

Research Article

A New Roper-Suffridge Extension Operator on a Reinhardt Domain

Jianfei Wang and Cailing Gao

Department of Mathematics and Physics, Information Engineering, Zhejiang Normal University, Zhejiang, Jinhua 321004, China

Correspondence should be addressed to Jianfei Wang, wjfustc@zjnu.cn

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We introduce a new Roper-Suffridge extension operator on the following Reinhardt domain $\Omega_{n,p_2,\dots,p_n} = \{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^n |z_j|^{p_j} < 1\}$ given by $F(z) = (f(z_1) + f'(z_1) \sum_{j=2}^n a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n)$, where f is a normalized locally biholomorphic function on the unit disc D , p_j are positive integer, a_j are complex constants, and $j = 2, \dots, n$. Some conditions for a_j are found under which the operator preserves almost starlike mappings of order α and starlike mappings of order α , respectively. In particular, our results reduce to many well-known results when all $\alpha_j = 0$.

1. Introduction

In 1995, Roper and Suffridge [1] introduced an extension operator. This operator is defined as follows:

$$\Phi_n(f)(z) = \left(f(z_1), \sqrt{f'(z_1)} z'_0 \right)', \quad (1.1)$$

where f is a normalized locally biholomorphic function on the unit disk D in $\mathbb{C}$, $z = (z_1, z'_0)'$ belonging to the unit ball $\mathcal{B}^n$ in $\mathbb{C}^n$, $z_0 = (z_2, \dots, z_n)' \in \mathbb{C}^{n-1}$ and the branch of the square root is chosen such that $\sqrt{f'(0)} = 1$.

It is well known that the Roper-Suffridge extension operator has the following remarkable properties:

- (i) if f is a normalized convex function on D , then $\Phi_n(f)$ is a normalized convex mapping on $\mathcal{B}^n$;

- (ii) if f is a normalized starlike function on D , then $\Phi_n(f)$ is a normalized starlike mapping on $\mathcal{B}^n$;
- (iii) if f is a normalized Bloch function on D , then $\Phi_n(f)$ is a normalized Bloch mapping on $\mathcal{B}^n$.

The above result (i) was proved by Roper and Suffridge [1] and the result (ii) and (iii) was proved by Graham and Kohr [2, 3]. Until now, it is difficult to construct the concrete convex mappings, starlike mappings, and Bloch mappings on $\mathcal{B}^n$. By making use of the Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator.

In 2005, Muir [4] modified the Roper-Suffridge extension operator as follows:

$$F(z) = \left(f(z_1) + f'(z_1)P(z_0), \sqrt{f'(z_1)z'_0} \right)', \quad (1.2)$$

where $P(z_0)$ is a homogeneous polynomial of degree 2 with respect to z_0 , and f , z_1 , and z_0 are defined as above. They proved that this operator preserves starlikeness and convexity if and only if $\|P\| \leq 1/4$ and $\|P\| \leq 1/2$, respectively. The modified operator plays a key role to study the extreme points of convex mappings on $\mathcal{B}^n$ (see [5, 6]). Later, Kohr [7] and Muir [8] used the Loewner chain to study the modified Roper-Suffridge extension operator. Recently, the modified Roper-Suffridge extension operator on the unit ball is also studied by Wang and Liu [9] and Feng and Yu [10].

On the other hand, people also considered the generalized Roper-Suffridge extension operator on the general Reinhardt domains. For example, Gong and Liu [11, 12] induced the definition of ε starlike mappings and obtained that the operator

$$\Phi_{n,(1/p)}(f)(z) = \left(f(z_1), (f'(z_1))^{1/p} z'_0 \right)' \quad (1.3)$$

maps the ε starlike functions on D to the ε starlike mappings on the Reinhardt domain $\Omega_{n,p} = \{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^n |z_j|^p < 1\}$, where $p \geq 1$, f , z_1 , and z_0 are defined as above. When $\varepsilon = 0$ and $\varepsilon = 1$, $\Phi_{n,(1/p)}$ maps the starlike function and convex function on D to the starlike mapping and the convex mapping on $\Omega_{n,p}$, respectively.

