |\Phi_\sigma(t)\rangle &= \sum_{\alpha,\beta} c_\alpha \bar{a}_{\beta,\sigma}|\mathrm{vac}\rangle\langle \mathrm{vac}|a_{\beta,\sigma}e^{-iHt}\bar{a}_{\alpha,\sigma}|\mathrm{vac}\rangle \\ &= \sum_{\alpha,\beta} c_\alpha U_{\beta\alpha}\bar{a}_{\beta,\sigma}|\mathrm{vac}\rangle, \end{aligned} \tag{14}$$

where

$$U_{\beta\alpha} = \langle \mathrm{vac}| a_{\beta,\sigma} e^{-iHt} \bar{a}_{\alpha,\sigma} |\mathrm{vac}\rangle \tag{15}$$

is the propagator in the framework of biorthogonal basis and satisfies

$$\sum_{\gamma} U_{\gamma\alpha} U^*_{\gamma\beta} = \delta_{\alpha\beta}, \tag{16}$$

because of feature (i) of H. Accordingly, the Dirac norm has the form

$$\begin{aligned} ||\Phi_\sigma(t)\rangle|^2 &= [|\Phi_\sigma(t)\rangle]^\dagger |\Phi_\sigma(t)\rangle \\ &= \left(\sum_{\alpha',\beta'} c^*_{\alpha'} U^*_{\beta'\alpha'} \langle \mathrm{vac}| \bar{a}^\dagger_{\beta',\sigma}\right)\left(\sum_{\alpha,\beta} c_\alpha U_{\beta\alpha} \bar{a}_{\beta,\sigma} |\mathrm{vac}\rangle\right) \\ &= \sum_\alpha |c_\alpha|^2 \Delta_\sigma = \Delta_\sigma, \end{aligned} \tag{17}$$

where the relation Eq. (16) is applied and the α-independent factor Δ_σ can be obtained from

$$\langle \mathrm{vac}| \bar{a}^\dagger_{\alpha,\sigma} \bar{a}_{\beta,\sigma} |\mathrm{vac}\rangle = \Delta_\sigma \delta_{\alpha\beta}. \tag{18}$$

It follows that although $|\Phi_\sigma(t)\rangle$ is not the eigenstate of the entire network system, the time evolution is Dirac norm-conserving. This is a direct consequence of the quasicanonical commutation relations. The result presented here for the evolution of an arbitrary state involving an identical isomorphic cluster eigenmode provides a new way to connect the pseudo-Hermitian and Hermitian systems.

It is worth mentioning that this probability-preserving evolution can also occur for a state involving multiple eigenmodes. This is because states whose parts belong to different eigenmodes are orthogonal in terms of Dirac inner product and hence preserve the Dirac probability. For instance, for a state involving two eigenmodes σ_1 and σ_2, its parts on σ_1 and σ_2 are spatially separated local states with respect to the coordinate space α. Then the two parts of the state are orthogonal in terms of Dirac inner product, and the evolution of such a state is probability preserving because of the quasicanonical commutation relations. We demonstrate this point explicitly via the following illustrative example.

III. PSEUDO-HERMITIAN LADDER

Now we investigate a concrete example to demonstrate the application of the previous result. We consider a system of a two-legged ladder [Fig. 1(b)], consisting of N dimers as pseudo-Hermitian clusters. The Hamiltonian reads

$$H_{\mathrm{Ladd}} = \sum_{\alpha=1}^{N} H_\alpha + \sum_{\alpha=1}^{N} H_{\alpha,\alpha+1}, \tag{19}$$

$$H_\alpha = -J(a^\dagger_{\alpha,1} a_{\alpha,2} + \mathrm{H.c.}) + i\gamma(n_{\alpha,1} - n_{\alpha,2}), \tag{20}$$

$$H_{\alpha,\alpha+1} = -\kappa \sum_{\ell=1}^{2} (a^\dagger_{\alpha,\ell} a_{\alpha+1,\ell} + \mathrm{H.c.}), \tag{21}$$

where $n_{\alpha,\ell} = a^\dagger_{\alpha,\ell} a_{\alpha,\ell}$ is the particle number operator and the operators obey the periodic boundary condition $a^\dagger_{N+1,\ell} = a^\dagger_{1,\ell}$, with $\ell = 1, 2$. κ (J) is the hopping integral along legs (rungs) and γ denotes the norm of the imaginary onsite potential. Note that the ladder is a $\mathcal{PT}$-symmetric Hamiltonian, where $\mathcal{P}$ is the parity and $\mathcal{T}$ denotes time reversal. The simple structure of this model makes it an ideal testing ground for a more profound understanding of the Hermitian dynamics in a pseudo-Hermitian system. When we take the transformations

