DOD and DOA Estimation in Bistatic Non-Uniform Multiple-Input Multiple-Output Radar Systems

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***Abstract*—This letter investigates the joint estimation of the direction of departure (DOD) and direction of arrival (DOA) for multi-input multi-output (MIMO) radar systems. A novel estimation method based on non-uniform array configuration is proposed and the practical identifiability of the corresponding parameter is analyzed. The key idea is to use the Doppler diversity to construct a virtual MIMO array. Through the theoretical proof, we demonstrate that the proposed method can provide much stronger parameter identifiability than the conventional ones, and also can improve the parameter estimation performance. Numerical simulations verify the effectiveness of the proposed algorithm.**

***Index Terms*—Multiple-input multiple-output radar, non-uniform array, parameter identifiability, angle estimation.**

I. Introduction

JOINT estimation of direction of departure (DOD) and direction of arrival (DOA), as an important method for exploiting the opportunistic space-division multiple access (OSDMA) in wireless communication system [1] or moving target localization in multiple-input multiple-output (MIMO) radar system [2], [3], [4], [5], has been attracted lots of attention. Two-dimension multiple signal classification (MUSIC) algorithm [3] and its reduced dimension version [4] were exploited separately, both of which have almost the same performance. The rotational invariance technique have been studied in [5], however the performance is inferior to the MUSIC based algorithms. Moreover, the parameter identifiability also needs to be discussed elaborately, which is a natural pre-requisite for a well-posed estimation problem. The authors in [6] proved that the maximum number of targets that can be uniquely identified by co-located uniform MIMO radar (i.e., uniform linear array is utilized on both transmit and receive ends) is up to M times as that of the conventional uniform phased radar, where M is the number of transmit antennas. It also shows in [7] that the bistatic uniform MIMO radar with single pulse in a coherent process interval provides a identifiability upper bound $0.34M(N+1)$, where N is the number of receive antennas.

To further improve the parameter identifiability and the estimation performance, we utilize a bistatic non-uniform MIMO array in this letter. The non-uniform means that the antenna locations normalized by half carrier wavelength are not a series of consecutive integers. The typical examples are the minimum redundancy (MR) array [8], nested array [9] and coprime array [10]. We herein develop a novel joint DOD and DOA estimation algorithm, the kernel of which is to use the Doppler diversity to construct a large virtual MIMO array with more degree of freedom (DOF). The proposed scheme is proved to be possessed of much stronger parameter identifiability and much better estimation performance than the conventional uniform MIMO array. Meanwhile, it is also with lower computational complexity and requires no parameter pairing.

Notation: $(\cdot)^*$, $(\cdot)^T$, $(\cdot)^H$, $(\cdot)^\dagger$ denote the complex conjugate, transpose, Hermitian transpose, pseudo-inverse, respectively. Symbol "$\otimes$" denotes Kronecker product and "$\odot$" stands for Khatri-Rao product (column-wise Kronecker product). $\mathbf{I}_M$ is a $M \times M$ identity matrix and $\mathbf{0}$ symbolizes zero matrix. $\mathbf{A}^{(m)}$ is a submatrix of $\mathbf{A}$ formed by its last m rows.

II. Non-Uniform MIMO Radar for Target Localization

A. Data Model

Consider a bistatic MIMO radar system with M-antenna transmit array and N-antenna receive array, both of which are in non-uniform configuration. It is also assumed that there are K targets, and the output baseband signal of the matched filters at the receive array can be written as [4] [5] [7]

$$\mathbf{y}(t) = [\mathbf{b}(\phi_1)\otimes\mathbf{a}(\theta_1), \cdots, \mathbf{b}(\phi_K)\otimes\mathbf{a}(\theta_K)]\mathbf{h}(t) + \mathbf{z}(t) \quad (1)$$

where the transmit steering vector $\mathbf{a}(\theta_k)$ and the receive steering vector $\mathbf{b}(\phi_k)$, for $k = 1, 2, \cdots, K$, are assumed to be unchanged during a coherent processing interval (CPI), and θ_k, ϕ_k are the DOD and DOA of the kth target, respectively. The vector $\mathbf{h}(t) = [\gamma_1(t), \cdots, \gamma_K(t)]^T$ relies on the Doppler frequency f_{dk} and the radar cross section (RCS) coefficient β_k, i.e., $\gamma_k(t) = \beta_k e^{j2\pi f_{dk}(t-1)}$. Note that $\mathbf{a}(\theta_k) = [e^{j\pi l_1 \sin\theta_k}, \cdots, e^{j\pi l_M \sin\theta_k}]^T$ and $\mathbf{b}(\phi_k) = [e^{j\pi l_1 \sin\phi_k}, \cdots, e^{j\pi l_N \sin\phi_k}]^T$ are mutilated Vandemonde vectors, where l is the antenna location normalized by half carrier wavelength. $\mathbf{z}(t)$ is the additive zero-mean Gaussian noise with covariance σ_n^2.