Furthermore, Gong and Liu [13] proved that the operator

$$\Phi_{n,(1/p_2), \dots, 1/p_n}(f)(z) = \left(f(z_1), (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n \right)' \quad (1.4)$$

maps the ε starlike functions on D to the ε starlike mappings on the domain $\Omega_{n,p_2, \dots, p_n} = \{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^n |z_j|^{p_j} < 1\}$, where $p_j \geq 1$, $j = 2, \dots, n$, f , z_1 , and z_0 are defined as above. Liu and Liu [14] proved that this operator preserves starlikeness of order α on the domain $\Omega_{n,p_2, \dots, p_n}$. On the other hand, Feng and Liu [15] proved that this operator preserves almost starlikeness of order α on the domain $\Omega_{n,p_2, \dots, p_n}$.

In contrast to the modified Roper-Suffridge extension operator in the unit ball, it is natural to ask if we can modify the Roper-Suffridge extension operator on the Reinhardt domains. In this paper, we will introduce the following modified operator:

$$F(z) = \left(f(z_1) + f'(z_1) \sum_{j=2}^n a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n \right)' \quad (1.5)$$

on the Reinhardt domain $\Omega_{n,p_2,\dots,p_n}$. We will give some sufficient conditions for a_j under which the above Roper-Suffridge operator preserves an almost starlike mappings of order α and starlike mappings of order α , respectively.

In the following, we give some notation and definitions. Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \dots, z_n)'$ with the Euclidean inner product $\langle z, w \rangle = \sum_{i=1}^n z_i \bar{w}_i$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$, where $z, w \in \mathbb{C}^n$ and the symbol $'$ means transpose. The unit ball of $\mathbb{C}^n$ is the set $\mathcal{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$, and the unit sphere is denoted by $\partial\mathcal{B}^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$. In the case of one complex variable, $\mathcal{B}^1$ is the unit disk, usually denoted by D . Let Ω be a domain in $\mathbb{C}^n$. Denote $H(\Omega)$ by the space of all holomorphic mappings from Ω into $\mathbb{C}^n$. A mapping $f \in H(\mathcal{B}^n)$ is called normalized if $f(0) = 0$ and $J_f(0) = I_n$, where $J_f(0)$ is the complex Jacobian matrix of f at the origin and I_n is the identity operator on $\mathbb{C}^n$. A mapping $f \in H(\mathcal{B}^n)$ is said to be locally biholomorphic if $\det J_f(z) \neq 0$ for every $z \in \mathcal{B}^n$. A normalized mapping $f \in H(\mathcal{B}^n)$ is said to be convex if $\lambda\omega_1 + (1-\lambda)\omega_2 \in f(\mathcal{B}^n)$ for arbitrary $\omega_1, \omega_2 \in f(\mathcal{B}^n)$ and $0 \leq \lambda \leq 1$. A normalized mapping $f \in H(\mathcal{B}^n)$ is said to be starlike with respect to the origin if $\lambda f(\mathcal{B}^n) \subset f(\mathcal{B}^n)$, $0 \leq \lambda \leq 1$. A normalized mapping $f \in H(\mathcal{B}^n)$ is said to be ε starlike if there exists a positive number ε , $0 \leq \varepsilon \leq 1$, such that $f(\mathcal{B}^n)$ is starlike with respect to every point in $\varepsilon f(\mathcal{B}^n)$.

A domain Ω is called a Reinhardt domain if $(e^{i\theta_1} z_1, e^{i\theta_2} z_2, \dots, e^{i\theta_n} z_n)' \in \Omega$ holds for any $z = (z_1, z_2, \dots, z_n)' \in \Omega$ and $\theta_1, \theta_2, \dots, \theta_n \in \mathbb{R}$. A domain Ω is called a circular domain if $e^{i\theta} z \in \Omega$ holds for any $z \in \Omega$ and $\theta \in \mathbb{R}$. The Minkowski functional $\rho(z)$ of the Reinhardt domain

$$\Omega_{n,p_2,\dots,p_n} = \left\{ z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^n |z_j|^{p_j} < 1 \right\}, \quad p_j \geq 1, \quad j = 2, \dots, n \quad (1.6)$$

is defined as

$$\rho(z) = \inf \left\{ t > 0, \frac{z}{t} \in \Omega_{n,p_2,\dots,p_n} \right\}, \quad z \in \mathbb{C}^n. \quad (1.7)$$