$$\bar{a}_{\alpha,\sigma} = \frac{1}{\sqrt{2\cos\theta}} (e^{i\sigma\theta/2} a^\dagger_{\alpha,1} - \sigma e^{-i\sigma\theta/2} a^\dagger_{\alpha,2}), \tag{22}$$

$$a_{\alpha,\sigma} = \frac{1}{\sqrt{2\cos\theta}} (e^{i\sigma\theta/2} a_{\alpha,1} - \sigma e^{-i\sigma\theta/2} a_{\alpha,2}), \tag{23}$$

where ($\alpha \in [1,N]$, $\sigma = \pm$), which are obtained from the solution of the dimer (a general solution of N_d-dimension cluster is shown in Ref. [36]), the ladder Hamiltonian can be written as

$$H_{\mathrm{Ladd}} = \sum_{\alpha=1,\sigma=\pm}^{N} (-\kappa \bar{a}_{\alpha,\sigma} a_{\alpha+1,\sigma} - \kappa \bar{a}_{\alpha+1,\sigma} a_{\alpha,\sigma} + \sigma\Delta \bar{a}_{\alpha,\sigma} a_{\alpha,\sigma}). \tag{24}$$

which is illustrated in Fig. 1(c). Here $\Delta = \sqrt{J^2 - \gamma^2}$ and $\sin\theta = \gamma/J$, $\theta \in [0,\pi/2]$. The biorthogonal structure of the solution for a dimer admits the following canonical commutation relations: Eq. (8) and

$$[\bar{a}^\dagger_{\alpha,\sigma}, \bar{a}_{\alpha',\sigma}]_\pm = [a_{\alpha,\sigma}, a^\dagger_{\alpha,\sigma}]_\pm = \sec\theta\, \delta_{\alpha\alpha'}, \tag{25}$$

$$[\bar{a}^\dagger_{\alpha,-\sigma}, \bar{a}_{\alpha',\sigma}]_\pm = [a_{\alpha,\sigma}, a^\dagger_{\alpha,-\sigma}]_\pm = i\sigma \tan\theta\, \delta_{\alpha\alpha'}. \tag{26}$$

Obviously, the Hamiltonian in Eq. (24) represents a two-band model, which has an interesting feature compared to a Hermitian two-band model: Although there are no interband transitions, the two bands are not independent. It is due to the pseudo-Hermiticity of the clusters, which allows $[a_{\alpha,\sigma}, \bar{a}_{\alpha,-\sigma}]_\pm = 0$ but $[a_{\alpha,\sigma}, a^\dagger_{\alpha,-\sigma}]_\pm \neq 0$. This characteristic is further demonstrated through the following quasicanonical commutation relations Eq. (33) and the time evolution for various Gaussian wave packets. Figure 1(c) schematically illustrates such an equivalent two-band structure. Nevertheless, Hamiltonian Eq. (24) can be diagonalized as a Hermitian one; that is, we have

$$H_{\mathrm{Ladd}} = \sum_{k,\sigma=\pm} \varepsilon_{k,\sigma} \bar{a}_{k,\sigma} a_{k,\sigma}, \tag{27}$$

$$\varepsilon_{k,\pm} = -2\kappa\cos k \pm \Delta, \tag{28}$$

and by using the linear transformations,

$$\bar{a}_{k,\sigma} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ikj} \bar{a}_{j,\sigma}, \tag{29}$$

$$a_{k,\sigma} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-ikj} a_{j,\sigma}, \tag{30}$$

where $k = 2n\pi/N$, $n \in [1,N]$. The linearity of the transformations allows

$$[a_{k,\sigma}, \bar{a}_{k',\sigma'}]_\pm = \delta_{kk'} \delta_{\sigma\sigma'}, \tag{31}$$

$$[\bar{a}_{k,\sigma}, \bar{a}_{k',\sigma'}]_\pm = [a_{k,\sigma}, a_{k',\sigma'}]_\pm = 0. \tag{32}$$

