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ORIGINAL PAPER

Stability and Hopf bifurcation of a neural network model with distributed delays and strong kernel

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Abstract In this paper, the dynamical behaviors of a two-neuron network model with distributed delays and strong kernel are investigated. Considering the mean delay as a bifurcation parameter, explicit algorithms for determining the conditions of Hopf bifurcation are derived. A family of periodic solutions bifurcate from an equilibrium when the bifurcation parameter exceeds a critical value. The direction and stability of the bifurcating periodic solutions are determined in detail by using the theory of normal form and center manifold. Numerical simulations are performed to illustrate the effectiveness of the results found.

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1 Introduction

Neural network is the abstraction and modeling of the human brain or biological neural network. In many practical models, it is important to take into account time delay, and dynamical characteristics of neural network models with time delay have become an intense subject in research activities [1]. Considering that most neural network models are large-scale and nonlinear systems, it is difficult to analyze the dynamical behaviors of these network models. Many simplified neural network models are developed, and the corresponding bifurcating dynamics have been investigated (see [1–7]). These studies are necessary for theoretical and practical use.

In neural network, the periodic activities of neural impulses are of fundamental significance in the control of regular dynamical behaviors. It is very important to understand these periodic activities of neural network [1]. The stability and periodic phenomenon of neuron models and predator–prey systems have also been widely studied [8–16].

Recently, several simple neuron models with discrete or distributed delays are proposed [1,5–7,17]. For

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example, Liao et al. [1] proposed a two-neuron system with distributed delays and without self-connections,

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_{12} f \\ \qquad \times \left[x_2(t) - b_2 \int_0^\infty F(s)x_2(t-s)\mathrm{d}s - c_2\right], \\ \dot{x}_2(t) = -x_2(t) + a_{21} f \\ \qquad \times \left[x_1(t) - b_1 \int_0^\infty F(s)x_1(t-s)\mathrm{d}s - c_1\right], \end{cases} \tag{1}$$

which displays a Hopf bifurcation.

Cheng et al. [4] proposed a two-neuron system with self-connections and weak kernel,

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_{11} f \left[x_1(t) - b_1 \int_0^\infty F(s)x_1(t-s)\mathrm{d}s - c_1\right] \\ \qquad + a_{12} f \left[x_2(t) - b_2 \int_0^\infty F(s)x_2(t-s)\mathrm{d}s - c_2\right], \\ \dot{x}_2(t) = -x_2(t) + a_{21} f \left[x_1(t) - b_1 \int_0^\infty F(s)x_1(t-s)\mathrm{d}s - c_1\right] \\ \qquad + a_{22} f \left[x_2(t) - b_2 \int_0^\infty F(s)x_2(t-s)\mathrm{d}s - c_2\right]. \end{cases} \tag{2}$$

with

$F(s) = \alpha e^{-\alpha s}$ $(\alpha > 0)$,

which means the neural network is with weak kernel. Parameters a_{ij}, b_i and c_i $(i, j = 1, 2)$ are nonnegative; x_i denotes the mean soma potential of the neuron; a_{ij} corresponds to the range of continuous variable x_i; b_i is the measure of the inhibitory influence of the past history; c_i denotes the neuronal threshold; x_i in the argument of the function f represents local positive feedback. The weight function $F(\cdot)$, nonnegative bounded, is defined on $[0, \infty)$ and reflects the past state on the current dynamics.

However, considering delay network models with self-connections and strong kernel, little work can be found in the existing literature. Our aim in this paper is to study the stability and Hopf bifurcations of the system (2) with strong kernel:

$F(s) = \alpha^2 s e^{-\alpha s}$ $(\alpha > 0)$.

The rest of this paper is organized as follows. In Sect. 2, local stability and the existence of Hopf bifurcation are investigated. In Sect. 3, by means of the normal form method and the theory of center manifold introduced by Hassard et al. [18], the direction, stability and the period of the bifurcating periodic solutions are determined at the critical value. In Sect. 4, numerical simulations are performed to verify the theoretic results. The conclusion is given in Sect. 5.

2 Local stability and Hopf bifurcation

In this section, system (2) with a strong kernel will be discussed, that is

$F(s) = \alpha^2 s e^{-\alpha s}$ $(\alpha > 0)$.

For simplicity, set $c_1 = c_2 = 0$. Similar conclusions can also be given if $c_1 \neq 0$ or $c_2 \neq 0$.

Let

$$y_1(t) = x_1(t) - b_1 \int_0^\infty F(s)x_1(t-s)\mathrm{d}s,$$
$$y_2(t) = x_2(t) - b_2 \int_0^\infty F(s)x_2(t-s)\mathrm{d}s.$$

System (2) becomes the following form:

$$\begin{cases} \dot{y}_1(t) = -y_1(t) + a_{11} f[y_1(t)] \\ \qquad - a_{11}b_1 \int_{-\infty}^0 F(-s) f[y_1(t+s)]\mathrm{d}s \\ \qquad + a_{12} f[y_2(t)] \\ \qquad - a_{12}b_1 \int_{-\infty}^0 F(-s) f[y_2(t+s)]\mathrm{d}s, \\ \dot{y}_2(t) = -y_2(t) + a_{21} f[y_1(t)] \\ \qquad - a_{21}b_2 \int_{-\infty}^0 F(-s) f[y_1(t+s)]\mathrm{d}s \\ \qquad + a_{22} f[y_2(t)] \\ \qquad - a_{22}b_2 \int_{-\infty}^0 F(-s) f[y_2(t+s)]\mathrm{d}s. \end{cases} \tag{3}$$

We investigate system (3) under the hypothesis:

(H1) $f \in C^4$, $f(0) = 0$, $f'(0) \neq 0$, and $\mu f(\mu) > 0$ for $\mu \neq 0$.