Then, the Minkowski functional $\rho(z)$ is a norm of $\mathbb{C}^n$ and $\Omega_{n,p_2,\dots,p_n}$ is the unit ball in the Banach space $\mathbb{C}^n$ with respect to this norm. The Minkowski functional $\rho(z)$ is C^1 on $\bar{\Omega}_{n,p_2,\dots,p_n}$

except for a lower-dimensional manifold Ω_0 . Moreover, we give the following properties of the Minkowski functional $\rho(z)$ (see [16]):

$$\begin{aligned} 2 \frac{\partial \rho}{\partial z}(z)z &= \rho(z), \quad \forall z \in \mathbb{C}^n \setminus \Omega_0, \\ 2 \frac{\partial \rho}{\partial z}(z)z &= 1, \quad \forall z \in \partial\Omega_{n,p_2,\dots,p_n} \setminus \Omega_0, \\ \frac{\partial \rho}{\partial z}(\lambda z) &= \frac{\partial \rho}{\partial z}(z), \quad \forall \lambda \in [0, \infty), \quad z \in \mathbb{C}^n \setminus \Omega_0, \\ \frac{\partial \rho}{\partial z}(e^{i\theta}z) &= e^{-i\theta} \frac{\partial \rho}{\partial z}(z), \quad \forall z \in \mathbb{C}^n \setminus \Omega_0, \quad \theta \in \mathbb{R}. \end{aligned} \tag{1.8}$$

Definition 1.1 (see [17]). Suppose that Ω is a bounded starlike circular domain in $\mathbb{C}^n$. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $0 \leq \alpha < 1$. We say that a normalized locally biholomorphic mapping $f \in H(\Omega)$ is an almost starlike mapping of order α if the following condition holds:

$$\Re \frac{2}{\rho(z)} \frac{\partial \rho}{\partial z}(z) J_f^{-1}(z) f(z) \geq \alpha, \quad z \in \Omega \setminus \{0\}. \tag{1.9}$$

When $\Omega = \mathcal{B}^n$, its Minkowski functional $\rho(z) = \|z\|$, the above inequality becomes

$$\Re \overline{z}^T J_f^{-1}(z) f(z) \geq \alpha \|z\|^2, \quad z \in \mathcal{B}^n. \tag{1.10}$$

In particular, when $\alpha = 0$, f reduces to a starlike mapping on Ω .

Definition 1.2 (see [18]). Suppose that $\Omega \in \mathbb{C}^n$ is a bounded starlike circular domain. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $0 < \alpha < 1$. We say that a normalized locally biholomorphic mapping $f \in H(\Omega)$ is a starlike mapping of order α if the following condition holds:

$$\left| \frac{2}{\rho(z)} \frac{\partial \rho}{\partial z}(z) J_f^{-1}(z) f(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad z \in \Omega \setminus \{0\}. \tag{1.11}$$

When $\Omega = \mathcal{B}^n$, the above inequality reduces to

$$\left| \frac{1}{\|z\|^2} \overline{z}^T J_f^{-1}(z) f(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad z \in \mathcal{B}^n \setminus \{0\}. \tag{1.12}$$

2. Some Lemmas

In order to prove the main results, we need the following three lemmas.

Lemma 2.1 (see [19]). *Let p be a holomorphic function on D . If $\Re p(z) > 0$ and $p(0) > 0$, then*

$$|p'(z)| \leq \frac{2\Re p(z)}{1 - |z|^2}. \quad (2.1)$$

Lemma 2.2 (see [19]). *Let f be a normalized biholomorphic function on D . Then,*

$$\left| \left(1 - |z|^2\right) \frac{f''(z)}{f'(z)} - 2\bar{z} \right| \leq 4 \quad (2.2)$$

holds for all $z \in D$.