Defining $\mathbf{A} = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_K)] \in \mathbb{C}^{M\times K}$, $\mathbf{B} = [\mathbf{b}(\phi_1), \cdots, \mathbf{b}(\phi_K)] \in \mathbb{C}^{N\times K}$, and after collecting Q consecutive pulses, (1) can be rewritten by the following compact form

$$\mathbf{Y} = (\mathbf{B}\odot\mathbf{A})\mathbf{H} + \mathbf{Z} \quad (2)$$

where $\mathbf{Y} = [\mathbf{y}(1), \cdots, \mathbf{y}(Q)]$, $\mathbf{H} = [\mathbf{h}(1), \cdots, \mathbf{h}(Q)]$ and $\mathbf{Z} = [\mathbf{z}(1), \cdots, \mathbf{z}(Q)]$.

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B. Virtual MIMO Array

Different from the conventional subspace-based algorithms, i.e., directly extracting noise subspace or signal subspace of estimated covariance matrix to perform MUSIC or ESPRIT algorithms, we adopt a new operation based on Doppler diversity, which is later proved to be greatly effective in improving parameter identifiability and angle estimation accuracy.

Assuming the Doppler frequencies satisfy $f_{d1} \neq \cdots \neq f_{dK}$ and are all well separated so that the target signals sampled by a sufficient large snapshot rate (at least twice the maximum Doppler frequency) are uncorrelated with each other. The correlation matrix of the observed data is given by

$$\begin{aligned}\mathbf{R_{YY}} &= E[\mathbf{YY}^H] \\ &= (\mathbf{B}\odot\mathbf{A})\mathbf{\Lambda}(\mathbf{B}\odot\mathbf{A})^H + \sigma_n^2\mathbf{I}_{MN}.\end{aligned} \tag{3}$$

Then we vectorize $\mathbf{R_{YY}}$ to get the following vector

$$\begin{aligned}\mathbf{r} &= vec(\mathbf{R_{YY}}) \\ &= [(\mathbf{B}\odot\mathbf{A})^*\odot(\mathbf{B}\odot\mathbf{A})]\mathbf{p} + \sigma_n^2\mathbf{1}\end{aligned} \tag{4}$$

where $vec(\cdot)$ is vectorizing operator. $\mathbf{p} = [\beta_1^2, \cdots, \beta_K^2]^T$, diagonal matrix $\mathbf{\Lambda} = diag[\mathbf{p}]$ and $\mathbf{1} = [\mathbf{e}_1^T, \cdots, \mathbf{e}_{MN}^T]^T$ with $\mathbf{e}_i$ being a vector of all zeros except a 1 at the ith position.

In order to discuss the parameter identifiability, we introduce the following lemma with respect to the Khatri-Rao product.

Lemma 1: For two matrices, $\mathbf{C}_1 \in \mathbb{C}^{N\times K}$ and $\mathbf{C}_2 \in \mathbb{C}^{M\times K}$, by defining the selection matrix, $\mathbf{\Pi} = \mathbf{I}_N \otimes \mathbf{\Gamma} \otimes \mathbf{I}_M$, where

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{\Gamma}_1 & & & \\ \mathbf{0}_{M\times1} & \mathbf{\Gamma}_1 & & \\ \vdots & \vdots & \ddots & \\ \mathbf{0}_{M\times1} & \mathbf{0}_{M\times1} & \dots & \mathbf{\Gamma}_1 \end{bmatrix}^{MN\times MN},$$

$$\mathbf{\Gamma}_1 = \begin{bmatrix} 1 & & & \\ \mathbf{0}_{1\times N} & 1 & & \\ \vdots & \vdots & \ddots & \\ \mathbf{0}_{1\times N} & \mathbf{0}_{1\times N} & \dots & 1 \end{bmatrix}^{M\times[N(M-1)+1]}$$

then the following equation holds,

$$\mathbf{\Pi}[(\mathbf{C}_1\odot\mathbf{C}_2)^*\odot(\mathbf{C}_1\odot\mathbf{C}_2)] = (\mathbf{C}_1^*\odot\mathbf{C}_1)\odot(\mathbf{C}_2^*\odot\mathbf{C}_2). \tag{5}$$

The above equation utilizes some properties of the Khatri-Rao product of multiple matrices, one can reference [12] for more detailed derivation.