The equilibrium (0,0) of system (2) exists if

$$[1 - a_{11}(1 - b_1) f'(0)][1 - a_{22}(1 - b_2) f'(0)] \\ > a_{12}a_{21}(1 - b_1)(1 - b_2)[f'(0)]^2.$$

Under the hypothesis (H_1), the linearized form of system (3) at equilibrium $(0, 0)$ is

$$\begin{cases} \dot{y}_1(t) = -y_1(t) + c_{11}y_1(t) \\ \qquad - c_{11}b_1 \int_{-\infty}^{0} F(-s)y_1(t+s)\mathrm{d}s \\ \qquad + c_{12}y_2(t) - c_{12}b_1 \int_{-\infty}^{0} F(-s)y_2(t+s)\mathrm{d}s, \\ \dot{y}_2(t) = -y_2(t) + c_{21}y_1(t) \\ \qquad - c_{21}b_2 \int_{-\infty}^{0} F(-s)y_1(t+s)\mathrm{d}s \\ \qquad + c_{22}y_2(t) - c_{22}b_2 \int_{-\infty}^{0} F(-s)y_2(t+s)\mathrm{d}s, \end{cases} \tag{4}$$

where $c_{ij} = a_{ij} f'(0), i, j = 1, 2$.

The associated characteristic equation of system (4) is

$$det \begin{pmatrix} \lambda+1-c_{11}\left[1-b_1\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] & -c_{12}\left[1-b_1\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] \\ -c_{21}\left[1-b_2\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] & \lambda+1-c_{22}\left[1-b_2\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] \end{pmatrix} = 0. \tag{5}$$

The equilibrium solution is asymptotically stable if and only if all roots of (5) have negative real parts. The characteristic equation is equivalent to the following form:

$$\lambda^6 + m_1(\alpha)\lambda^5 + m_2(\alpha)\lambda^4 + m_3(\alpha)\lambda^3 + m_4(\alpha)\lambda^2 + m_5(\alpha)\lambda + m_6(\alpha) = 0, \tag{6}$$

where

$$\begin{cases} m_1(\alpha) = 4\alpha + 2 - c_{11} - c_{22}, \\ m_2(\alpha) = 6\alpha^2 + 4(2 - c_{11} - c_{22})\alpha \\ \qquad + (1 - c_{11})(1 - c_{22}) - c_{12}c_{21}, \\ m_3(\alpha) = 4\alpha^3 + (12 - 6c_{11} - 6c_{22} + c_{11}b_1 + c_{22}b_2)\alpha^2 \\ \qquad + 4[(1 - c_{11})(1 - c_{22}) - c_{12}c_{21}]\alpha, \\ m_4(\alpha) = \alpha^4 + 2(4 - 2c_{11} - 2c_{22} + c_{11}b_1 + c_{22}b_2)\alpha^3 \\ \qquad + [6(1 - c_{11})(1 - c_{22}) + (1 - c_{11})c_{22}b_2 \\ \qquad + (1 - c_{22})c_{11}b_1 - c_{12}c_{21}(6 - b_1 - b_2)]\alpha^2, \\ m_5(\alpha) = (2 - c_{11} - c_{22} + c_{11}b_1 + c_{22}b_2)\alpha^4 \\ \qquad + 2[2(1 - c_{11})(1 - c_{22}) \\ \qquad + (1 - c_{11})c_{22}b_2 + (1 - c_{22})c_{11}b_1 \\ \qquad - c_{12}c_{21}(2 - b_1 - b_2)]\alpha^3, \\ m_6(\alpha) = [(1 - c_{11} + c_{11}b_1)(1 - c_{22} \\ \qquad + c_{22}b_2) - c_{12}c_{21}(1 - b_1)(1 - b_2)]\alpha^4. \end{cases} \tag{7}$$

