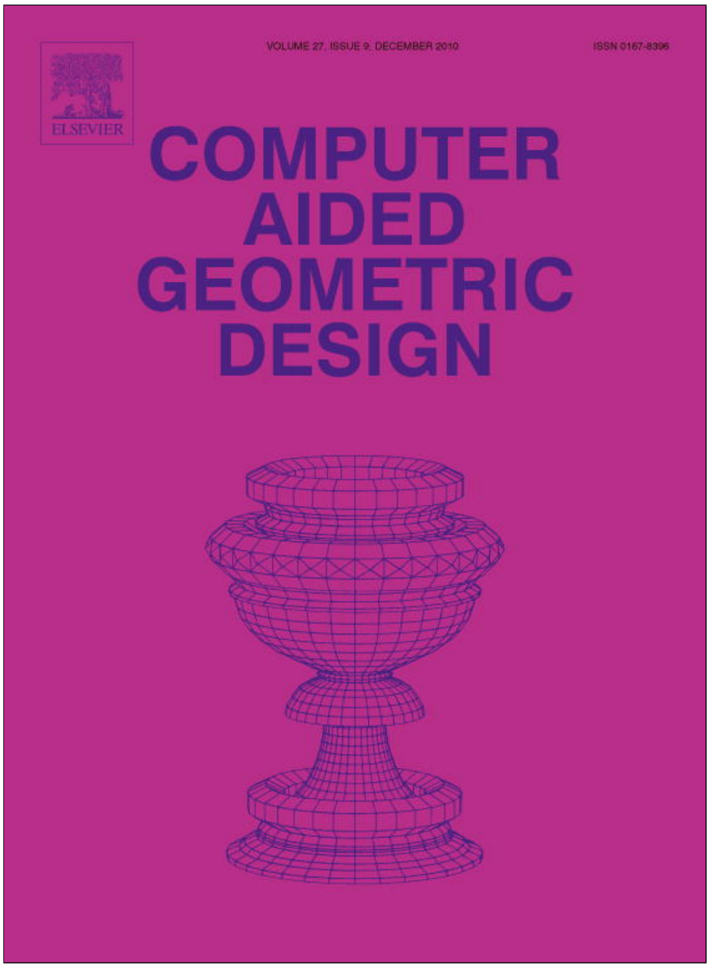
VOLUME 27, ISSUE 9, DECEMBER 2010
ISSN 0167-8396
ELSEVIER
COMPUTER
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GEOMETRIC
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Computer Aided Geometric Design 27 (2010) 746–755

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Minimizing the maximal ratio of weights of rational Bézier curves and surfaces

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ARTICLE INFO

Article history:
Received 21 May 2009
Received in revised form 25 June 2010
Accepted 22 August 2010
Available online 16 September 2010

Keywords:
Rational Bézier curve
Rational Bézier surface
Bound
Derivative
Invariant

ABSTRACT

Applying the Möbius transformation to rational Bézier curves and surfaces, the weights can be modified whereas the control points remain unchanged. With appropriate transformation parameters, the maximal ratio of the weights of rational Bézier curves and surfaces can be minimized, which have applications in improving the bounds of derivatives, optimizing degree reduction of rational Bézier curves. In the surface case, there has not yet been a solution for the problem of finding transformation parameters such that the maximal ratio of the weights reaches its minimum. In this paper, a new method for the problem in the curve case is presented, and the uniqueness of the solution can be easily proved; then the method is generalized to the surface case with geometric perception. Some numerical examples are given for showing our results in improving the bounds of derivatives of rational Bézier curves and surfaces.

1. Introduction

In geometric design, it is often needed to estimate the bounds of derivatives of rational curves and surfaces. Given a rational Bézier curve and a rational Bézier surface

$$\boldsymbol{R}(t)=\frac{\sum_{i=0}^{n}B_i^n(t)\omega_i\boldsymbol{R}_i}{\sum_{i=0}^{n}B_i^n(t)\omega_i},\quad 0\leqslant t\leqslant 1,\tag{1}$$

$$\boldsymbol{R}(u,v)=\frac{\sum_{i=0}^{m}\sum_{j=0}^{n}B_i^m(u)B_j^n(v)\omega_{ij}\boldsymbol{R}_{ij}}{\sum_{i=0}^{m}\sum_{j=0}^{n}B_i^m(u)B_j^n(v)\omega_{ij}},\quad 0\leqslant u,v\leqslant 1,\tag{2}$$

where ω_i, ω_{ij} are positive weights of the curve and surface respectively. Floater (1992) and Wang et al. (1997) provided the following estimations for the bounds of derivatives of the curve and surface respectively,

$$\left\|d\boldsymbol{R}(t)/dt\right\|\leqslant n\Big(\max_{0\leqslant i\leqslant n}\omega_i\Big/\min_{0\leqslant i\leqslant n}\omega_i\Big)\max_{0\leqslant i,j\leqslant n}\|\boldsymbol{R}_i-\boldsymbol{R}_j\|,\tag{3}$$

$$\left\|d\boldsymbol{R}(t)/dt\right\|\leqslant n\Big(\max_{0\leqslant i\leqslant n}\omega_i\Big/\min_{0\leqslant i\leqslant n}\omega_i\Big)^2\max_{0\leqslant i\leqslant n-1}\|\boldsymbol{R}_i-\boldsymbol{R}_{i+1}\|;\tag{4}$$

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0167-8396/$ – see front matter
doi:10.1016/j.cagd.2010.08.001

$$\left\| \frac{\partial \boldsymbol{R}(u,v)}{\partial u} \right\| \leqslant m \Big(\max_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} \omega_{ij} \Big/ \min_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} \omega_{ij} \Big)^3 \max_{\substack{0 \leqslant i,k \leqslant m \\ 0 \leqslant j,l \leqslant n}} \|\boldsymbol{R}_{ij} - \boldsymbol{R}_{kl}\|, \tag{5}$$

where neither of inequalities (3) and (4) is stronger than the other. To further decrease the bounds of derivatives of the curve and surface, Selimovic (2005) used the intermediate weights and control points generated by the de Casteljau algorithm and improved inequalities (3) and (5). With the techniques of fractional inequalities, Zhang and Ma (2006) gave a better result than that of Selimovic (2005), and improved inequality (4) greatly when the degree of the rational Bézier curve is smaller than or equal to six. Applying the degree elevation algorithm to rational Bézier curve, Huang and Su (2006) also got a better result than that of Selimovic (2005).

However the expressions of these improved results are quite complicated and more and more variables are present. It is very difficult for further improvement. Fortunately, there is another way to decrease the bounds of derivatives of rational Bézier curves and surfaces. Firstly applying the Möbius transformation $t = \gamma s/(\gamma s + 1 - s)$ to the rational Bézier curve, Eq. (1) is changed to

$$\boldsymbol{R}(t) = \boldsymbol{R}(s) = \frac{\sum_{i=0}^{n} B_i^n(s)\omega_i \gamma^i \boldsymbol{R}_i}{\sum_{i=0}^{n} B_i^n(s)\omega_i \gamma^i}, \quad 0 \leqslant s \leqslant 1,$$

where γ (> 0) is the transformation parameter and the weights ω_i are changed to $\gamma^i \omega_i$; similarly applying the Möbius transformations $u = \alpha s/(\alpha s + 1 - s)$, $v = \beta t/(\beta t + 1 - t)$ to the rational Bézier surface, Eq. (2) is changed to

$$\boldsymbol{R}(u,v) = \boldsymbol{R}(s,t) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(s) B_j^n(t)\omega_{ij}\alpha^i \beta^j \boldsymbol{R}_{ij}}{\sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(s) B_j^n(t)\omega_{ij}\alpha^i \beta^j}, \quad 0 \leqslant s,\ t \leqslant 1,$$

where α, β (> 0) are transformation parameters and the weights ω_{ij} are changed to $\alpha^i \beta^j \omega_{ij}$. Secondly, apply certain algorithms to find appropriate transformation parameters such that the maximal ratio of weights of the curve and surface can be minimized, and then the bounds of the derivatives of the curve and surface can be decreased via inequalities (3), (4) and (5).