However, when dealing with the Dirac inner product, the quasicanonical commutation relations

$$[\bar{a}_{k,\sigma}^{\dagger},\bar{a}_{k',\sigma}]_{\pm}=[a_{k,\sigma},a_{k',\sigma}^{\dagger}]_{\pm}=\sec\theta\delta_{kk'}, \tag{33a}$$

$$[\bar{a}_{k,-\sigma}^{\dagger},\bar{a}_{k',\sigma}]_{\pm}=[a_{k,\sigma},a_{k',-\sigma}^{\dagger}]_{\pm}=i\sigma\tan\theta\delta_{kk'}, \tag{33b}$$

will be taken into account. Such quasicanonical commutation relations reflect the subtle features of the system: When dealing with different k, $a_{k,\sigma}$ and $a_{k',\sigma'}^{\dagger}$ act as canonical conjugate pairs and the system displays Hermitian behavior.

We can gain some insight regarding the role of the quasicanonical statistics. Such a model displays dynamics similar to those of a Hermitian ladder. We start our investigation from the quantum dynamics of various initial wave packets. In the situation of a Hermitian ladder, any two wave packets propagate independently and the total probability is preserving.

Consider an arbitrary state involving both upper and lower bands:

$$|\Phi(0)\rangle=\sum_{k,\sigma=\pm}f_{k,\sigma}\bar{a}_{k,\sigma}|\mathrm{vac}\rangle. \tag{34}$$

With $\sum_{k,\sigma=\pm}|f_{k,\sigma}|^2=1$, we have

$$\begin{aligned}||\Phi(t)\rangle|^2&=\sum_{k,\sigma}|f_{k,\sigma}|^2\langle\mathrm{vac}|[\bar{a}_{k,\sigma}^{\dagger},\bar{a}_{k,\sigma}]_{\pm}|\mathrm{vac}\rangle\\&\quad+\sum_{k,\sigma}f_{k,-\sigma}^{*}f_{k,\sigma}e^{-i2\sigma\Delta t}\langle\mathrm{vac}|[\bar{a}_{k,-\sigma}^{\dagger},\bar{a}_{k,\sigma}]_{\pm}|\mathrm{vac}\rangle\\&=\sec\theta+i\tan\theta\sum_{k,\sigma}\sigma f_{k,-\sigma}^{*}f_{k,\sigma}e^{-i\sigma2\pi(t/T_{\mathrm{D}})},\end{aligned} \tag{35}$$

where $T_{\mathrm{D}}=\pi/\Delta$ denotes the period of the oscillation. The first term gives the contribution from single band, while the second term captures the influence of the non-Hermiticity. For vanishing θ, we recover the unitary evolution in a Hermitian system. Evidently, $||\Phi(t)\rangle|^2=\sec\theta$ for a state with $f_{k,-\sigma}^{*}f_{k,\sigma}=0$, which involves only a single mode. Note, however, that mathematically speaking the time-dependent terms can vanish even in the case of $f_{k,-\sigma}^{*}f_{k,\sigma}\neq0$, for example, additional orthogonality of the wave packet with multiple eigenmodes. To demonstrate this, we study the evolution of initial wave packets of the form

$$\begin{aligned}&|\Psi(N_A,N_B,\phi_A,\phi_B,0)\rangle\\&\quad=\frac{1}{\sqrt{\Omega}}\sum_k[e^{-(k-\phi_A)^2/(2\rho^2)}e^{-i(k-\phi_A)N_A}\bar{a}_{k,+}\\&\qquad+e^{-(k-\phi_B)^2/(2\rho^2)}e^{-i(k-\phi_B)N_B}\bar{a}_{k,-}]|\mathrm{vac}\rangle,\end{aligned} \tag{36}$$

which is the superposition of wave packets A and B, where $\Omega=2\sum_k e^{-(k-\phi_A)^2/\rho^2}=2\sum_k e^{-(k-\phi_B)^2/\rho^2}$. The time evolution of a wave packet is a powerful tool for understanding the dynamical property of Hermitian quantum systems [37]. Recently, the propagation of wave packets in discrete systems has been utilized as a flying qubit for quantum state transfer [38–42]. In the Hermitian case, an initially Gaussian state stays Gaussian as it propagates for a long time, especially for the case of $|\phi_{A,B}|=\pi/2$ [43].