By using the Hurwitz criterion, all roots of the characteristic Eq. (5) have negative real parts if and only if the following conditions hold:

$$\begin{cases} D_1(\alpha) = m_1(\alpha) > 0, \\ D_2(\alpha) = m_1(\alpha)m_2(\alpha) - m_3(\alpha) > 0, \\ D_3(\alpha) = m_1^2(\alpha)m_4(\alpha) + m_1(\alpha)m_2(\alpha)m_3(\alpha) \\ \qquad + m_1(\alpha)m_5(\alpha) - m_3^2(\alpha) > 0, \\ D_4(\alpha) = m_1^2(\alpha)m_2(\alpha)m_6(\alpha) - m_1^2(\alpha)m_4^2(\alpha) \\ \qquad - m_1(\alpha)m_2^2(\alpha)m_5(\alpha) \\ \qquad + m_1(\alpha)m_2(\alpha)m_3(\alpha)m_4(\alpha) \\ \qquad - m_1(\alpha)m_3(\alpha)m_6(\alpha) \\ \qquad + 2m_1(\alpha)m_4(\alpha)m_5(\alpha) \\ \qquad + m_2(\alpha)m_3(\alpha)m_5(\alpha) - m_3^2(\alpha)m_4(\alpha) \\ \qquad - m_5^2(\alpha) > 0, \\ D_5(\alpha) = -m_1^3(\alpha)m_6^2(\alpha) + 2m_1^2(\alpha)m_2(\alpha)m_5(\alpha)m_6(\alpha) \\ \qquad + m_1^2(\alpha)m_3(\alpha)m_4(\alpha)m_6(\alpha) \\ \qquad - m_1^2(\alpha)m_4(\alpha)m_5(\alpha) - m_1(\alpha)m_2^2(\alpha)m_5^2(\alpha) \\ \qquad - m_1(\alpha)m_2(\alpha)m_3^2(\alpha)m_6(\alpha) \\ \qquad + m_1(\alpha)m_2(\alpha)m_3(\alpha)m_4(\alpha)m_5(\alpha) \\ \qquad - 3m_1(\alpha)m_3(\alpha)m_5(\alpha)m_6(\alpha) + m_3^3(\alpha)m_6(\alpha) \\ \qquad + m_2(\alpha)m_3(\alpha)m_5^2(\alpha) - m_3^2(\alpha)m_4(\alpha)m_5(\alpha) \\ \qquad - m_5^3(\alpha) + 2m_1(\alpha)m_4(\alpha)m_5^2(\alpha), \\ D_6(\alpha) = m_6(\alpha)D_5(\alpha) > 0. \end{cases} \tag{8}$$

Make the following hypothesis.

(H2) $m_1(\alpha) > 0, m_6(\alpha) > 0, m_3(\alpha) < m_1(\alpha)m_2(\alpha)$, $D_3(\alpha) > 0, D_4(\alpha) > 0, D_5(\alpha) > 0$.

Theorem 2.1 *If the parameters* $\alpha, c_{11}, c_{12}, c_{21}, c_{22}, b_1, b_2$ *take values such that (H2) is satisfied, the equilibrium* $(0, 0)$ *of system* (4) *is locally asymptotically stable.*

Let $\lambda = \omega_0 \mathrm{i}\ (\omega_0 > 0)$ *be the root of* (6). *We then have*

$$\omega_0^6 - \mathrm{i}\omega_0^5 m_1(\alpha) - \omega_0^4 m_2(\alpha) + \mathrm{i}\omega_0^3 m_3(\alpha) + \omega_0^2 m_4(\alpha) - \mathrm{i}\omega_0 m_5(\alpha) - m_6(\alpha) = 0. \tag{9}$$

Separating the real and imaginary parts gives

$$\begin{cases} \omega_0^4 m_1(\alpha) - \omega_0^2 m_3(\alpha) + m_5(\alpha) = 0, \\ \omega_0^6 - \omega_0^4 m_2(\alpha) + \omega_0^2 m_4(\alpha) - m_6(\alpha) = 0. \end{cases} \tag{10}$$

After some complicated calculation (more details are referred to "Appendix"), we get

$$D_5(\alpha) = 0. \tag{11}$$

If there exists $\alpha_0 \in \mathbf{R}^+$ satisfying $D_5(\alpha) = 0$, then Eq. (6) has a pair of purely imaginary roots $\pm i\omega_0$. Taking the derivative of $Re(\lambda)$, the real part of the root of (6), with respect to the parameter α at the value α_0 yields

$$\frac{d}{d\alpha}[Re\lambda]|_{\alpha_0} = -\frac{2\left[m_3(m_3+\sqrt{m_3^2-4m_1m_5})-4m_1m_5\right]}{m_1\cdot M(\alpha_0)\cdot N(\alpha_0)} \times \frac{dD_5(\alpha)}{d\alpha}|_{\alpha_0}, \tag{12}$$

where

$$\begin{cases} M(\alpha_0) = \left(2 - m_1^3m_6 + 3m_1m_3m_5 - 2m_1^2m_2m_5 - m_3^3 + m_1m_2m_3^2 - m_1^2m_3m_4\right) + \sqrt{m_3^2-4m_1m_5} \times \left(m_3^2 - m_1m_2m_3 + m_1^2m_4 - m_1m_5\right), \\ N(\alpha_0) = \left(5m_1\omega_0^4 - 3m_3\omega_0^2 + m_5\right)^2 + \left(6\omega_0^5 - 4m_2\omega_0^3 + 2m_4\omega_0\right)^2, \\ \omega_0^2 = \dfrac{m_3+\sqrt{m_3^2-4m_1m_5}}{2m_1}, \end{cases} \tag{13}$$

and $m_i = m_i(\alpha_0)$ $(i = 1, 2, 3, 4, 5, 6)$ [see "Appendix" for (11–13) in detail].