Lemma 2.3 (see [20]). *If $\rho(z)$ is a Minkowski function of the domain $\Omega_{n,p_2,\dots,p_n}$, $z \neq 0$, then*

$$\begin{aligned} \frac{\partial \rho}{\partial z_1}(z) &= \frac{\bar{z}_1}{\rho(z) \left[2|z_1/\rho(z)|^2 + \sum_{j=2}^n p_j |z_j/\rho(z)|^{p_j} \right]}, \\ \frac{\partial \rho}{\partial z_j}(z) &= \frac{p_j \bar{z}_j |z_j/\rho(z)|^{p_j-2}}{2\rho(z) \left[2|z_1/\rho(z)|^2 + \sum_{j=2}^n p_j |z_j/\rho(z)|^{p_j} \right]}, \quad j = 2, \dots, n. \end{aligned} \quad (2.3)$$

3. Main Results

Theorem 3.1. *Let $0 \leq \alpha < 1$ and let f be an almost starlike function of order α on the unit disc D . If complex numbers a_j satisfy the condition $|a_j| \leq (1 - \alpha)/4$, $j = 2, \dots, n$, then*

$$F(z) = \left(f(z_1) + f'(z_1) \sum_{j=2}^n a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n \right)' \quad (3.1)$$

is an almost starlike mapping of order α on the domain $\Omega_{n,p_2,\dots,p_n}$, where p_j are positive integer and $p_j \geq 2$; the branches are chosen such that $(f'(z_1))^{1/p_j}|_{z_1=0} = 1$.

Proof. By the definition of almost starlike mapping of order α , we need only to prove that the following inequality:

$$\Re \frac{2}{\rho(z)} \frac{\partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) \geq \alpha \quad (3.2)$$

holds for all $z \in \Omega_{n,p_2,\dots,p_n}$ and $z \neq 0$.

The case of $z_0 = 0$ is trivial. So, we need only consider that $z = (z_1, z'_0)' \in \overline{\Omega}_{n,p_2,\dots,p_n}$, $z_0 \neq 0$. Let $z = \zeta u = |\zeta|e^{i\theta}u$, $u \in \partial\Omega_{n,p_2,\dots,p_n}$, and $\zeta \in \overline{D} \setminus \{0\}$, then we have

$$\begin{aligned}
& \Re \frac{2}{\rho(z)} \frac{\partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) \geq \alpha \\
& \iff \Re \frac{2}{\rho(|\zeta|e^{i\theta}u)} \frac{\partial \rho}{\partial z}(|\zeta|e^{i\theta}u) J_F^{-1}(|\zeta|e^{i\theta}u) F(|\zeta|e^{i\theta}u) \geq \alpha \\
& \iff \Re \frac{2}{|\zeta|} \frac{e^{-i\theta} \partial \rho}{\partial z}(u) J_F^{-1}(|\zeta|e^{i\theta}u) F(|\zeta|e^{i\theta}u) \geq \alpha \\
& \iff \Re \frac{2 \partial \rho}{\partial z}(u) \frac{J_F^{-1}(\zeta u) F(\zeta u)}{\zeta} \geq \alpha.
\end{aligned} \tag{3.3}$$

For a fixed u , the expression $\Re(2\partial\rho/\partial z)(u)(J_F^{-1}(\zeta u)F(\zeta u)/\zeta) - \alpha$ is the real part of a holomorphic function with respect to ζ , so it is a harmonic function. By the minimum of harmonic function principle, we know that it attains its minimum on $|\zeta| = 1$, so we need only to prove for all $z \in \partial\Omega_{n,p_2,\dots,p_n}$ and $z_0 \neq 0$. Hence, $\rho(z) = 1$ and inequality (3.2) becomes

$$\Re \frac{2 \partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) \geq \alpha, \quad z \in \partial\Omega_{n,p_2,\dots,p_n}, \quad z_0 \neq 0. \tag{3.4}$$

In the following, we will prove inequality (3.4).

Since

$$F(z) = \left(f(z_1) + f'(z_1) \sum_{j=2}^n a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n \right)', \tag{3.5}$$

we have

$$J_F(z) = \begin{pmatrix} f'(z_1) + f''(z_1) \sum_{j=2}^n a_j z_j^{p_j} & a_2 p_2 f'(z_1) z_2^{p_2-1} & \cdots & a_n p_n f'(z_1) z_n^{p_n-1} \\ \frac{1}{p_2} (f'(z_1))^{(1/p_2)-1} f''(z_1) z_2 & (f'(z_1))^{1/p_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{p_n} (f'(z_1))^{(1/p_n)-1} f''(z_1) z_n & 0 & \cdots & (f'(z_1))^{1/p_n} \end{pmatrix}. \tag{3.6}$$