For a sufficient broad wave packet ($\rho\ll1$), we have $\Omega\approx\rho N/\sqrt{\pi}$. Equation (36) can also be expressed in the coordinate space spanned by $\{a_{\alpha,1}^{\dagger}|\mathrm{vac}\rangle, a_{\alpha,2}^{\dagger}|\mathrm{vac}\rangle\}$ as

$$\begin{aligned}&|\Psi(N_A,N_B,\phi_A,\phi_B,0)\rangle\\&\approx\sqrt{\frac{\rho}{4\sqrt{\pi}\cos\theta}}\sum_{\alpha=1}^{N}\big[e^{-\rho^2(j-N_A)^2/2}e^{i\phi_A j}\\&\quad\times(e^{i\theta/2}a_{\alpha,1}^{\dagger}-e^{-i\theta/2}a_{\alpha,2}^{\dagger})+e^{-\rho^2(j-N_B)^2/2}e^{i\phi_B j}\\&\quad\times(e^{-i\theta/2}a_{\alpha,1}^{\dagger}+e^{i\theta/2}a_{\alpha,2}^{\dagger})\big]|\mathrm{vac}\rangle,\end{aligned} \tag{37}$$

which involves both eigenmodes ($\sigma=\pm$) and is actually composed of four wave packets with centers at N_Ath and N_Bth sites of legs 1 and 2 with velocities ϕ_A and ϕ_B, respectively. To investigate the dynamics of the Dirac norm, by substituting

$$f_{k,+}=\frac{1}{\sqrt{\Omega}}e^{-(k-\phi_A)^2/(2\rho^2)}e^{-i(k-\phi_A)N_A}, \tag{38a}$$

$$f_{k,-}=\frac{1}{\sqrt{\Omega}}e^{-(k-\phi_B)^2/(2\rho^2)}e^{-i(k-\phi_B)N_B} \tag{38b}$$

into Eq. (35), we have

$$\begin{aligned}&||\Psi(N_A,N_B,\phi_A,\phi_B,t)\rangle|^2\\&\quad=\sec\theta+e^{-(\phi_A-\phi_B)^2/(4\rho^2)}e^{-\rho^2(N_B-N_A)^2/4}\\&\qquad\times\sin(2\pi t/T_{\mathrm{D}}-\varphi_{AB})\tan\theta,\end{aligned} \tag{39}$$

where $\varphi_{AB}=(N_A+N_B)(\phi_A-\phi_B)/2$. We note that if the two wave packets of Eq. (36) are well separated in k or α space initially (wave packets orthogonal in k or α space), the weighted exponential factor becomes zero and the probability is always conserved in the evolution, even if they meet each other in the coordinate space α. This indicates that for states having additional orthogonal modes, Hermitian-like behavior still holds even if multiple eigenmodes are involved.

To show more detailed propagation behavior, we study the profile of $P_{\ell}(j,t)$ ($\ell=1,2$), where

$$P_{\ell}(j,t)=|\langle\mathrm{vac}|a_{j,\ell}|\Psi(N_A,N_B,\phi_A,\phi_B,t)\rangle|^2. \tag{40}$$

It is a convenient way to investigate the dynamical properties from two typical cases: (a) $\phi_A=-\phi_B=\pi/2$, $|N_A-N_B|\gg2\sqrt{\ln2}/\rho$, and (b) $\phi_A=\phi_B=\pi/2$, $N_A=N_B$. In case (a), the situation corresponds to two counterpropagating wave packets, with the evolved wave function

$$\begin{aligned}&|\Psi(N_A,N_B,\pi/2,-\pi/2,t)\rangle\\&\quad=\frac{1}{\sqrt{\Omega}}\sum_k\big[e^{-i\Delta t}e^{-(k-\pi/2)^2/(2\rho^2)}e^{-i(k-\pi/2)(N_A+2\kappa t)}\bar{a}_{k,+}\\&\qquad+e^{i\Delta t}e^{-(k+\pi/2)^2/(2\rho^2)}e^{-i(k+\pi/2)(N_B-2\kappa t)}\bar{a}_{k,-}\big]|\mathrm{vac}\rangle\\&\quad=|\Psi'(N_A+2\kappa t,N_B-2\kappa t,\pi/2,-\pi/2,0)\rangle,\end{aligned} \tag{41}$$

where the approximation of Taylor expansions for $\cos k$ around $\pm\pi/2$ are used for two wave packets and $|\Psi'\rangle$ represents the superposition of two wave packets as state $|\Psi\rangle$ but with different overall phases. It shows that the evolved state is still the independent nonspreading wave packets. Similarly, the