Zheng (2005) reduced the problem of minimizing the maximal ratio of weights of a rational Bézier curve to a linear programming problem, and provided a direct expression for the appropriate transformation parameter. In this paper, a new method for obtaining the transformation parameter is presented for the problem in the curve case, which can be generalized to the surface case, whereas Zheng (2005)'s method is only suitable for the curve case.

The rest of the paper is organized as follows. In Sections 2 and 3, we describe the problem in a mathematical manner and give the solutions of the problem in the curve and surface cases respectively. In Section 4, some numerical examples are given for showing our methods in improving the bounds of derivatives of rational Bézier curves and surfaces. Finally, we give the conclusions of this paper in Section 5.

2. Description of the problem in curve case and its solution

From Section 1, we know that applying the Möbius transformation with parameter γ to the rational Bézier curve in Eq. (1), the weights of the curve is changed from ω_i to $\gamma^i \omega_i$, and the ratio of two weights is changed correspondingly from ω_i/ω_j to $\gamma^{i-j}\omega_i/\omega_j$. To minimize the maximal ratio of the weights, we should find the maximum of the set $\{\gamma^{i-j}\omega_i/\omega_j \mid 0 \leqslant i, j \leqslant n\}$, and vary the positive real number γ such that the maximum reaches its minimum. As the logarithmic function $y = \log(x)$ is monotonically increasing on its domain $x > 0$, the problem can be described as

Problem 2.1. Find a positive real number γ, such that the function

$$\rho(\gamma) = \max\big\{\log\big(\gamma^{i-j}\omega_i/\omega_j\big) \,\big|\, 0 \leqslant i, j \leqslant n\big\} \quad (\gamma > 0)$$

reaches its minimum.

In order to solve the problem, we firstly introduce $2n + 1$ constants

$$f_k = \max\big\{\log(\omega_i/\omega_j) \,\big|\, 0 \leqslant i, j \leqslant n,\ i - j = k\big\}, \quad -n \leqslant k \leqslant n,$$

and $2n + 1$ functions

$$f_k(\gamma) = \max\big\{\log(\gamma^{i-j}\omega_i/\omega_j) \,\big|\, 0 \leqslant i, j \leqslant n,\ i - j = k\big\} = k\log(\gamma) + f_k \quad (\gamma > 0),\ -n \leqslant k \leqslant n.$$

Then we get a compact expression for the target function

$$\rho(\gamma) = \max\big\{f_k(\gamma) \,\big|\, -n \leqslant k \leqslant n\big\}.$$

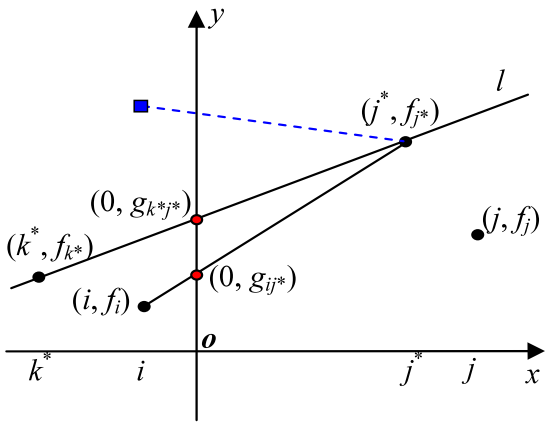

Fig. 1. Geometric perception of Theorems 2.3, 2.4.

With these functions and constants, we define a set of invariants which are independent of the transformation parameter γ,

$$g_{kj} := \frac{jf_k(\gamma) - kf_j(\gamma)}{j-k} = \frac{jf_k - kf_j}{j-k}, \quad -n \leqslant k < j \leqslant n.$$

There is a geometric perception for g_{kj}. In the planar Cartesian coordinate, $(0, g_{kj})$ is the intersection point of the y-axis and the line passing through the points (k, f_k) and (j, f_j).

We choose all of the invariants g_{kj} with $kj < 0$, and congregate them to a set

$$G = \{g_{kj} \mid kj < 0,\ -n \leqslant k < j \leqslant n\} = \bigl\{(jf_k - kf_j)/(j-k) \bigm| -n \leqslant k < 0 < j \leqslant n\bigr\}.$$

With the set, we give a lower bound of the function $\rho(\gamma)$ $(\gamma > 0)$.

Lemma 2.2. $0 \leqslant \max G \leqslant \min\{\rho(\gamma) \mid \gamma > 0\}$.

Proof. Firstly $0 = (\log(\omega_0/\omega_1) + \log(\omega_1/\omega_0))/2 \leqslant (f_{-1} + f_1)/2 = g_{-11} \leqslant \max G$.

And then for every $\gamma > 0$, $g_{kj} \in G$, $-n \leqslant k < 0 < j \leqslant n$, we have

$$g_{kj} = \bigl(jf_k(\gamma) - kf_j(\gamma)\bigr)/(j-k) \leqslant \bigl(j\rho(\gamma) - k\rho(\gamma)\bigr)/(j-k) = \rho(\gamma).$$

$\Rightarrow \max G \leqslant \rho(\gamma) \Rightarrow \max G \leqslant \min\{\rho(\gamma) \mid \gamma > 0\}$. $\square$

To prove $\min\{\rho(\gamma) \mid \gamma > 0\} = \max G$, we only need to find a γ^* (> 0), such that $\rho(\gamma^*) = \max G$.

Theorem 2.3. *Let k^*, j^* be integers, $-n \leqslant k^* < 0 < j^* \leqslant n$, γ^* be positive real number such that*

$$g_{k^*j^*} = (j^* f_{k^*} - k^* f_{j^*})/(j^* - k^*) = \max G, \qquad \log(\gamma^*) = (f_{k^*} - f_{j^*})/(j^* - k^*).$$

We then have $\rho(\gamma^*) = \max G = \min\{\rho(\gamma) \mid \gamma > 0\}$.

Proof. It can be readily shown that the equation of the line passing through the points (k^*, f_{k^*}) and (j^*, f_{j^*}) is l: $x\log(\gamma^*) + y = g_{k^*j^*}$, which leads to $f_{k^*}(\gamma^*) = k^*\log(\gamma^*) + f_{k^*} = g_{k^*j^*} = \max G$, $f_{j^*}(\gamma^*) = \max G$ (see Fig. 1).

We then need to prove that for every integer i, $-n \leqslant i \leqslant n$, $f_i(\gamma^*) \leqslant \max G$.

If $i = 0$, since $f_0(\gamma^*) = f_0 = 0$, according to Lemma 2.2, it's clear that $f_0(\gamma^*) \leqslant \max G$.