Suppose that $J_F^{-1}(z)F(z) = A = (x_1, x_2, \dots, x_n)'$, then $F(z) = J_F(z)A$; that is,

$$\begin{aligned} x_1 \left[f'(z_1) + f''(z_1) \sum_{j=2}^n a_j z_j^{p_j} \right] + f'(z_1) \sum_{j=2}^n a_j p_j x_j z_j^{p_j-1} &= f(z_1) + f'(z_1) \sum_{j=2}^n a_j z_j^{p_j}, \\ x_1 \frac{f''(z_1)}{p_2 f'(z_1)} z_2 + x_2 &= z_2, \\ &\vdots \\ x_1 \frac{f''(z_1)}{p_n f'(z_n)} z_n + x_n &= z_n. \end{aligned} \quad (3.7)$$

Some computation shows that

$$\begin{aligned} x_1 &= \frac{f(z_1)}{f'(z_1)} - \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j}, \\ x_2 &= \left[1 - \frac{f(z_1) f''(z_1)}{p_2 (f'(z_1))^2} + \frac{f''(z_1)}{p_2 f'(z_1)} \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \right] z_2, \\ &\vdots \\ x_n &= \left[1 - \frac{f(z_1) f''(z_1)}{p_n (f'(z_1))^2} + \frac{f''(z_1)}{p_n f'(z_1)} \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \right] z_n. \end{aligned} \quad (3.8)$$

From Lemma 2.3, we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial z_1}(z) &= \frac{\bar{z}_1}{\rho(z) \left[2|z_1/\rho(z)|^2 + \sum_{j=2}^n p_j |z_j/\rho(z)|^{p_j} \right]} = \frac{\bar{z}_1}{2|z_1|^2 + \sum_{j=2}^n p_j |z_j|^{p_j}}, \\ \frac{\partial \rho}{\partial z_j}(z) &= \frac{p_j \bar{z}_j |z_j|^{p_j-2}}{2\rho(z) \left[2|z_1/\rho(z)|^2 + \sum_{j=2}^n p_j |z_j|^{p_j} \right]} = \frac{p_j \bar{z}_j |z_j|^{p_j-2}}{2 \left[2|z_1|^2 + \sum_{j=2}^n p_j |z_j|^{p_j} \right]}. \end{aligned} \quad (3.9)$$

In terms of (3.8) and (3.9), we obtain

$$\frac{2\partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) = \frac{G(z)}{2|z_1|^2 + \sum_{j=2}^n p_j |z_j|^{p_j}}, \quad (3.10)$$

where

$$\begin{aligned}
 G(z) &= 2\bar{z}_1 \left[\frac{f(z_1)}{f'(z_1)} - \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \right] \\
 &\quad + \sum_{j=2}^n p_j |z_j|^{p_j} \left[1 - \frac{f(z_1)f''(z_1)}{p_j (f'(z_1))^2} + \frac{f''(z_1)}{p_j f'(z_1)} \sum_{k=2}^n a_k (p_k - 1) z_k^{p_k} \right] \\
 &= 2|z_1|^2 \frac{f(z_1)}{z_1 f'(z_1)} + \sum_{j=2}^n p_j |z_j|^{p_j} \left[1 - \frac{f(z_1)f''(z_1)}{p_j (f'(z_1))^2} \right] \\
 &\quad + \sum_{k=2}^n a_k (p_k - 1) z_k^{p_k} \left[\frac{f''(z_1)}{f'(z_1)} \sum_{j=2}^n |z_j|^{p_j} - 2\bar{z}_1 \right].
 \end{aligned} \tag{3.11}$$

By making use of the equality $|z_1|^2 + \sum_{j=2}^n |z_j|^{p_j} = 1$, we then get

$$\begin{aligned}
 G(z) &= 2|z_1|^2 \frac{f(z_1)}{z_1 f'(z_1)} + \sum_{j=2}^n p_j |z_j|^{p_j} \left[1 - \frac{f(z_1)f''(z_1)}{p_j (f'(z_1))^2} \right] \\
 &\quad + \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \left[\frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right].
 \end{aligned} \tag{3.12}$$