We consider the conditions so that $\frac{d}{d\alpha}[Re\lambda]|_{\alpha_0} \neq 0$ holds.

(H3) $m_3^2 - 4m_1m_5 \neq 0$, and $\frac{dD_5(\alpha)}{d\alpha}|_{\alpha_0} \neq 0$.

Thus, we have the following results.

Theorem 2.2 *Under the hypotheses (H1) and (H2), if there exists $\alpha_0 \in \mathbf{R}^+$ such that $D_5(\alpha_0) = 0$ and $\frac{dD_5(\alpha)}{d\alpha}|_{\alpha_0} \neq 0$, system (4) undergoes a Hopf bifurcation at the equilibrium $(0, 0)$ when α passes through α_0.*

Note that condition (H2) is complex, which is the direct deduction of Hurwitz criterion. For convenience, we give simple and readable conditions under the following hypothesis.

(H4) $c_{11}c_{22} = c_{12}c_{21}$.

The associated characteristic equation becomes (Please see "Appendix" for more details)

$$(\lambda+1)(\lambda+\alpha)^2\left[\lambda^3 + m_1(\alpha)\lambda^2 + m_2(\alpha)\lambda + m_3(\alpha)\right] = 0, \tag{14}$$

where

$$\begin{cases} m_1(\alpha) = 2\alpha + (1 - c_{11} - c_{22}), \\ m_2(\alpha) = \alpha[\alpha + 2(1 - c_{11} - c_{22})], \\ m_3(\alpha) = [(1 - c_{11} - c_{22}) + (c_{11}b_1 + c_{22}b_2)]\alpha^2. \end{cases} \tag{15}$$

According to the Hurwitz criterion, all roots of the characteristic Eq. (5) have negative real parts if and only if:

$$\begin{cases} D_1(\alpha) = m_1(\alpha) > 0, \\ D_2(\alpha) = m_1(\alpha)m_2(\alpha) - m_3(\alpha) > 0, \\ D_3(\alpha) = m_1(\alpha)D_3(\alpha) > 0. \end{cases} \tag{16}$$

We have

$$\begin{cases} D_1(\alpha) = 2\alpha + (1 - c_{11} - c_{22}), \\ D_2(\alpha) = \alpha\{2\alpha^2 + [4(1 - c_{11} - c_{22}) - (c_{11}b_1 + c_{22}b_2)]\alpha + 2(1 - c_{11} - c_{22})\}. \end{cases} \tag{17}$$

Consider

(H5) $c_{11}b_1 + c_{22}b_2 \geq 8(1 - c_{11} - c_{22}) > 0$, and then we have the following results.

Corollary 2.1 *If (H1), (H4) and (H5) hold, there exists $\alpha_0 \in \mathbf{R}^+$ such that $D_2(\alpha_0) = 0$. If $\frac{dD_2(\alpha)}{d\alpha}|_{\alpha_0} \neq 0$, system (4) undergoes a Hopf bifurcation at the equilibrium $(0, 0)$ when α passes through α_0, where*

$$\alpha_0 = \frac{(c_{11}b_1 + c_{22}b_2) - 4(1 - c_{11} - c_{22}) \pm \sqrt{(c_{11}b_1 + c_{22}b_2)^2 - 8(1 - c_{11} - c_{22})}}{4}. \tag{18}$$

Remark 1 Compared with conditions (H2, H3), it is found that (H4, H5) are more simple. One can choose suitable parameters to make Hopf bifurcation occur.

3 Direction and stability of Hopf bifurcation

In Sect. 2, we obtain the conditions of existence of Hopf bifurcation. In this section, we will investigate the direction of Hopf bifurcation and stability of periodic solutions bifurcating from the equilibrium (0, 0). The approaches used here are the normal form method and center manifold theorem introduced by Hassard et al. [18].

The nonlinear form of system (3) can be written as:

$$\dot{y}(t) = Ly(t) + \int_{-\infty}^{0} K(s)y(t+s)\mathrm{d}s + G(y), \tag{19}$$

$$L = \begin{pmatrix} -1 + c_{11} & c_{12} \\ c_{21} & -1 + c_{22} \end{pmatrix}, \tag{20}$$

$$K(s) = \begin{pmatrix} -c_{11}b_1F(-s) & -c_{12}b_1F(-s) \\ -c_{21}b_2F(-s) & -c_{22}b_2F(-s) \end{pmatrix}, \tag{21}$$

$$G(y) = \begin{pmatrix} c_{11}^{(2)}\left[y_1^2(t) - b_1\int_{-\infty}^{0} F(-s)y_1^2(t+s)\mathrm{d}s\right] \\ \quad + c_{12}^{(2)}\left[y_2^2(t) - b_1\int_{-\infty}^{0} F(-s)y_2^2(t+s)\mathrm{d}s\right] + \cdots \\ c_{21}^{(2)}\left[y_1^2(t) - b_2\int_{-\infty}^{0} F(-s)y_1^2(t+s)\mathrm{d}s\right] \\ \quad + c_{22}^{(2)}\left[y_2^2(t) - b_2\int_{-\infty}^{0} F(-s)y_2^2(t+s)\mathrm{d}s\right] + \cdots \end{pmatrix}, \tag{22}$$