If $i \neq 0$, without loss of generality, we assume that $i < 0$. Suppose the point (i, f_i) is above the line l (see the square point in Fig. 1), since $(0, g_{ij^*})$ is the intersection point of the y-axis and the line passing through the points (i, f_i) and (j^*, f_{j^*}), we must have $g_{ij^*} > g_{k^*j^*} = \max G$, which is a contradiction. So the point (i, f_i) must be under or on the line l, which means $f_i(\gamma^*) = i\log(\gamma^*) + f_i \leqslant g_{k^*j^*} = \max G$.

With Lemma 2.2, we thus have $\rho(\gamma^*) = \max G = \min\{\rho(\gamma) \mid \gamma > 0\}$. $\square$

Theorem 2.3 provides the solution of Problem 2.1, which is the same as the result given in Zheng (2005). In the following, we can prove the uniqueness of the solution.

Theorem 2.4. *There is only one γ^*, given in Theorem* 2.3, *such that $\rho(\gamma^*) = \min\{\rho(\gamma) \mid \gamma > 0\}$.*

Proof. Let k^*, j^* be the integers and γ^* be the positive real number given in Theorem 2.3. If there exists a positive real number γ_0 such that $\rho(\gamma_0) = \rho(\gamma^*)$, then we have $f_i(\gamma_0) \leqslant \rho(\gamma_0) = \rho(\gamma^*) = g_{k^* j^*}$, $-n \leqslant i \leqslant n$, or equivalently, every point (i, f_i) is under or on the line l_0: $x \log(\gamma_0) + y = g_{k^* j^*}$. Since the line l_0 passes through the point $(0, g_{k^* j^*})$ and covers the points (k^*, f_{k^*}) and (j^*, f_{j^*}), we conclude that l_0 is just the line l in the proof of Theorem 2.3 (see Fig. 1), which leads to $\gamma_0 = \gamma^*$. □

3. Description of the problem in surface case and its solutions

From Section 1, we know that applying the Möbius transformations with parameters α, β (> 0) to the rational Bézier surface in Eq. (2), the weights of the surface are changed from ω_{ij} to $\alpha^i \beta^j \omega_{ij}$. Like the curve case, the problem can be described as

Problem 3.1. Find two positive real numbers α, β, such that the function

$$\rho(\alpha, \beta) = \max\bigl\{\log\bigl(\alpha^{i-k}\beta^{j-l}\omega_{ij}/\omega_{kl}\bigr) \bigm| 0 \leqslant i, k \leqslant m,\ 0 \leqslant j, l \leqslant n\bigr\} \quad (\alpha, \beta > 0)$$

reaches its minimum.

For simplification, we firstly define a set

$$\Omega = \bigl\{(k, l) \in \mathbb{Z} \times \mathbb{Z} \bigm| -m \leqslant k \leqslant m,\ -n \leqslant l \leqslant n\bigr\}, \quad \text{where } \mathbb{Z} \text{ is the set of all integers.}$$

Then for every $\boldsymbol{p} = (k, l) \in \Omega$, we define a constant

$$f_{\boldsymbol{p}} = \max\bigl\{\log(\omega_{i_1, j_1}/\omega_{i_2, j_2}) \bigm| 0 \leqslant i_1, i_2 \leqslant m,\ i_1 - i_2 = k;\ 0 \leqslant j_1, j_2 \leqslant n,\ j_1 - j_2 = l\bigr\},$$

and a function

$$f_{\boldsymbol{p}}(\alpha, \beta) = k \log(\alpha) + l \log(\beta) + f_{\boldsymbol{p}} = \boldsymbol{p} \cdot \bigl(\log(\alpha), \log(\beta)\bigr) + f_{\boldsymbol{p}},$$

where $\cdot$ is the sign of inner product of two vectors. Then we have

$$\rho(\alpha, \beta) = \max\bigl\{f_{\boldsymbol{p}}(\alpha, \beta) \bigm| \boldsymbol{p} \in \Omega\bigr\} \quad (\alpha, \beta > 0).$$

Let $\boldsymbol{o} = (0, 0)$ be the origin of the planar Cartesian coordinate. We define two sets as

$$A_2 = \bigl\{(\boldsymbol{p}_1, \boldsymbol{p}_2) \bigm| \boldsymbol{p}_1, \boldsymbol{p}_2 \in \Omega - \{\boldsymbol{o}\},\ \|\boldsymbol{p}_2\|\boldsymbol{p}_1 + \|\boldsymbol{p}_1\|\boldsymbol{p}_2 = \boldsymbol{o}\bigr\}, \quad \text{where } \|\cdot\| \text{ is the Euclidean norm;}$$

$A_3 = \{(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) \mid \boldsymbol{p}_i \in \Omega - \{\boldsymbol{o}\},\ T(\boldsymbol{o}\boldsymbol{p}_i\boldsymbol{p}_{i+1}) > 0,\ i = 1, 2, 3,\ \boldsymbol{p}_4 = \boldsymbol{p}_1\}$, where $T = 2S$, and $S(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3)$ is the signed area of the triangle $\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3$ (Pogorelov, 1984). If $\boldsymbol{p}_i = (k_i, l_i)$, $i = 1, 2, 3$, the expression of T is

$$T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3) = \begin{vmatrix} 1 & 1 & 1 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{vmatrix} = k_1 l_2 + k_2 l_3 + k_3 l_1 - k_2 l_1 - k_3 l_2 - k_1 l_3.$$

It can be readily verified that $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$ are collinear $\Leftrightarrow$ $T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3) = 0$. If $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$ are not collinear and in counter clock order, then $T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3) > 0$. Else if they are in clockwise order, then $T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3) < 0$. By direct computations, we can get two identities

$$T(\boldsymbol{o}\boldsymbol{p}_2\boldsymbol{p}_3) + T(\boldsymbol{o}\boldsymbol{p}_3\boldsymbol{p}_1) + T(\boldsymbol{o}\boldsymbol{p}_1\boldsymbol{p}_2) = T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3), \tag{6}$$

$$T(\boldsymbol{o}\boldsymbol{p}_2\boldsymbol{p}_3)\boldsymbol{p}_1 + T(\boldsymbol{o}\boldsymbol{p}_3\boldsymbol{p}_1)\boldsymbol{p}_2 + T(\boldsymbol{o}\boldsymbol{p}_1\boldsymbol{p}_2)\boldsymbol{p}_3 = \boldsymbol{o}. \tag{7}$$

Thus every element $(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3)$ of the set A_3 can be identified as the three vertices of a triangle, arranged in counterclockwise order, in the planar Cartesian coordinate, and the origin $\boldsymbol{o}$ is inside the triangle $\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3$ (excluding the edges).

The elements of the sets A_2 and A_3 can be visualized from Figs. 2 and 3 respectively.