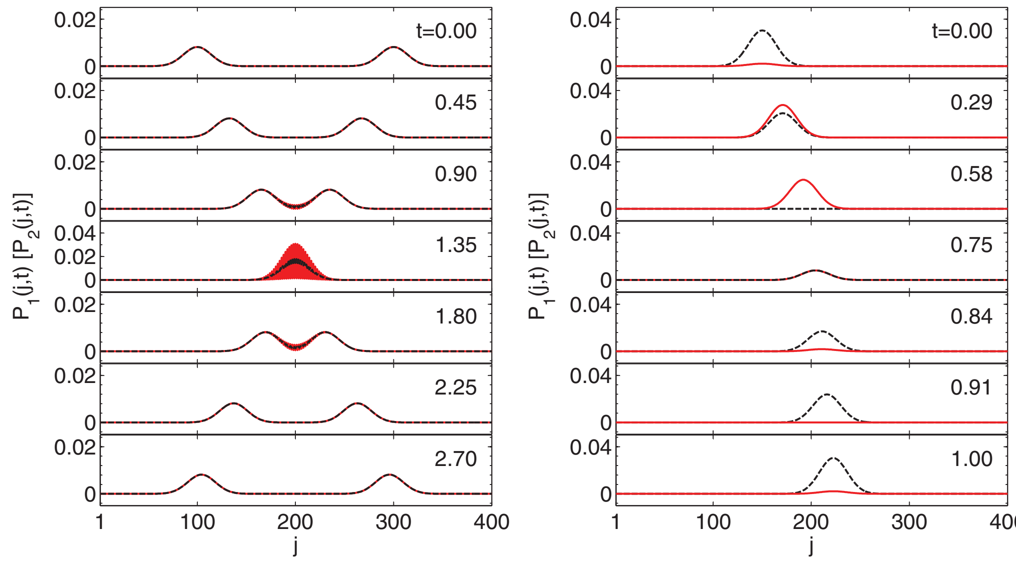

FIG. 2. (Color online) The Dirac probabilities $P_1(j,t)$ (black dashed line) and $P_2(j,t)$ (red solid line) of a particle, initially located in the state $|\Psi(N_A,N_B,\phi_A,\phi_B,0)\rangle$ for a system with $N=400$, $\gamma=0.05$, $J=0.10$, $\kappa=1.00$, and $\rho=0.05$. We obtain $\theta=\pi/6$ and the time t is in units of $T_{\mathrm{D}}\approx 36.276\ \kappa^{-1}$. We plot Eq. (40) for two cases with (a) $\phi_A=-\phi_B=\pi/2$, $N_A=100$, $N_B=300$ and (b) $\phi_A=\phi_B=\pi/2$, $N_A=N_B=150$. The shapes of all the curves are in agreement with the analysis in the text.

evolved wave function for case (b) has the form

$$|\Psi(N_A,N_A,\pi/2,\pi/2,t)\rangle = \frac{1}{\sqrt{\Omega}}\sum_k \left[e^{-(k-\pi/2)^2/(2\rho^2)}e^{-i(k-\pi/2)(N_A+2\kappa t)} \times (\bar{a}_{k,+}e^{-i\Delta t}+\bar{a}_{k,-}e^{i\Delta t})\right]|\mathrm{vac}\rangle. \tag{42}$$

It has more clear profile in the coordinate space ℓ, that is,

$$|\Psi(N_A,N_A,\pi/2,\pi/2,t)\rangle \approx \sum_{\ell=1,2} g_\ell(t)\sum_{j=1}^{N} e^{-\rho^2[j-(N_A+2\kappa t)]^2/2}e^{ij\pi/2}a_{j,\ell}^{\dagger}|\mathrm{vac}\rangle, \tag{43}$$

where

$$g_\ell(t)=\sqrt{\frac{\rho}{\sqrt{\pi}\cos\theta}}\times\begin{cases}\cos(\pi t/T_{\mathrm{D}}-\theta/2), & \ell=1\\ i\sin(\pi t/T_{\mathrm{D}}+\theta/2), & \ell=2\end{cases}. \tag{44}$$