where $c_{ij}^{(2)} = \frac{1}{2}a_{ij}f''(0), i, j = 1, 2.$

We can rewrite (19) into an operator equation

$$\dot{y}_t = \mathcal{A}(\mu)y_t + \mathcal{R}(\mu)y_t, \tag{23}$$

where $y_t = (y_1(t), y_2(t))^T y_t = y(t + \theta), \theta \in (-\infty, 0], \mu = \alpha - \alpha_0$. We define the involved operators $\mathcal{A}$ and $\mathcal{R}$ as

$$\mathcal{A}(\mu)\phi(\theta) = \begin{cases} \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in (-\infty, 0), \\ L\phi(0) + \int_{-\infty}^{0} K(s)\phi(s)\mathrm{d}s, & \theta = 0, \end{cases} \tag{24}$$

and

$$\mathcal{R}(\mu)\phi(\theta) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \theta \in (-\infty, 0), \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, & \theta = 0, \end{cases} \tag{25}$$

where

$$\begin{cases} f_1 = c_{11}^{(2)}\left[\phi_1^2(0) - b_1\int_{-\infty}^{0} F(-s)\phi_1^2(s)\mathrm{d}s\right] \\ \quad + c_{12}^{(2)}\left[\phi_2^2(0) - b_1\int_{-\infty}^{0} F(-s)\phi_2^2(s)\mathrm{d}s\right] \\ \quad + c_{11}^{(3)}\left[\phi_1^3(0) - b_1\int_{-\infty}^{0} F(-s)\phi_1^3(s)\mathrm{d}s\right] \\ \quad + c_{12}^{(3)}\left[\phi_2^3(0) - b_1\int_{-\infty}^{0} F(-s)\phi_2^3(s)\mathrm{d}s\right] + \cdots, \\ f_2 = c_{21}^{(2)}\left[\phi_1^2(0) - b_2\int_{-\infty}^{0} F(-s)\phi_1^2(s)\mathrm{d}s\right] \\ \quad + c_{22}^{(2)}\left[\phi_2^2(0) - b_2\int_{-\infty}^{0} F(-s)\phi_2^2(s)\mathrm{d}s\right] \\ \quad + c_{21}^{(3)}\left[\phi_1^3(0) - b_2\int_{-\infty}^{0} F(-s)\phi_1^3(s)\mathrm{d}s\right] \\ \quad + c_{22}^{(3)}\left[\phi_2^3(0) - b_2\int_{-\infty}^{0} F(-s)\phi_2^3(s)\mathrm{d}s\right] + \cdots, \end{cases} \tag{26}$$

where $c_{ij}^{(k)} = a_{ij}f^{(k)}(0), i, j = 1, 2, k = 2, 3, \ldots$.

Furthermore, we define $\mathcal{A}^*$, the adjoint operator of $\mathcal{A}$, as

$$\mathcal{A}^*(\mu)\psi(\delta) = \begin{cases} -\frac{\mathrm{d}\psi(\delta)}{\mathrm{d}\delta}, & \delta \in (0, \infty), \\ L^T\psi(0) + \int_{-\infty}^{0} K^T(s)\psi(-s)\mathrm{d}s, & \delta = 0, \end{cases} \tag{27}$$

and define a bilinear inner product

$$\langle\psi(s), \phi(\theta)\rangle = \overline{\psi}^T(0)\phi(0) - \int_{\theta=-\infty}^{0}\int_{\xi=0}^{\theta} \overline{\psi}^T(\xi - \theta)K(\theta)\phi(\xi)\mathrm{d}\xi\mathrm{d}\theta, \tag{28}$$

where L^T, K^T, $\overline{\psi}^T$ are the transposes of L, K, ψ.

Because $\mathcal{A}$ and $\mathcal{A}^*$ are the adjoint operators, the eigenvalues of $\mathcal{A}$ are also eigenvalues of $\mathcal{A}^*$. If $q(\theta) = (1, B_1)^T e^{\mathrm{i}\omega_0\theta}$ is the eigenvector of $\mathcal{A}(0)$ corresponding to $\mathrm{i}\omega_0$, then

$$\mathcal{A}(0)q(\theta) = \mathrm{i}\omega_0 q(\theta),$$

we have

$$\begin{pmatrix} \mathrm{i}\omega_0 + 1 - c_{11}\left[1 - b_1\left(\frac{\alpha}{\alpha+\mathrm{i}\omega_0}\right)^2\right] & -c_{12}\left[1 - b_1\left(\frac{\alpha}{\alpha+\mathrm{i}\omega_0}\right)^2\right] \\ -c_{21}\left[1 - b_2\left(\frac{\alpha}{\alpha+\mathrm{i}\omega_0}\right)^2\right] & \mathrm{i}\omega_0 + 1 - c_{22}\left[1 - b_2\left(\frac{\alpha}{\alpha+\mathrm{i}\omega_0}\right)^2\right] \end{pmatrix} \begin{pmatrix} 1 \\ B_1 \end{pmatrix} e^{\mathrm{i}\omega_0\theta} = 0. \tag{29}$$

it follows that

$$B_1 = \frac{(\mathrm{i}\omega_0 + 1)(\alpha + \mathrm{i}\omega_0)^2 - c_{11}[(\alpha + \mathrm{i}\omega_0)^2 - b_1\alpha^2]}{c_{12}\left[(\alpha + \mathrm{i}\omega_0)^2 - b_1\alpha^2\right]}. \tag{30}$$