Then we define two classes of invariants which are independent of α, β. For every $(\boldsymbol{p}_1, \boldsymbol{p}_2) \in A_2$,

$$\begin{aligned} g_{\boldsymbol{p}_1\boldsymbol{p}_2} &:= \frac{\|\boldsymbol{p}_2\| f_{\boldsymbol{p}_1}(\alpha, \beta) + \|\boldsymbol{p}_1\| f_{\boldsymbol{p}_2}(\alpha, \beta)}{\|\boldsymbol{p}_2\| + \|\boldsymbol{p}_1\|} = \frac{(\|\boldsymbol{p}_2\|\boldsymbol{p}_1 + \|\boldsymbol{p}_1\|\boldsymbol{p}_2) \cdot (\log(\alpha), \log(\beta)) + \|\boldsymbol{p}_2\| f_{\boldsymbol{p}_1} + \|\boldsymbol{p}_1\| f_{\boldsymbol{p}_2}}{\|\boldsymbol{p}_2\| + \|\boldsymbol{p}_1\|} \\ &= \frac{\|\boldsymbol{p}_2\| f_{\boldsymbol{p}_1} + \|\boldsymbol{p}_1\| f_{\boldsymbol{p}_2}}{\|\boldsymbol{p}_2\| + \|\boldsymbol{p}_1\|}. \end{aligned} \tag{8}$$

For every triple $(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3)$, $\boldsymbol{p}_i = (k_i, l_i)$, $i = 1, 2, 3$, $T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3) \neq 0$,

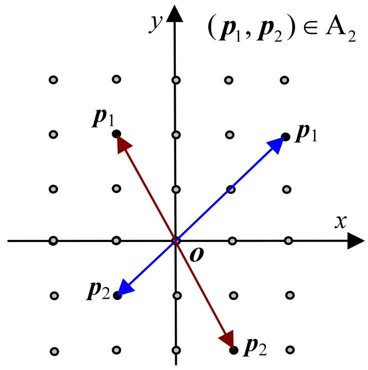

Fig. 2. Elements of the set A_2.

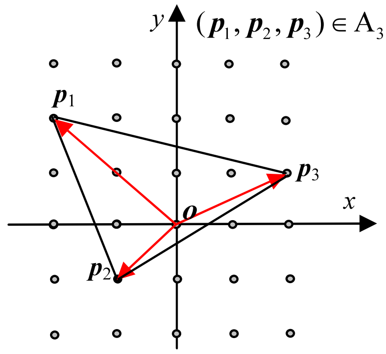

Fig. 3. Element of the set A_3.

$$\begin{aligned} g_{\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3} &:= \frac{T(\boldsymbol{op}_2\boldsymbol{p}_3)f_{\boldsymbol{p}_1}(\alpha,\beta) + T(\boldsymbol{op}_3\boldsymbol{p}_1)f_{\boldsymbol{p}_2}(\alpha,\beta) + T(\boldsymbol{op}_1\boldsymbol{p}_2)f_{\boldsymbol{p}_3}(\alpha,\beta)}{T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3)} \\ &= \big\{\big(T(\boldsymbol{op}_2\boldsymbol{p}_3)\boldsymbol{p}_1 + T(\boldsymbol{op}_3\boldsymbol{p}_1)\boldsymbol{p}_2 + T(\boldsymbol{op}_1\boldsymbol{p}_2)\boldsymbol{p}_3\big)\cdot\big(\log(\alpha), \log(\beta)\big) + T(\boldsymbol{op}_2\boldsymbol{p}_3)f_{\boldsymbol{p}_1} + T(\boldsymbol{op}_3\boldsymbol{p}_1)f_{\boldsymbol{p}_2} \\ &\quad + T(\boldsymbol{op}_1\boldsymbol{p}_2)f_{\boldsymbol{p}_3}\big\}\big\{T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3)\big\}^{-1} \\ &= \frac{T(\boldsymbol{op}_2\boldsymbol{p}_3)f_{\boldsymbol{p}_1} + T(\boldsymbol{op}_3\boldsymbol{p}_1)f_{\boldsymbol{p}_2} + T(\boldsymbol{op}_1\boldsymbol{p}_2)f_{\boldsymbol{p}_3}}{T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3)} = \frac{1}{T(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3)}\begin{vmatrix} k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \\ f_{\boldsymbol{p}_1} & f_{\boldsymbol{p}_2} & f_{\boldsymbol{p}_3} \end{vmatrix}. \end{aligned} \tag{9}$$

There are geometric perceptions for these invariants. In the spatial Cartesian coordinate, $(\boldsymbol{o}, g_{\boldsymbol{pq}})$ is the intersection point of the z-axis and the line passing through the points $(\boldsymbol{p}, f_{\boldsymbol{p}})$ and $(\boldsymbol{q}, f_{\boldsymbol{q}})$; $(\boldsymbol{o}, g_{\boldsymbol{pqr}})$ is the intersection point of the z-axis and the plane passing through the points $(\boldsymbol{p}, f_{\boldsymbol{p}})$, $(\boldsymbol{q}, f_{\boldsymbol{q}})$ and $(\boldsymbol{r}, f_{\boldsymbol{r}})$.

Let $G_2 = \{g_{\boldsymbol{p}_1\boldsymbol{p}_2} \mid (\boldsymbol{p}_1, \boldsymbol{p}_2) \in A_2\}$, $G_3 = \{g_{\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3} \mid (\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) \in A_3\}$, we can have

Lemma 3.2. $0 \leqslant \max G_2 \cup G_3 \leqslant \min\{\rho(\alpha,\beta) \mid \alpha, \beta > 0\}$.

Proof. Let $\boldsymbol{p} = (0, 1)$, $\boldsymbol{q} = (0, -1)$, it is clear that

$$0 = \big(\log(\omega_{01}/\omega_{00}) + \log(\omega_{00}/\omega_{01})\big)/2 \leqslant (f_{\boldsymbol{p}} + f_{\boldsymbol{q}})/2 = g_{\boldsymbol{pq}} \leqslant \max G_2 \leqslant \max G_2 \cup G_3.$$

For every $\alpha, \beta > 0$, $(\boldsymbol{p}_1, \boldsymbol{p}_2) \in A_2$, $(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) \in A_3$, with Eqs. (6), (8) and (9), we can obtain that

$$g_{\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3} = \frac{T(\boldsymbol{oq}_2\boldsymbol{q}_3)f_{\boldsymbol{q}_1}(\alpha,\beta) + T(\boldsymbol{oq}_3\boldsymbol{q}_1)f_{\boldsymbol{q}_2}(\alpha,\beta) + T(\boldsymbol{oq}_1\boldsymbol{q}_2)f_{\boldsymbol{q}_3}(\alpha,\beta)}{T(\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3)} \leqslant \rho(\alpha,\beta) \Rightarrow \max G_3 \leqslant \rho(\alpha,\beta);$$

$$g_{\boldsymbol{p}_1\boldsymbol{p}_2} = \frac{\|\boldsymbol{p}_2\| f_{\boldsymbol{p}_1}(\alpha,\beta) + \|\boldsymbol{p}_1\| f_{\boldsymbol{p}_2}(\alpha,\beta)}{\|\boldsymbol{p}_2\| + \|\boldsymbol{p}_1\|} \leqslant \rho(\alpha,\beta) \Rightarrow \max G_2 \leqslant \rho(\alpha,\beta).$$

$\Rightarrow \max G_2 \cup G_3 \leqslant \rho(\alpha,\beta) \Rightarrow \max G_2 \cup G_3 \leqslant \min\{\rho(\alpha,\beta) \mid \alpha, \beta > 0\}$. □

In addition to Lemma 3.2, one more lemma is needed for the solutions of the surface case, which is so easy that we omit the proof of it.