According to the lemma above, after left-multiplying the selection matrix $\mathbf{\Pi}$ on (4), we can acquire a new observed data with the following form,

$$\tilde{\mathbf{r}} = \mathbf{\Pi}\mathbf{r} = [(\mathbf{B}^*\odot\mathbf{B})\odot(\mathbf{A}^*\odot\mathbf{A})]\mathbf{p} + \sigma_n^2\mathbf{\Pi}\mathbf{1}. \tag{6}$$

The vector $\tilde{\mathbf{r}}$ can be viewed as one-pulse baseband observation in a deterministic noise environment of a virtual MIMO radar with equivalent transmitting steering matrix $(\mathbf{A}^*\odot\mathbf{A})$ and equivalent receiving steering matrix $(\mathbf{B}^*\odot\mathbf{B})$.

Remark 1: The equivalent steering matrix $[(\mathbf{B}^*\odot\mathbf{B})\odot(\mathbf{A}^*\odot\mathbf{A})]$ is a kind of two dimension difference co-array, whose one dimension version was used in [9]. Taking the transmit steering vector $\mathbf{a}(\theta_k)$ for example, $[\mathbf{a}^*(\theta_k)\otimes\mathbf{a}(\theta_k)]$ implies that the virtual antennas locate in the integer set $\{l_i - l_j\}, i, j = 1, \cdots, M$. In order to obtain the maximum DOF through the non-uniform array, we use the minimum redundancy configuration on both transmit and receive ends. For example, if let $M = 4$ with antenna locations $l_i \in [0\ 1\ 4\ 6]$ at transmit end, the virtual antennas appear from -6 to 6 with only four redundancy items at location 0.

We define $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ as the virtual transmit and the virtual receive steering matrix after deleting the redundant items, which can be achieved by deleting the corresponding row observations in $\tilde{\mathbf{r}}$. Therefore, (6) can be further rewritten as

$$\bar{\mathbf{r}} = [\bar{\mathbf{B}}\odot\bar{\mathbf{A}}]\mathbf{p} + \sigma_n^2\mathbf{e}, \tag{7}$$

where the kth column of $\bar{\mathbf{B}}$ and $\bar{\mathbf{A}}$ have the following forms, $\bar{\mathbf{b}}(\phi_k) = [e^{-j\pi\bar{N}sin\phi_k}, \cdots, 1, \cdots, e^{j\pi\bar{N}sin\phi_k}]^T$ and $\bar{\mathbf{a}}(\theta_k) = [e^{-j\pi\bar{M}sin\theta_k}, \cdots, 1, \cdots, e^{j\pi\bar{M}sin\theta_k}]^T$. $\mathbf{e}$ is a zero column vector except a 1 in the middle.

C. Angle Estimation

When exploiting the subspace algorithm to achieve angle estimation, it is necessary to perform two dimension smoothing on (7). So we define the following $(\bar{M}+1)(\bar{N}+1)\times(\bar{M}+1)(\bar{N}+1)$ selection operator

$$\begin{aligned}\mathbf{\Xi}_{n,m} = {} & [\mathbf{0}_{(\bar{N}+1)\times(\bar{N}+1-n)}\ \mathbf{I}_{(\bar{N}+1)}\ \mathbf{0}_{(\bar{N}+1)\times(n-1)}] \\ & \otimes[\mathbf{0}_{(\bar{M}+1)\times(\bar{M}+1-m)}\ \mathbf{I}_{\bar{M}+1}\ \mathbf{0}_{(\bar{M}+1)\times(m-1)}]\end{aligned} \tag{8}$$

where $1 \le n \le \bar{N}+1$, $1 \le m \le \bar{M}+1$. Then stacking the observation vector $\bar{\mathbf{r}}$ after using the selecting operator, we have

$$\begin{aligned}\mathcal{L}(\bar{\mathbf{r}}) = {} & [\mathbf{\Xi}_{1,1}\bar{\mathbf{r}}\ \cdots\ \mathbf{\Xi}_{1,\bar{M}+1}\bar{\mathbf{r}}\ \ \mathbf{\Xi}_{2,1}\bar{\mathbf{r}} \\ & \cdots\ \mathbf{\Xi}_{2,\bar{M}+1}\bar{\mathbf{r}}\ \cdots\ \mathbf{\Xi}_{\bar{N}+1,\bar{M}+1}\bar{\mathbf{r}}].\end{aligned} \tag{9}$$