Similarly, suppose that $q^*(\delta) = D(1, B_2)^T e^{-\mathrm{i}\omega_0\delta}$, which is the eigenvector corresponding to $-\mathrm{i}\omega_0$ of $\mathcal{A}^*(0)$, and by $\mathcal{A}^*(0)q^*(\delta) = -\mathrm{i}\omega_0 q^*(\delta)$, we get

$$B_2 = \frac{(-\mathrm{i}\omega_0+1)(\alpha - \mathrm{i}\omega_0)^2 - c_{11}[(\alpha-\mathrm{i}\omega_0)^2 - b_1\alpha^2]}{c_{21}\left[(\alpha - \mathrm{i}\omega_0)^2 - b_2\alpha^2\right]}. \tag{31}$$

According to Eq. (28), we have

$$\begin{aligned}
&\langle q^*(\delta), q(\theta)\rangle \\
&\quad = \overline{q^*}^T(0)q(0) - \int_{-\infty}^{0}\int_{\xi=0}^{\theta} \overline{q^*}^T(\xi - \theta)K(\theta)q(\xi)\mathrm{d}\xi\mathrm{d}\theta \\
&\quad = \overline{D}(1 + B_1\overline{B}_2) - \overline{D}\int_{-\infty}^{0}(1, \overline{B}_2)e^{\mathrm{i}\omega_0\xi}K(\theta)\begin{pmatrix} 1 \\ B_1 \end{pmatrix}\mathrm{d}\xi\mathrm{d}\theta \\
&\quad = \overline{D}\Big[1 + B_1\overline{B}_2 - \int_{-\infty}^{0}(1, \overline{B}_2)\begin{pmatrix} -c_{11}b_1 & -c_{12}b_1 \\ -c_{21}b_2 & -c_{22}b_2 \end{pmatrix}\begin{pmatrix} 1 \\ B_1 \end{pmatrix} \\
&\qquad \times F(-\theta)\theta e^{\mathrm{i}\omega_0\theta}\mathrm{d}\theta\Big] \\
&= \overline{D}\Big[1 + B_1\overline{B}_2 + (c_{11}b_1 + c_{21}b_2\overline{B}_2 + c_{12}b_1B_1 + c_{22}b_2B_1\overline{B}_2) \\
&\qquad \times \int_{-\infty}^{0} -\alpha^2\theta^2 e^{(\alpha+\mathrm{i}\omega_0)\theta}\mathrm{d}\theta\Big] \\
&= \overline{D}\Big[1 + B_1\overline{B}_2 - \frac{2\alpha^2}{(\alpha + \mathrm{i}\omega_0)^3}(c_{11}b_1 \\
&\qquad + c_{21}b_2\overline{B}_2 + c_{12}b_1B_1 + c_{22}b_2B_1\overline{B}_2)\Big].
\end{aligned} \tag{32}$$

Because $\langle q^*(\delta), q(\theta)\rangle = 1$, we can choose

$$\begin{aligned}\overline{D} = {} & [1 + B_1\overline{B}_2 - \frac{2\alpha^2}{(\alpha + \mathrm{i}\omega_0)^3}(c_{11}b_1 + c_{21}b_2\overline{B}_2 \\ & + c_{12}b_1B_1 + c_{22}b_2B_1\overline{B}_2)]^{-1}.\end{aligned} \tag{33}$$

On the other hand, since $\langle \psi, \mathcal{A}\phi\rangle = \langle \mathcal{A}^*\psi, \phi\rangle$, we have

$$\begin{aligned}\mathrm{i}\omega_0\langle q^*, \overline{q}\rangle &= \langle q^*, \mathcal{A}\overline{q}\rangle = \langle \mathcal{A}^*q^*, \overline{q}\rangle \\ &= \langle -\mathrm{i}\omega_0 q^*, \overline{q}\rangle = -\mathrm{i}\omega_0\langle q^*, \overline{q}\rangle,\end{aligned}$$

which imply

$$\langle q^*(\delta), \overline{q}(\theta)\rangle = 0.$$

Next, we compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let y_t be the solution of (19) when $\mu = 0$. Define

$$\begin{cases} z(t) = \langle q^*, y_t\rangle, \\ W(t, \theta) = y_t - zq - \overline{z}(t)\overline{q}(\theta). \end{cases} \tag{34}$$