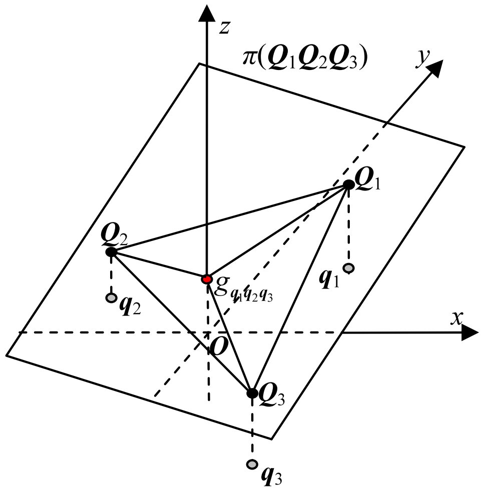

Fig. 4. Geometric perception of Theorem 3.4.

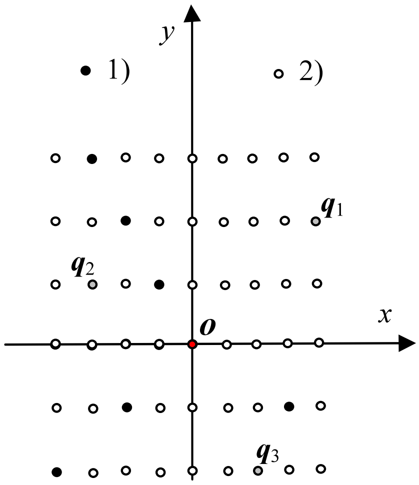

Fig. 5. Classification of a point $\boldsymbol{p} \in \Omega$.

Lemma 3.3. *If $\boldsymbol{P}_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, are non-collinear points in the spatial Cartesian coordinate, then the equation of the plane passing through $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$ can be written as*

$$\pi(\boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_3):\quad \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} x + \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \end{vmatrix} y + \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} z = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

To prove $\min\{\rho(\alpha, \beta) \mid \alpha, \beta > 0\} = \max G_2 \cup G_3$, we need to find α^*, β^* (> 0) such that $\rho(\alpha^*, \beta^*) = \max G_2 \cup G_3$. There are two cases, $\max G_3 \geqslant \max G_2$ and $\max G_2 > \max G_3$.

Theorem 3.4. *If $\max G_3 \geqslant \max G_2$, there exists $(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) \in A_3$, $\boldsymbol{q}_i = (k_i, l_i)$, $i = 1, 2, 3$, such that $g_{\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3} = \max G_3$. Let*

$$\big(\log(\alpha^*), \log(\beta^*)\big) = \frac{1}{T(\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3)} \left(\begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ f_{\boldsymbol{q}_1} & f_{\boldsymbol{q}_2} & f_{\boldsymbol{q}_3} \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 \\ f_{\boldsymbol{q}_1} & f_{\boldsymbol{q}_2} & f_{\boldsymbol{q}_3} \\ k_1 & k_2 & k_3 \end{vmatrix} \right),$$

we then have $\rho(\alpha^*, \beta^*) = \max G_2 \cup G_3 = \min\{\rho(\alpha, \beta) \mid \alpha, \beta > 0\}$.

Proof. According to Lemma 3.3 and Eq. (9), we know that the equation of the plane passing through the points $\mathbf{Q}_i = (\boldsymbol{q}_i, f_{\boldsymbol{q}_i})$, $i = 1, 2, 3$, is

$$\pi(\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3):\quad x\log(\alpha^*) + y\log(\beta^*) + z = g_{\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3} = \max G_3 = \max G_2 \cup G_3,$$

which leads to $f_{\boldsymbol{q}}(\alpha^*, \beta^*) = \boldsymbol{q} \cdot (\log(\alpha^*), \log(\beta^*)) + f_{\boldsymbol{q}} = \max G_2 \cup G_3$, $\boldsymbol{q} = \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$. (See Fig. 4.)

Then we need to prove that for every $\boldsymbol{p} = (k, l) \in \Omega$, $f_{\boldsymbol{p}}(\alpha^*, \beta^*) \leqslant \max G_2 \cup G_3$.

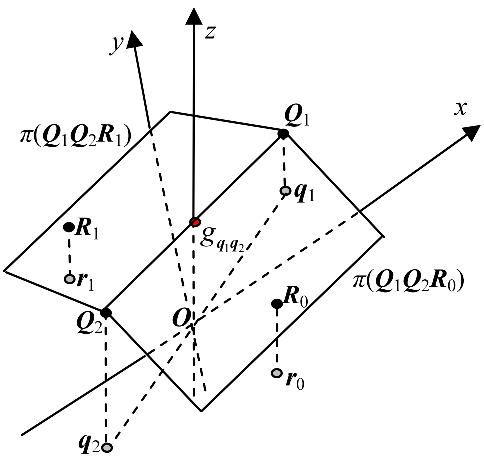

Fig. 6. Geometric perception of Theorem 3.6.

If $\boldsymbol{p} = (0, 0)$, then $f_{(0,0)}(\alpha^*, \beta^*) = f_{(0,0)} = 0 \leqslant \max G_2 \cup G_3$ (Lemma 3.2).

If $\boldsymbol{p} \neq (0, 0)$, then one of the following statements must be true (see Fig. 5): 1) there is a $\boldsymbol{q}_i$, such that $(\boldsymbol{p}, \boldsymbol{q}_i) \in A_2$; 2) there are $\boldsymbol{q}_j$, $\boldsymbol{q}_k$, such that $(\boldsymbol{p}, \boldsymbol{q}_j, \boldsymbol{q}_k) \in A_3$, $i, j, k \in \{1, 2, 3\}$. Suppose a point $(\boldsymbol{p}, f_{\boldsymbol{p}})$ is above the plane $\pi(\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3)$, with the geometric perceptions of $(\boldsymbol{o}, g_{\boldsymbol{p}\boldsymbol{q}_i})$ and $(\boldsymbol{o}, g_{\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3})$, we can deduce that the validity of statement 1) leads to $g_{\boldsymbol{p}\boldsymbol{q}_i} > g_{\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3} = \max G_3 \geqslant \max G_2$, a contradiction, and the validity of statement 2) leads to $g_{\boldsymbol{p}\boldsymbol{q}_j\boldsymbol{q}_k} > g_{\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3} = \max G_3$, also a contradiction. Hence every point $(\boldsymbol{p}, f_{\boldsymbol{p}})$, $\boldsymbol{p} \in \Omega$, is under or on the plane $\pi(\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3)$, that means $f_{\boldsymbol{p}}(\alpha^*, \beta^*) = \boldsymbol{p} \cdot (\log(\alpha^*), \log(\beta^*)) + f_{\boldsymbol{p}} \leqslant \max G_2 \cup G_3$.

With Lemma 3.2, we finally get $\rho(\alpha^*, \beta^*) = \max G_2 \cup G_3 = \min\{\rho(\alpha, \beta) \mid \alpha, \beta > 0\}$. $\square$

In a similar way as the proof of Theorem 2.4, we can get

Theorem 3.5. *If* $\max G_3 \geqslant \max G_2$, *there is only one pair of parameters* α^*, β^*, *given in Theorem* 3.4, *such that* $\rho(\alpha^*, \beta^*) = \min\{\rho(\alpha, \beta) \mid \alpha, \beta > 0\}$.