Then the two dimension smoothing operation yields

$$\bar{\mathbf{Y}} = \mathcal{L}(\bar{\mathbf{r}}) = [\bar{\mathbf{B}}^{(\bar{N}+1)}\odot\bar{\mathbf{A}}^{(\bar{M}+1)}]\bar{\mathbf{S}} + \sigma_n^2\mathbf{I}_{(\bar{M}+1)(\bar{N}+1)} \tag{10}$$

where the equivalent signal matrix $\bar{\mathbf{S}} = \mathbf{\Lambda}[\bar{\mathbf{B}}^{(\bar{N}+1)}\odot\bar{\mathbf{A}}^{(\bar{M}+1)}]^H$.

For convenience, we define $\tilde{\mathbf{A}} = \bar{\mathbf{B}}^{(\bar{N}+1)}\odot\bar{\mathbf{A}}^{(\bar{M}+1)}$. Illuminating by the almost surely full column rank of the Khatri-Rao product of two Vandermonde matrices [11], i.e., $rank[\mathbf{C}_1^{M_1\times K}\odot\mathbf{C}_2^{M_2\times K}] = \min(M_1M_2, K)$, we know that providing $(\bar{N}+1)(\bar{M}+1)\ge K$, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^H$ are all almost surely K rank. Considering the eigenvalue decomposition

$$\bar{\mathbf{Y}} = \mathbf{U\Sigma U}^H, \tag{11}$$

we can extract the $(\bar{N}+1)(\bar{M}+1)\times K$ signal subspace $\mathbf{U}_s$ according to the K largest eigenvalues. When disregarding of noise, $\mathbf{U}_s$ and $\tilde{\mathbf{A}}$ span the same signal subspace, namely $\mathbf{U}_s = \tilde{\mathbf{A}}\mathbf{T}$, where $\mathbf{T}$ is a nonsingular matrix. Without loss of generality, assume $\bar{N}\ge\bar{M}$, so we can partition $\mathbf{U}_s$ by the first $\bar{N}(\bar{M}+1)$ rows as $\mathbf{U}_1$ and the last $\bar{N}(\bar{M}+1)$ rows as $\mathbf{U}_2$, then we have $\mathbf{U}_1^\dagger\mathbf{U}_2 = \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}$. Obviously, $\mathbf{T}^{-1}$ can be calculated by the eigenvalue decomposition of $\mathbf{U}_1^\dagger\mathbf{U}_2$, and furthermore the least square solution of $\tilde{\mathbf{A}}$ can be given by

$$\tilde{\mathbf{A}}_{LS} = \mathbf{U}_s\mathbf{T}^{-1}. \tag{12}$$

The corresponding auto-paired angle estimations are

$$\hat{\theta}_k = asin\left(\frac{1}{\pi L_1}\sum_{p=1}^{\bar{N}+1}\sum_{q=2}^{\bar{M}+1} angle\left[\frac{\tilde{\mathbf{a}}_k(p,q)}{\tilde{\mathbf{a}}_k(p,q-1)}\right]\right) \quad (13)$$

$$\hat{\phi}_k = asin\left(\frac{1}{\pi L_2}\sum_{q=1}^{\bar{M}+1}\sum_{p=2}^{\bar{N}+1} angle\left[\frac{\tilde{\mathbf{a}}_k(p,q)}{\tilde{\mathbf{a}}_k(p-1,q)}\right]\right) \quad (14)$$

where $angle(\cdot)$ is the operation of getting the phase. $\tilde{\mathbf{a}}_k$ is the kth column of $\tilde{\mathbf{A}}_{LS}$, and $\tilde{\mathbf{a}}_k(p,q)$ is the corresponding $[(p-1)(\bar{M}+1)+q]$th element. $L_1 = \bar{M}(\bar{N}+1)$ and $L_2 = \bar{N}(\bar{M}+1)$.