Obviously, it represents two breathing shape-invariant wave packets propagating along two legs of the ladder with the breathing period T_{D}. Furthermore, the Dirac norm $P_\ell^s=\sum_j P_\ell(j,t)$ ($\ell=1,2$) and $P_{\mathrm{T}}^s=P_1^s+P_2^s$ can be obtained as the form

$$P_1^s=\cos^2(\pi t/T_{\mathrm{D}}-\theta/2)/\cos\theta, \tag{45}$$

$$P_2^s=\sin^2(\pi t/T_{\mathrm{D}}+\theta/2)/\cos\theta, \tag{46}$$

$$P_{\mathrm{T}}^s=\sec\theta+\tan\theta\sin(2\pi t/T_{\mathrm{D}}). \tag{47}$$

As mentioned in the introduction, the profile of the evolved wave function $P_\ell(j,t)$ can be observed in experiments. In practice, the quantum-optical analogy has been employed to visualize the dynamics in the non-Hermitian system [18–20]. In this context, the light intensity corresponds to $P_\ell(j,t)$ (for a review, see [16]) and the profile corresponds to the light intensity distribution along its propagation direction.

It follows that a manifestation of the non-Hermitian nature of H_{Ladd} is represented by the relative phase θ between the breathing oscillations of the two legs, which also leads to the time-dependent Dirac probability. The profiles of the evolved wave functions and the Dirac norms are plotted in Figs. 2 and 3. We can see that in case (a) the evolved wave packets propagate independently and the Dirac norms are preserved. It indicates that although the Hamiltonian is non-Hermitian, due to the quasicanonical commutation relations, which are a direct consequence of the Hermitian connection structure

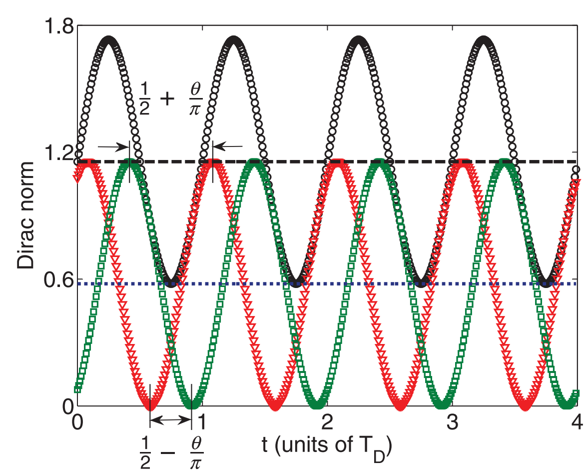

FIG. 3. (Color online) The Dirac norms $P_1^s(t)$, $P_2^s(t)$ (blue dotted line), and $P_{\mathrm{T}}^s(t)$ (black dashed line) for the case of $\phi_A=-\phi_B=\pi/2$ [as in Fig. 2(a)]. The Dirac norms $P_1^s(t)$ (red triangle), $P_2^s(t)$ (green square), and $P_{\mathrm{T}}^s(t)$ (black circle) for the case of $\phi_A=\phi_B=\pi/2$ [as in Fig. 2(b)]. All the parameters are the same as in Fig. 2. The phase difference $\theta=\pi/6$ and also the quasicanonical commutation relations ($\sec\theta\approx 1.155$) are indicated. The shapes of all the curves are in agreement with the analysis in the text.

between clusters, it acts as a Hermitian ladder for some initial state. In contrast, the dynamics of case (b) differs drastically from the Hermitian case and the Dirac norms are no longer preserved. Further, the phase difference between the breathing oscillations on the two legs can also be observed in case (b).

IV. SUMMARY AND DISCUSSION

In summary, within the context of a class of non-Hermitian lattice systems, which consist of a set of isomorphic pseudo-Hermitian clusters combined in a Hermitian manner, we show that Hermitian-like dynamics could be observed in non-Hermitian systems, including the property that the time evolution preserves Dirac probability. We investigate a concrete network, a $\mathcal{PT}$-symmetric ladder, composed of many pseudo-Hermitian dimers, and show that it acts as a Hermitian system in the following sense: Besides the reality of the spectrum and probability preservation, the propagation of certain wave packets exhibits the same behavior as that in a Hermitian ladder. Our finding indicates that the spectrum as well as the Dirac-probability-preserving dynamics can occur in a system that violates the axiom of Hermiticity. This paves the way for the development of descriptions of quantum systems.

ACKNOWLEDGMENTS

We acknowledge the support of the CNSF (Grant No. 10874091) and the National Basic Research Program (973 Program) of China under Grant No. 2012CB921900.

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