Before going to the next case, $\max G_2 > \max G_3$, we need two numbers a, b. Given $(\boldsymbol{q}_1, \boldsymbol{q}_2) \in A_2$, it's true that $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$, so there always exist two numbers a, b such that

$$f_{\boldsymbol{q}_1}(a, b) = \boldsymbol{q}_1 \cdot \big(\log(a), \log(b)\big) + f_{\boldsymbol{q}_1} = \boldsymbol{q}_2 \cdot \big(\log(a), \log(b)\big) + f_{\boldsymbol{q}_2} = f_{\boldsymbol{q}_2}(a, b). \tag{10}$$

For example, we can choose $(\log(a), \log(b)) = (f_{\boldsymbol{q}_1} - f_{\boldsymbol{q}_2})(\boldsymbol{q}_2 - \boldsymbol{q}_1)/\|\boldsymbol{q}_2 - \boldsymbol{q}_1\|^2$.

Theorem 3.6. *If* $\max G_2 > \max G_3$, *there exists* $(\boldsymbol{q}_1, \boldsymbol{q}_2) \in A_2$, $\boldsymbol{q}_i = (k_i, l_i)$, $i = 1, 2$, *such that* $g_{\boldsymbol{q}_1\boldsymbol{q}_2} = \max G_2$. *Let a, b be two numbers satisfying Eq.* (10), *and* $\boldsymbol{r}_i = (r_i, s_i) \in \Omega$, $i = 0, 1$, *such that*

$$\big(f_{\boldsymbol{r}_0}(a, b) - g_{\boldsymbol{q}_1\boldsymbol{q}_2}\big)/T(\boldsymbol{r}_0\boldsymbol{q}_1\boldsymbol{q}_2) = \max_{\substack{\boldsymbol{p}\in\Omega\\ T(\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2)>0}} \big\{\big(f_{\boldsymbol{p}}(a, b) - g_{\boldsymbol{q}_1\boldsymbol{q}_2}\big)/T(\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2)\big\}, \tag{11}$$
$$\big(f_{\boldsymbol{r}_1}(a, b) - g_{\boldsymbol{q}_1\boldsymbol{q}_2}\big)/T(\boldsymbol{r}_1\boldsymbol{q}_2\boldsymbol{q}_1) = \max_{\substack{\boldsymbol{p}\in\Omega\\ T(\boldsymbol{p}\boldsymbol{q}_2\boldsymbol{q}_1)>0}} \big\{\big(f_{\boldsymbol{p}}(a, b) - g_{\boldsymbol{q}_1\boldsymbol{q}_2}\big)/T(\boldsymbol{p}\boldsymbol{q}_2\boldsymbol{q}_1)\big\}.$$

Let $\alpha^*(i)$, $\beta^*(i)$, $i = 0, 1$, *be real numbers such that*

$$\big(\log\big(\alpha^*(i)\big), \log\big(\beta^*(i)\big)\big) = \frac{1}{T(\boldsymbol{r}_i\boldsymbol{q}_1\boldsymbol{q}_2)}\left(\begin{vmatrix} 1 & 1 & 1 \\ s_i & l_1 & l_2 \\ f_{\boldsymbol{r}_i} & f_{\boldsymbol{q}_1} & f_{\boldsymbol{q}_2} \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 \\ f_{\boldsymbol{r}_i} & f_{\boldsymbol{q}_1} & f_{\boldsymbol{q}_2} \\ r_i & k_1 & k_2 \end{vmatrix}\right), \quad i = 0, 1, \tag{12}$$

and let $\alpha^*(\lambda)$, $\beta^*(\lambda)$, $0 \leqslant \lambda \leqslant 1$, *be functions such that*

$$\big(\log\big(\alpha^*(\lambda)\big), \log\big(\beta^*(\lambda)\big)\big) = (1 - \lambda)\big(\log\big(\alpha^*(0)\big), \log\big(\beta^*(0)\big)\big) + \lambda\big(\log\big(\alpha^*(1)\big), \log\big(\beta^*(1)\big)\big). \tag{13}$$

We then have $\rho(\alpha^*(\lambda), \beta^*(\lambda)) = \max G_2 \cup G_3 = \min\{\rho(\alpha, \beta) \mid \alpha, \beta > 0\}$, $0 \leqslant \lambda \leqslant 1$.

Proof. With Lemma 3.3 and a few computations, we can obtain that the equation of the plane passing through the points $\boldsymbol{R}_0 = (\boldsymbol{r}_0, f_{\boldsymbol{r}_0})$, $\boldsymbol{Q}_i = (\boldsymbol{q}_i, f_{\boldsymbol{q}_i})$, $i = 1, 2$, is (see Fig. 6)

$$\pi(\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{R}_0):\quad x\log\big(\alpha^*(0)\big) + y\log\big(\beta^*(0)\big) + z = g_{\boldsymbol{q}_1\boldsymbol{q}_2} = \max G_2 = \max G_2 \cup G_3,$$

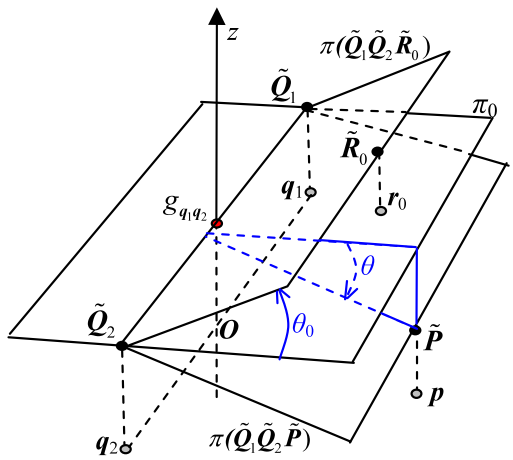

Fig. 7. Illustration of Eq. (11).

which leads to $f_{\boldsymbol{q}}(\alpha^*(0), \beta^*(0)) = \boldsymbol{q} \cdot (\log(\alpha^*(0)), \log(\beta^*(0))) + f_{\boldsymbol{q}} = \max G_2 \cup G_3$, $\boldsymbol{q} = \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_0$.

Then we want to prove that for every $\boldsymbol{p} = (k, l) \in \Omega$, $f_{\boldsymbol{p}}(\alpha^*(0), \beta^*(0)) \leqslant \max G_2 \cup G_3$.

If $\boldsymbol{p} = (0, 0)$, then $f_{(0,0)}(\alpha^*(0), \beta^*(0)) = f_{(0,0)} = 0 \leqslant \max G_2 \cup G_3$ (Lemma 3.2).

If $\boldsymbol{p} \neq (0, 0)$, then one of the following statements must be true: a) there is a $\boldsymbol{q} \in \{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_0\}$, such that $(\boldsymbol{p}, \boldsymbol{q}) \in A_2$; b) there are $\boldsymbol{q}, \boldsymbol{r} \in \{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_0\}$, such that $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}) \in A_3$; c) $T(\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2) > 0$.

In a similar way as the proof of Theorem 3.4, we can prove by apagoge that $f_{\boldsymbol{p}}(\alpha^*(0), \beta^*(0)) \leqslant \max G_2 \cup G_3$ if statement a) or b) is true. Thus we only need to show that $f_{\boldsymbol{p}}(\alpha^*(0), \beta^*(0)) = \boldsymbol{p} \cdot (\log(\alpha^*(0)), \log(\beta^*(0))) + f_{\boldsymbol{p}} \leqslant \max G_2 \cup G_3$, provided $\boldsymbol{p} \in \Omega$ and $T(\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2) > 0$.