III. Parameter Identifiability Analysis

Actually, (7) manifests the problem of two dimension frequency estimation, which has been discussed in [11]. However its conclusion is not appropriate to our situation because the special structure of (7), i.e., $\bar{\mathbf{r}} = \bar{\mathbf{r}}^*$, cannot be further used to exploit the backward permutation matrix. We herein provide the parameter identifiability by the following theorem.

Theorem 1: Given the non-uniform MIMO radar model (2) with K distinct Doppler frequencies, after the transformation to (6) by lemma 1, and further to (10), the parameter set (θ_k, ϕ_k), for $k = 1, 2, \cdots, K$, can be uniquely identified if

$$K \leq \bar{N}(\bar{M}+1) \quad (15)$$

where the parameters set is drawn from a continuous distribution with respect to the Lebesgue measure in $\mathbb{L}^{2K}$, $\mathbb{L} := [-\pi/2, \pi/2]$.

Proof: Suppose the parameters are all drawn from a continuous distribution. Retrospecting the model described in (10), according to the almost surely full column rank of the Khatri-Rao product of two Vandermonde matrices, we know that the requirement of $rank(\tilde{\mathbf{A}}) = K$ is $K \leq (\bar{M}+1)(\bar{N}+1)$. In addition, the calculation of $\mathbf{T}$ in (12) by partitioning the signal subspace $\mathbf{U_s}$ requires $K \leq \bar{N}(\bar{M}+1)$. Therefore, the proposed joint DOD and DOA estimation algorithm requires $K \leq \bar{N}(\bar{M}+1)$ to guarantee the parameter identifiability with probability one. ∎

Remark2: In some very special cases, it has $rank(\tilde{\mathbf{A}}) < K$ even though $K \leq \bar{N}(\bar{M}+1)$, however the theorem tell us that such cases are measure-zero events, that is to say, $\tilde{\mathbf{A}}$ is almost surely full column rank.

Previous result on the maximum upper bound of the parameter identifiability such as MUSIC-like or rotational invariance algorithms is $MN-1$ when the uniform array configuration is used. Due to the non-uniform array configuration in the proposed method, it usually satisfies $\bar{M} > M$ and $\bar{N} > N$, showing a much stronger identifiability. It is worth mentioning that although the upper bound of parameter identifiability is derived under the noise-less case, it still works under the limited snapshot number and lower signal-to-noise ratio (SNR) case.

IV. Numerical Examples

We present some numerical simulations to demonstrate the effectiveness and advantages of the proposed algorithm under the non-uniform MIMO radar configuration. The average root mean squared error (RMSE) is defined as $(1/K)\sum_{k=1}^{K}\sqrt{E\left[(\hat{\theta}_k-\theta_k)^2+(\hat{\phi}_k-\phi_k)^2\right]}$ to assess the estimation performance. All the numerical results are obtained from 1000 independent trials.

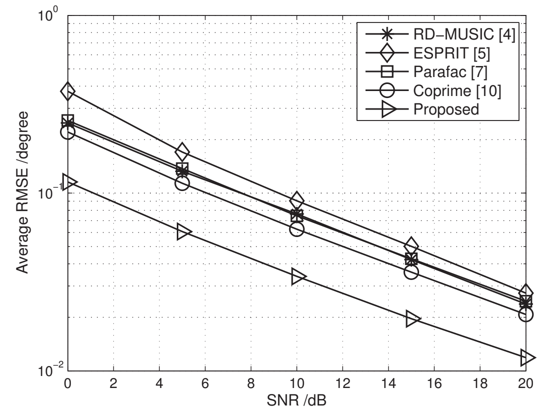


Fig. 1: The average RMSE performance comparison between different methods.