On the center manifold C_0, we have

$$\begin{aligned}W(t, \theta) = W(z, \overline{z}, \theta) = {} & W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} \\ & + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots,\end{aligned} \tag{35}$$

where z and $\overline{z}$ are local coordinates of center manifold C_0 in direction of q^* and $\overline{q}^*$. Since W is real if y_t is real, one only needs to consider real solutions of y_t. Since $\mu = 0$, by (23) and (28), we have

$$\begin{aligned}\dot{z}(t) &= \langle q^*, y_t\rangle = \langle q^*, \mathcal{A}(0)y_t + \mathcal{R}(0)y_t\rangle \\ &= \langle \mathcal{A}^*(0)q^*, y_t\rangle + \langle \overline{q}^*, \mathcal{R}(0)(W(z, \overline{z}, \theta) \\ &\quad + 2Rezq(\theta))\rangle \\ &= \mathrm{i}\omega_0\langle \mathcal{A}^*(0)q^*, y_t\rangle + \overline{q}^{*^T}(0)\mathcal{R}(0)(W(z, \overline{z}, 0) \\ &\quad + 2Rezq(0)) \\ &\doteq \mathrm{i}\omega_0 z(t) + \overline{q}^{*^T}(0)\mathcal{R}_0(z, \overline{z}),\end{aligned} \tag{36}$$

or

$$\dot{z}(t) = \mathrm{i}\omega_0 z(t) + g(z, \overline{z}), \tag{37}$$

where

$$g(z,\overline{z}) = \overline{q}^{*T}(0)\mathcal{R}_0(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{38}$$

By comparing the coefficients of (36) and (38), we get

$$\begin{aligned}
g_{20} &= 2\overline{D}\Bigg[\left(c_{11}^{(2)} + B_1^2 c_{12}^{(2)}\right)\left(1 - \frac{b_1\alpha^2}{(\alpha+2\mathrm{i}\omega_0)^2}\right) \\
&\quad + \overline{B}_2\left(c_{21}^{(2)} + B_1^2 c_{22}^{(2)}\right)\left(1 - \frac{b_2\alpha^2}{(\alpha+2\mathrm{i}\omega_0)^2}\right)\Bigg], \\
g_{02} &= 2\overline{D}\Bigg[\left(c_{11}^{(2)} + \overline{B}_2^2 c_{12}^{(2)}\right)\left(1 - \frac{b_1\alpha^2}{(\alpha-2\mathrm{i}\omega_0)^2}\right) \\
&\quad + \overline{B}_2\left(c_{21}^{(2)} + \overline{B}_2^2 c_{22}^{(2)}\right)\left(1 - \frac{b_2\alpha^2}{(\alpha-2\mathrm{i}\omega_0)^2}\right)\Bigg], \\
g_{11} &= 2\overline{D}\Big[(1-b_1)\left(c_{11}^{(2)} + c_{12}^{(2)}B_1\overline{B}_2\right) \\
&\quad + \overline{B}_2(1-b_2)\left(c_{21}^{(2)} + c_{22}^{(2)}B_1\overline{B}_2\right)\Big], \\
g_{21} &= 2\overline{D}\Bigg\{\Bigg[c_{11}^{(2)}\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) \\
&\quad + c_{12}^{(2)}\left(W_{20}^{(2)}(0)\overline{B}_2 + 2W_{11}^{(2)}(0)B_1\right) \\
&\quad - c_{11}^{(2)}b_1\int_{-\infty}^{0} F(-s)\left(W_{20}^{(1)}(s)e^{-\mathrm{i}\omega_0 s} + 2W_{11}^{(1)}(s)e^{\mathrm{i}\omega_0 s}\right)\mathrm{d}s \\
&\quad - c_{12}^{(2)}b_1\int_{-\infty}^{0} F(-s)\left(\overline{B}_2 W_{20}^{(1)}(s)e^{-\mathrm{i}\omega_0 s} + 2B_1 W_{11}^{(1)}(s)e^{\mathrm{i}\omega_0 s}\right)\mathrm{d}s \\
&\quad + 3\left(c_{11}^{(3)} + c_{12}^{(3)}B_1^2\overline{B}_2\right)\left(1 - \frac{b_1\alpha^2}{(\alpha+\mathrm{i}\omega_0)^2}\right)\Bigg] \\
&\quad + \overline{B}_2\Bigg[c_{21}^{(2)}\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) \\
&\quad + c_{22}^{(2)}\left(W_{20}^{(2)}(0)\overline{B}_2 + 2W_{11}^{(2)}(0)B_1\right) \\
&\quad - c_{21}^{(2)}b_2\int_{-\infty}^{0} F(-s)\left(W_{20}^{(1)}(s)e^{-\mathrm{i}\omega_0 s} + 2W_{11}^{(1)}(s)e^{\mathrm{i}\omega_0 s}\right)\mathrm{d}s \\
&\quad - c_{22}^{(2)}b_2\int_{-\infty}^{0} F(-s)\left(\overline{B}_2 W_{20}^{(1)}(s)e^{-\mathrm{i}\omega_0 s} + 2B_1 W_{11}^{(1)}(s)e^{\mathrm{i}\omega_0 s}\right)\mathrm{d}s \\
&\quad + 3\left(c_{21}^{(3)} + c_{22}^{(3)}B_1^2\overline{B}_2\right)\left(1 - \frac{b_2\alpha^2}{(\alpha+\mathrm{i}\omega_0)^2}\right)\Bigg]\Bigg\}.
\end{aligned} \tag{39}$$