Let φ be an affine transformation defined as $\varphi(x, y, z) = (x, y, z + x\log(a) + y\log(b))$, then for every $\boldsymbol{P} = (\boldsymbol{p}, f_{\boldsymbol{p}})$, $\boldsymbol{p} \in \Omega$, $\tilde{\boldsymbol{P}} = \varphi(\boldsymbol{P}) = (\boldsymbol{p}, f_{\boldsymbol{p}}(a, b))$. Since $f_{\boldsymbol{q}_1}(a, b) = f_{\boldsymbol{q}_2}(a, b)$, we can draw a plane π_0 that passes through $\tilde{\boldsymbol{Q}}_i = (\boldsymbol{q}_i, f_{\boldsymbol{q}_i}(a, b))$, $i = 1, 2$, and is perpendicular to the z-axis. For every $\tilde{\boldsymbol{P}}$ with $T(\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2) > 0$, let θ be the signed dihedral angle from π_0 to $\pi(\tilde{\boldsymbol{Q}}_1\tilde{\boldsymbol{Q}}_2\tilde{\boldsymbol{P}})$($\theta > 0$, if $\tilde{\boldsymbol{P}}$ is above π_0; $\theta < 0$, if $\tilde{\boldsymbol{P}}$ is under π_0). As the signed distance from $\tilde{\boldsymbol{P}}$ to π_0 is $f_{\boldsymbol{p}}(a, b) - g_{\boldsymbol{q}_1\boldsymbol{q}_2}$, and the height of the triangle $\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2$ on the edge $\boldsymbol{q}_1\boldsymbol{q}_2$ is $T(\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2)/\|\boldsymbol{q}_1\boldsymbol{q}_2\|$, the formula in brace on the right hand side of Eq. (11) is equivalent to $\tan\theta/\|\boldsymbol{q}_1\boldsymbol{q}_2\|$. Hence θ_0, the dihedral angle from π_0 to $\pi(\tilde{\boldsymbol{Q}}_1\tilde{\boldsymbol{Q}}_2\tilde{\boldsymbol{R}}_0)$, is the maximal one among these θ (see Fig. 7). So every $\tilde{\boldsymbol{P}} = \varphi(\boldsymbol{P})$ with $T(\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2) > 0$, must be under or on the plane $\pi(\tilde{\boldsymbol{Q}}_1\tilde{\boldsymbol{Q}}_2\tilde{\boldsymbol{R}}_0)$. Then from the definition of φ, we can correspondingly deduce that every $\boldsymbol{P} = (\boldsymbol{p}, f_{\boldsymbol{p}})$ with $T(\boldsymbol{p}\boldsymbol{q}_1\boldsymbol{q}_2) > 0$, is under or on the plane $\pi(\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{R}_0)$, which means $f_{\boldsymbol{p}}(\alpha^*(0), \beta^*(0)) = \boldsymbol{p} \cdot (\log(\alpha^*(0)), \log(\beta^*(0))) + f_{\boldsymbol{p}} \leqslant \max G_2 \cup G_3$.

With Lemma 3.2 we can obtain that $\rho(\alpha^*(0), \beta^*(0)) = \max G_2 \cup G_3 = \min\{\rho(\alpha, \beta) \mid \alpha, \beta > 0\}$. Similarly we can get that $\rho(\alpha^*(1), \beta^*(1)) = \max G_2 \cup G_3 = \min\{\rho(\alpha, \beta) \mid \alpha, \beta > 0\}$.

With Eq. (13), for every $\lambda \in [0, 1]$, $\boldsymbol{p} \in \Omega$, it is obvious that

$$f_{\boldsymbol{p}}\big(\alpha^*(\lambda), \beta^*(\lambda)\big) = \boldsymbol{p} \cdot \big(\log\big(\alpha^*(\lambda)\big), \log\big(\beta^*(\lambda)\big)\big) + f_{\boldsymbol{p}} = (1-\lambda) f_{\boldsymbol{p}}\big(\alpha^*(0), \beta^*(0)\big) + \lambda f_{\boldsymbol{p}}\big(\alpha^*(1), \beta^*(1)\big),$$

then we can obtain that $f_{\boldsymbol{q}}(\alpha^*(\lambda), \beta^*(\lambda)) = \max G_2 \cup G_3, \boldsymbol{q} = \boldsymbol{q}_1, \boldsymbol{q}_2$, and $f_{\boldsymbol{p}}(\alpha^*(\lambda), \beta^*(\lambda)) \leqslant \max G_2 \cup G_3$. With Lemma 3.2, we finally get the result $\rho(\alpha^*(\lambda), \beta^*(\lambda)) = \max G_2 \cup G_3 = \min\{\rho(\alpha, \beta) \mid \alpha, \beta > 0\}$. $\square$

Theorems 3.4, 3.6 provide the solutions of Problem 3.1. From Eq. (13), we can see that the solutions of the case $\max G_2 > \max G_3$ is generally lack of uniqueness, which is different from that of the case $\max G_3 \geqslant \max G_2$. Actually, according to the following Theorem 3.7, all the solutions of the case $\max G_2 > \max G_3$ are included in Eq. (13), for which we leave the proof to the readers.

Theorem 3.7. *If* $\max G_2 > \max G_3$, *let* $\alpha^*(i)$, $\beta^*(i)$, $i = 0, 1$, *be the positive real numbers given in Theorem* 3.6, *then we have*:

1) *If* $(\alpha^*(0), \beta^*(0)) = (\alpha^*(1), \beta^*(1))$, *the solution of Problem* 3.1 *is unique.*
2) *If* $(\alpha^*(0), \beta^*(0)) \neq (\alpha^*(1), \beta^*(1))$, *all the solutions of Problem* 3.1 *are included in Eq.* (13), *which are of infinite number.*

4. Numerical examples and discussions

In this section, we provide some numerical examples for showing the results of Theorems 2.3, 3.4, and 3.6 in improving the bounds of derivatives of rational Bézier curves and surfaces. In practical terms, computing max $G_2 \cup G_3$ for the surface case by mere comparisons is time consuming, so we present an efficient algorithm in the following for computing the value.

Algorithm 4.1 (*Computing* $\max G_2 \cup G_3$).

Step 1: Find a $\boldsymbol{p}_1$ such that $\boldsymbol{p}_1 = \arg\max\{f_{\boldsymbol{p}} \mid \boldsymbol{p} \in \Omega - \{\boldsymbol{o}\}\}$.
Step 2: Find a $\boldsymbol{p}_2$ such that $\boldsymbol{p}_2 = \arg\max_{\boldsymbol{p}\in\Omega-\{\boldsymbol{o}\}}\{(\sigma_1 f_{\boldsymbol{p}} - \sigma f_{\boldsymbol{p}_1})/(\sigma_1 - \sigma) \mid \sigma_1 = \boldsymbol{p}_1 \cdot \boldsymbol{p}_1 > \sigma = \boldsymbol{p} \cdot \boldsymbol{p}_1\}$.
Step 3: If $T(\boldsymbol{o}\boldsymbol{p}_1\boldsymbol{p}_2) \neq 0$, exchange the subscripts of $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ if needed to assure that $T(\boldsymbol{o}\boldsymbol{p}_1\boldsymbol{p}_2) > 0$.
Else if $(\boldsymbol{p}_1, \boldsymbol{p}_2) \in A_2$, then $\max G_2 \cup G_3 = \max G_2 = g_{\boldsymbol{p}_1\boldsymbol{p}_2}$, stop.
Else replace $\boldsymbol{p}_1$ by $\boldsymbol{p}_2$, and go to Step 2 to find a new $\boldsymbol{p}_2$.
Step 4: Find a $\boldsymbol{p}_3$ such that $\boldsymbol{p}_3 = \arg\max_{\boldsymbol{p}\in\Omega-\{\boldsymbol{o}\}}\{g_{\boldsymbol{p}\boldsymbol{p}_1\boldsymbol{p}_2} \mid T(\boldsymbol{p}\boldsymbol{p}_1\boldsymbol{p}_2) > 0\}$.
Step 5: If $(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) \in A_3$, then $\max G_2 \cup G_3 = \max G_3 = g_{\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3}$, stop.
Else replace $\boldsymbol{p}_1$ by $\boldsymbol{p}_3$ if $T(\boldsymbol{o}\boldsymbol{p}_2\boldsymbol{p}_3) \leqslant 0$ or replace $\boldsymbol{p}_2$ by $\boldsymbol{p}_3$ if $T(\boldsymbol{o}\boldsymbol{p}_3\boldsymbol{p}_1) \leqslant 0$, and go to Step 3.