Let $h(z_1) = (f(z_1)/z_1 f'(z_1)) - \alpha$. Notice that f is an almost starlike function of order α on the unit disc; hence, $\Re h(z_1) > 0$ and $h(0) = 1 - \alpha > 0$. By Lemma 2.1, we can obtain that

$$|h'(z)| \leq \frac{2\Re h(z)}{1 - |z|^2}. \tag{3.13}$$

Furthermore, we get

$$\frac{f(z_1)f''(z_1)}{[f'(z_1)]^2} = 1 - \alpha - h(z_1) - z_1 h'(z_1). \tag{3.14}$$

Substituting (3.14) into (3.12), we have

$$\begin{aligned}
 G(z) &= 2|z_1|^2 (h(z_1) + \alpha) + \sum_{j=2}^n p_j |z_j|^{p_j} \left(1 - \frac{1}{p_j} + \frac{\alpha}{p_j} + \frac{h(z_1)}{p_j} + \frac{z_1}{p_j} h'(z_1) \right) \\
 &\quad + \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \left[\frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right] \\
 &= h(z_1) \left(2|z_1|^2 + \sum_{j=2}^n |z_j|^{p_j} \right) + 2\alpha |z_1|^2 + \sum_{j=2}^n (p_j - 1 + \alpha) |z_j|^{p_j} \\
 &\quad + \sum_{j=2}^n z_1 |z_j|^{p_j} h'(z_1) + \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \left[\frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right].
 \end{aligned} \tag{3.15}$$

Hence,

$$\begin{aligned} \Re G(z) \geq & \left(2|z_1|^2 + \sum_{j=2}^n |z_j|^{p_j} \right) \Re h(z_1) + 2\alpha|z_1|^2 + \sum_{j=2}^n (p_j - 1 + \alpha) |z_j|^{p_j} \\ & - \sum_{j=2}^n |z_j|^{p_j} |z_1 h'(z_1)| - \sum_{j=2}^n |a_j| (p_j - 1) |z_j|^{p_j} \left| \frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right|. \end{aligned} \quad (3.16)$$

By Lemma 2.2 and (3.13), we can get that

$$\begin{aligned} \Re G(z) \geq & (1 + |z_1|^2) \Re h(z_1) + 2\alpha|z_1|^2 + \sum_{j=2}^n (p_j - 1 + \alpha) |z_j|^{p_j} - (1 - |z_1|^2) \frac{2|z_1| \Re h(z_1)}{1 - |z_1|^2} \\ & - 4 \sum_{j=2}^n |a_j| (p_j - 1) |z_j|^{p_j} \\ = & (1 + |z_1|^2) \Re h(z_1) + 2\alpha|z_1|^2 + \sum_{j=2}^n (p_j - 1 + \alpha) |z_j|^{p_j} - 2|z_1| \Re h(z_1) \\ & - 4 \sum_{j=2}^n |a_j| (p_j - 1) |z_j|^{p_j} \\ = & (1 - |z_1|)^2 \Re h(z_1) + 2\alpha|z_1|^2 + \sum_{j=2}^n |z_j|^{p_j} [\alpha + (1 - 4|a_j|)(p_j - 1)]. \end{aligned} \quad (3.17)$$

Hence, when $|a_j| \leq (1 - \alpha)/4$, we have

$$\Re G(z) \geq (1 - |z_1|)^2 \Re h(z_1) + 2\alpha|z_1|^2 + \alpha \sum_{j=2}^n p_j |z_j|^{p_j} \geq 2\alpha|z_1|^2 + \alpha \sum_{j=2}^n p_j |z_j|^{p_j}. \quad (3.18)$$

In terms of (3.10) and (3.18), we obtain

$$\Re \frac{2\partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) \geq \alpha, \quad (3.19)$$

which completes the proof of Theorem 3.1. $\square$

Remark 3.2. When $a_2 = a_3 = \dots = a_n = 0$, the result of Theorem 3.1 has been obtained by Liu and Liu [14].

Corollary 3.3. Let f be a normalized biholomorphic starlike function on the unit disc D . If $|a_j| \leq 1/4$, $j = 2, \dots, n$, then

$$F(z) = \left(f(z_1) + f'(z_1) \sum_{j=2}^n a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n \right)' \quad (3.20)$$

is a normalized biholomorphic starlike mapping on the domain $\Omega_{n,p_2,\dots,p_n}$, where p_j are positive integer and $p_j \geq 2$; the branches are chosen such that $(f'(z_1))^{1/p_j}|_{z_1=0} = 1$.