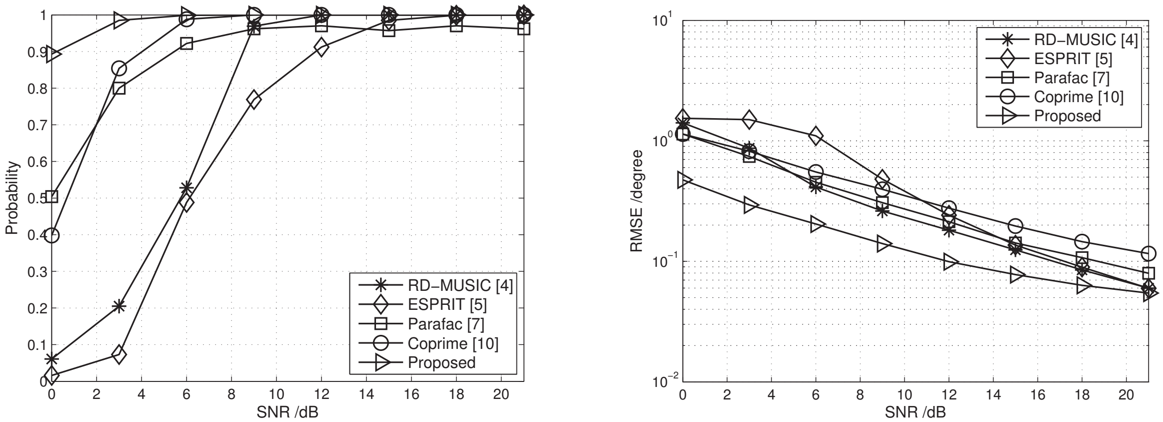
First, the performance comparison between different methods is shown in Fig. 1, where three targets locate at $\theta \in \{10°, 20°, 30°\}$, $\phi \in \{15°, 25°, 35°\}$ with $\beta \in \{1, 0.8, 0.5\}$ and $Q = 100$. The Doppler frequencies are well separated. We use $M = N = 7$ MR configuration in our method, $M = N = 8$ comprime configuration in [10] and $M = N = 8$ uniform configuration in [4], [5], and [7]. The results manifest that the non-uniform MIMO radar outperforms the uniform one. Furthermore, the MR configuration used in our method performs the best. Such performance improvement mainly benefits from the extended DOFs of the non-uniform array.

For large target number case, we consider ten targets with equal angle spacing $\theta \in \{45°, \cdots, -30°\}$ and $\phi \in \{-35°, \cdots, 40°\}$. The β and f_d hold linearly spaced values from 0.8 to 0.3 and 230Hz to 3170Hz, respectively. The snapshot number $Q = 200$. Other parameters are similar to the first example. A target is said to be localized successfully in a given trial if $|\hat{\gamma}-\gamma| \leq \epsilon, \gamma \in \{\theta, \phi\}$. When focussing on the minimum power target, see Fig. 2(a), the proposed method keeps a much higher successful estimation probability than others, especially at lower SNR. Fig. 2(b) depicts the RMSE after excluding the failed trials. Here we choose $\epsilon = 2°$.

Following the second example, we show the potential identifying ability under two cases: $M = N = 4$ and $M = 5, N = 6$, each one of which has MR and uniform configuration, and $Q = 500$, SNR= 20dB. Fig. 3 proves that when the target number increases, the uniform MIMO radar breaks down faster than the proposed non-uniform MIMO radar. Besides, we also notice that both of them cannot attain their theoretical identifying bound in this scenario, however the improvement of the proposed method is salient.

V. Conclusion

In this letter, we have discussed the joint DOD and DOA estimation in bistatic non-uniform MIMO radar system. We proved in theory that the proposed algorithm based on non-uniform MIMO radar can unveil a much stronger parameter



(a) (b)

Fig. 2: Performance comparison. (a) The probability of successful estimation; (b) the corresponding RMSE.

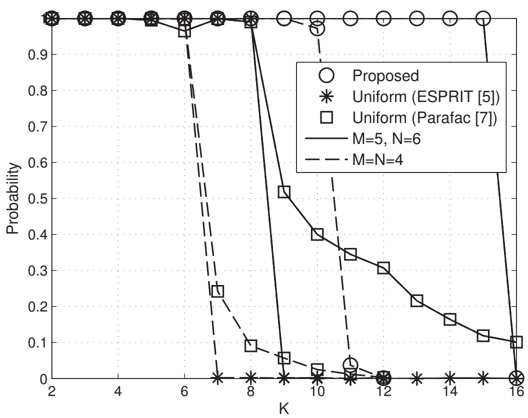


Fig. 3: The successful estimation probability with different target number.

identifiability than the conventional uniform MIMO radar. In addition, compared with the previous estimation algorithms, it achieved a superior performance when the same number of transmit/receive antennas is used.

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