Remark. The algorithm is a simulation of the process that a plane, which is originally perpendicular to the z-axis and above all points $\boldsymbol{P} = (\boldsymbol{p}, f_{\boldsymbol{p}})$, $\boldsymbol{p} \in \Omega$, is freely dropped down until it reaches the stable position that still covers all these points and passes through the points $\boldsymbol{Q} = (\boldsymbol{q}, f_{\boldsymbol{q}})$, $\boldsymbol{q} = \boldsymbol{p}_1, \boldsymbol{p}_2$, $(\boldsymbol{p}_1, \boldsymbol{p}_2) \in A_2$, or $\boldsymbol{q} = \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$, $(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) \in A_3$. From Theorems 3.4, 3.6, we know that the intersection point of the z-axis and the plane at the stable position is $(\boldsymbol{o}, \max G_2 \cup G_3)$.

Example 1. Given a degree 6 rational Bézier curve as Eq. (1) with weights $\{\omega_i \mid i = 0, 1, \ldots, 6\} = \{5, 2, 8, 56, 80, 96, 64\}$, from inequalities (3) and (4) we know that

$$\left\| d\boldsymbol{R}(t)/dt \right\| \leqslant 288 \max_{0\leqslant i,j\leqslant 6} \|\boldsymbol{R}_i - \boldsymbol{R}_j\|, \qquad \left\| d\boldsymbol{R}(t)/dt \right\| \leqslant 13\,824 \max_{0\leqslant i\leqslant 5} \|\boldsymbol{R}_i - \boldsymbol{R}_{i+1}\|.$$

From Theorem 2.3, we can derive the appropriate transformation parameter $\gamma^* = 0.5$. Applying the Möbius transformation with parameter γ^* to the rational Bézier curve, the weights of the curve is changed to $\{5, 1, 2, 7, 5, 3, 1\}$. Then according to inequalities (3) and (4) we can derive

$$\left\| d\boldsymbol{R}(s)/ds \right\| \leqslant 42 \max_{0\leqslant i,j\leqslant 6} \|\boldsymbol{R}_i - \boldsymbol{R}_j\|, \qquad \left\| d\boldsymbol{R}(s)/ds \right\| \leqslant 294 \max_{0\leqslant i\leqslant 5} \|\boldsymbol{R}_i - \boldsymbol{R}_{i+1}\|.$$

Example 2. Given a degree 3×3 rational Bézier surface as Eq. (2) with weights

$$\begin{bmatrix} \omega_{00} & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{20} & \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} = \begin{bmatrix} 5 & 18 & 27 & 27 \\ 2 & 42 & 36 & 270 \\ 8 & 60 & 72 & 216 \\ 32 & 72 & 144 & 216 \end{bmatrix},$$

from inequality (5), we can get that $\|\partial \boldsymbol{R}(u,v)/\partial u\| \leqslant 7\,381\,125 \max_{0\leqslant i,j,k,l\leqslant 3} \|\boldsymbol{R}_{ij} - \boldsymbol{R}_{kl}\|$.

From Algorithm 4.1, we can obtain that $\max G_2 \cup G_3 = \max G_3 = 1.9459$, then according to Theorem 3.4 we can derive the parameters $\alpha^* = 1/2$, $\beta^* = 1/3$. Applying the Möbius transformations with α^*, β^* to the rational Bézier surface, we can obtain from inequality (5) that $\|\partial \boldsymbol{R}(s,t)/\partial s\| \leqslant 1029 \max_{0\leqslant i,j,k,l\leqslant 3} \|\boldsymbol{R}_{ij} - \boldsymbol{R}_{kl}\|$.

Example 3. Given a degree 2×3 rational Bézier surface as Eq. (2) with weights

$$\begin{bmatrix} \omega_{00} & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{20} & \omega_{21} & \omega_{22} & \omega_{23} \end{bmatrix} = \begin{bmatrix} 1 & 7 & 5 & 3 \\ 12 & 16 & 12 & 4 \\ 8 & 16 & 4 & 8 \end{bmatrix},$$

according to inequality (5), we can get that $\|\partial \boldsymbol{R}(u,v)/\partial u\| \leqslant 8192 \max_{\substack{0\leqslant i,k\leqslant 2\\ 0\leqslant j,l\leqslant 3}} \|\boldsymbol{R}_{ij} - \boldsymbol{R}_{kl}\|$.

Appling Algorithm 4.1, we can obtain that $\max G_2 \cup G_3 = \max G_2 = 2.0794$, then we can derive the parameters $(\alpha^*(0), \beta^*(0)) = (2/3, 3/4)$, $(\alpha^*(1), \beta^*(1)) = (7/16, 8/7)$ from Theorem 3.6. Applying the Möbius transformations with parameters $\alpha^*(\lambda)$, $\beta^*(\lambda)$, $0 \leqslant \lambda \leqslant 1$ (derived from Eq. (13)) to the rational Bézier surface, we can obtain from inequality (5) that $\|\partial \boldsymbol{R}(s,t)/\partial s\| \leqslant 1024 \max_{\substack{0\leqslant i,k\leqslant 2\\ 0\leqslant j,l\leqslant 3}} \|\boldsymbol{R}_{ij} - \boldsymbol{R}_{kl}\|$.

From the three examples above, we can see that the bounds of the derivatives of rational Bézier curves and surfaces can be improved by applying the Möbius transformations with proposed parameters in Theorem 2.3, Theorems 3.4 and 3.6 respectively. Furthermore, the bounds of higher derivatives given in Wang and Tai (2008) can also be improved in a similar way. With sharper bounds of derivatives, some algorithms such as subdivision algorithm for the intersection of surfaces in Filip et al. (1986) can be improved. The proposed Möbius transformation can also be used to optimize the degree reduction of rational Bézier curves by evening the weights before applying degree reduction algorithm of polynomial curves to the rational ones in homogeneous coordinates (Cai and Wang, 2007).

5. Conclusions

In this paper, we provide a new algorithm for solving the problem of minimizing the maximal ratio of weights of a rational Bézier curve or surface. With geometric perception, the proof of the curve or surface case would be easily followed. What's more, for practical use, we present an efficient algorithm for computing max $G_2 \cup G_3$ in the surface case and give some numerical examples which demonstrate our results in improving the bounds of derivatives of rational Bézier curves and surfaces.

Acknowledgements

We would like to thank the reviewers for their valuable comments and suggestions. This work was supported by the National Natural Science Foundations of China under Grant No. 60873111 and No. 60933007.

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