Theorem 3.4. Let $0 < \alpha < 1$ and let f be a starlike function of order α on the unit disc D . If complex numbers a_j satisfy the condition $|a_j| \leq (1 - |2\alpha - 1|)/8\alpha$, $j = 2, \dots, n$, then

$$F(z) = \left(f(z_1) + f'(z_1) \sum_{j=2}^n a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n \right)' \quad (3.21)$$

is a starlike mapping of order α on the domain $\Omega_{n,p_2,\dots,p_n}$, where p_j are positive integer and $p_j \geq 1$; the branches are chosen such that $(f'(z_1))^{1/p_j}|_{z_1=0} = 1$.

Proof. By the definition of starlike mapping of order α , we need only to prove that the following inequality:

$$\left| \frac{2}{\rho(z)} \frac{\partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (3.22)$$

holds for all $z \in \Omega_{n,p_2,\dots,p_n}$ and $z_0 \neq 0$.

Similar to the proof of Theorem 3.1, we need only to prove that (3.22) holds for $\rho(z) = 1$ and $z_0 \neq 0$ according to the maximum modulus theorem for analytic functions. So, it is suffice to show that

$$\left| \frac{2\partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}. \quad (3.23)$$

From the proof of Theorem 3.1, we can get

$$\begin{aligned} & \frac{\partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) \\ &= \frac{1}{2[2|z_1|^2 + \sum_{j=2}^n p_j |z_j|^{p_j}]} \left\{ 2|z_1|^2 \frac{f(z_1)}{z_1 f'(z_1)} + \sum_{j=2}^n p_j |z_j|^{p_j} \left[1 - \frac{f(z_1) f''(z_1)}{p_j (f'(z_1))^2} \right] \right. \\ & \quad \left. + \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \left[\frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right] \right\}. \end{aligned} \quad (3.24)$$

Hence,

$$\frac{2\partial \rho}{\partial z}(z) J_F^{-1}(z) F(z) - \frac{1}{2\alpha} = \frac{H(z)}{2\alpha [2|z_1|^2 + \sum_{j=2}^n p_j |z_j|^{p_j}]}, \quad (3.25)$$

where

$$\begin{aligned}
 H(z) = & 2|z_1|^2 \left[2\alpha \frac{f(z_1)}{z_1 f'(z_1)} - 1 \right] + 2\alpha \sum_{j=2}^n p_j |z_j|^{p_j} \left[1 - \frac{1}{2\alpha} - \frac{f(z_1) f''(z_1)}{p_j (f'(z_1))^2} \right] \\
 & + 2\alpha \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \left[\frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right].
 \end{aligned} \tag{3.26}$$

Let $h(z_1) = 2\alpha(f(z_1)/z_1 f'(z_1)) - 1$. Then, $|h(z_1)| < 1$ because f is a starlike function of order α on the unit disc D . By the Schwarz-Pick lemma, we obtain that

$$|h'(z_1)| \leq \frac{1 - |h(z_1)|^2}{1 - |z_1|^2}. \tag{3.27}$$

On the other hand, we can get

$$\frac{f(z_1) f''(z_1)}{[f'(z_1)]^2} = 1 - \frac{1}{2\alpha} - \frac{h(z_1)}{2\alpha} - \frac{z_1 h'(z_1)}{2\alpha}. \tag{3.28}$$

Substituting (3.28) into (3.26), we have

$$\begin{aligned}
 H(z) = & 2|z_1|^2 \left[2\alpha \frac{f(z_1)}{z_1 f'(z_1)} - 1 \right] + 2\alpha \sum_{j=2}^n p_j |z_j|^{p_j} \left[1 - \frac{1}{2\alpha} - \left(\frac{1}{p_j} - \frac{1}{2\alpha p_j} - \frac{h(z_1)}{2\alpha p_j} - \frac{z_1 h'(z_1)}{2\alpha p_j} \right) \right] \\
 & + 2\alpha \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \left[\frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right] \\
 = & 2|z_1|^2 h(z_1) + h(z_1) \sum_{j=2}^n |z_j|^{p_j} + \sum_{j=2}^n z_1 h'(z_1) |z_j|^{p_j} + (2\alpha - 1) \sum_{j=2}^n (p_j - 1) |z_j|^{p_j} \\
 & + 2\alpha \sum_{j=2}^n a_j (p_j - 1) z_j^{p_j} \left[\frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right].
 \end{aligned} \tag{3.29}$$

Hence,

$$\begin{aligned}
 |H(z)| \leq & \left(1 + |z_1|^2 \right) |h(z_1)| + \sum_{j=2}^n |z_1 h'(z_1)| |z_j|^{p_j} + |2\alpha - 1| \sum_{j=2}^n (p_j - 1) |z_j|^{p_j} \\
 & + 2\alpha \sum_{j=2}^n |a_j| (p_j - 1) |z_j|^{p_j} \left| \frac{f''(z_1)}{f'(z_1)} (1 - |z_1|^2) - 2\bar{z}_1 \right|.
 \end{aligned} \tag{3.30}$$

By Lemma 2.2 and (3.27), we have

$$\begin{aligned}
|H(z)| &\leq \left(1 + |z_1|^2\right) |h(z_1)| + |z_1| \frac{1 - |h(z_1)|^2}{1 - |z_1|^2} \left(1 - |z_1|^2\right) + |2\alpha - 1| \sum_{j=2}^n (p_j - 1) |z_j|^{p_j} \\
&\quad + 8\alpha \sum_{j=2}^n |a_j| (p_j - 1) |z_j|^{p_j} \\
&\leq \left(1 + |z_1|^2\right) (|h(z_1)| - 1) + \left(1 + |z_1|^2\right) + 2|z_1|(1 - |h(z_1)|) \\
&\quad + |2\alpha - 1| \sum_{j=2}^n (p_j - 1) |z_j|^{p_j} + 8\alpha \sum_{j=2}^n |a_j| (p_j - 1) |z_j|^{p_j} \\
&\leq \left(1 + |z_1|^2\right) + (|h(z_1)| - 1)(1 - |z_1|^2) + \sum_{j=2}^n (|2\alpha - 1| + 8\alpha |a_j|) (p_j - 1) |z_j|^{p_j}.
\end{aligned} \tag{3.31}$$

If $|a_j| \leq (1 - |2\alpha - 1|)/8\alpha$, then we obtain

$$\begin{aligned}
|H(z)| &< 1 + |z_1|^2 + \left(|2\alpha - 1| + 8\alpha \frac{1 - |2\alpha - 1|}{8\alpha}\right) \sum_{j=2}^n (p_j - 1) |z_j|^{p_j} \\
&\leq 1 + |z_1|^2 + \sum_{j=2}^n (p_j - 1) |z_j|^{p_j} \\
&= 2|z_1|^2 + \sum_{j=2}^n p_j |z_j|^{p_j}.
\end{aligned} \tag{3.32}$$

The equality (3.25) and (3.32) show that

$$\left| \frac{2\partial\rho}{\partial z}(z) J_F^{-1}(z) F(z) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \tag{3.33}$$

which completes the proof of Theorem 3.4. $\square$

4. Problem

In 2003, Gong and Liu [13] proved that the Roper-Suffridge extension operator

$$\Phi_{n,(1/p_2),\dots,(1/p_n)}(f)(z) = \left(f(z_1), (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n\right)' \tag{4.1}$$

does preserve convexity on $\Omega_n, p_2, \dots, p_n$, which solved the open problem posed by Graham and Kohr [2]. Naturally, we will propose the following problem on the new Roper-Suffridge extension operator.

Problem 1. Let p_j be positive integer. Under what conditions for a_j such that if f is a convex function in the disc D , then the mapping defined by the new Roper-Suffridge extension operator

$$F(z) = \left(f(z_1) + f'(z_1) \sum_{j=2}^n a_j z_j^{p_j}, (f'(z_1))^{1/p_2} z_2, \dots, (f'(z_1))^{1/p_n} z_n \right)' \quad (4.2)$$

is a convex mapping in the Reinhardt domain $\Omega_n, p_2, \dots, p_n$?

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