In order to determine g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (23) and (34), we have

$$\begin{aligned}
\dot{W} &= \dot{y}_t - \dot{z}q - \dot{\overline{z}}\overline{q} \\
&= \mathcal{A}(0)y_t + \mathcal{R}(0)y_t - (\mathrm{i}\omega_0 z + g)q - (-\mathrm{i}\omega_0\overline{z} + \overline{g})\overline{q} \\
&= \mathcal{A}(0)x_t + \mathcal{R}(0)x_t - 2\Re(gq) \\
&= \begin{cases} \mathcal{A}(0)W - 2\Re(\overline{q}^{*T}(0)\mathcal{R}_0 q(\theta)), & \theta \in [-\infty, 0), \\ \mathcal{A}(0)W - 2\Re(\overline{q}^{*T}(0)\mathcal{R}_0 q(0) + \mathcal{R}_0, & \theta = 0, \end{cases} \\
&\doteq \mathcal{A}(0)W + H(z,\overline{z},\theta),
\end{aligned} \tag{40}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots. \tag{41}$$

On the center manifold C_0 near the origin,

$$\dot{W} = W_z\dot{z} + W_{\overline{z}}\dot{\overline{z}}. \tag{42}$$

Substituting (35) and (37) in (42), and comparing the coefficients of the above equation, we get

$$\begin{cases} (\mathcal{A}(0) - 2\mathrm{i}\omega_0)W_{20}(\theta) = -H_{20}(\theta), \\ \mathcal{A}(0)W_{11} = -H_{11}(\theta). \end{cases} \tag{43}$$

By (40), we know that for $\theta \in (-\infty, 0)$,

$$\begin{aligned}
&H(z,\overline{z},\theta) \\
&\quad = -2\Re(\overline{q}^{*T}(0)\mathcal{R}_0 q(\theta)) \\
&\quad = -g(z,\overline{z})q(\theta) - \overline{g}(z,\overline{z})\overline{q}(\theta) \\
&\quad = -\left(g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots\right)q(\theta) \\
&\qquad - \left(\overline{g}_{20}\frac{\overline{z}^2}{2} + g_{11}z\overline{z} + \overline{g}_{02}\frac{z^2}{2} + \overline{g}_{21}\frac{\overline{z}^2 z}{2} + \cdots\right)\overline{q}(\theta).
\end{aligned} \tag{44}$$

Comparing the coefficients of the above equation with that in (41), we have

$$\begin{cases} H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \\ H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \end{cases} \tag{45}$$

From (43), (45), and the definition of $\mathcal{A}(0)$, we have

$$\dot{W}_{20}(\theta) = 2\mathrm{i}\omega_0 W_{20}(\theta) + g_{20}q(0)e^{\mathrm{i}\omega_0\theta} + \overline{g}_{02}\overline{q}(0)e^{-\mathrm{i}\omega_0\theta}. \tag{46}$$

Solving for $W_{20}(\theta)$, we have

$$W_{20}(\theta) = \frac{\mathrm{i}g_{20}q(0)}{\omega_0}e^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\overline{g}_{02}\overline{q}(0)}{3\omega_0}e^{-\mathrm{i}\omega_0\theta} + E_1 e^{2\mathrm{i}\omega_0\theta}. \tag{47}$$

By the similar way,

$$\dot{W}_{11}(\theta) = g_{11}q(0)e^{\mathrm{i}\omega_0\theta} + \overline{g}_{11}\overline{q}(0)e^{-\mathrm{i}\omega_0\theta}, \tag{48}$$

and we get

$$W_{11}(\theta) = -\frac{\mathrm{i}g_{11}q(0)}{\omega_0}e^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\overline{g}_{11}\overline{q}(0)}{\omega_0}e^{-\mathrm{i}\omega_0\theta} + E_2. \tag{49}$$

E_1, E_2 in the above formulae are both constant vectors and can be determined by setting $\theta = 0$ in $H(z, \overline{z}, \theta)$.

From (40), we have

$$\begin{aligned} H(z, \overline{z}, 0) &= -2\Re\left(\overline{q}^{*^T}(0)\mathcal{R}_0 q(0) + \mathcal{R}_0\right) \\ &= -g(z,\overline{z})q(0) - \overline{g}(z,\overline{z})\overline{q}(0) + \mathcal{R}_0(z,\overline{z}) \\ &= -\left(g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots\right)q(0) \\ &\quad -\left(\overline{g}_{20}\frac{\overline{z}^2}{2} + g_{11}z\overline{z} + \overline{g}_{02}\frac{z^2}{2} + \overline{g}_{21}\frac{\overline{z}^2 z}{2} + \cdots\right)\overline{q}(0) \\ &\quad +\mathcal{R}_0. \end{aligned} \tag{50}$$

For $\mathcal{R}_0(z, \overline{z})$, suppose that

$$E_1 = 2\begin{pmatrix} k_{11}\left[2\mathrm{i}\omega_0 + 1 - c_{22}\left(1 - \frac{b_2\alpha^2}{(\alpha+2\mathrm{i}\omega_0)^2}\right)\right] - k_{21}c_{12}\left(1 - \frac{b_1\alpha^2}{(\alpha+2\mathrm{i}\omega_0)^2}\right) \\ k_{21}\left[2\mathrm{i}\omega_0 + 1 - c_{11}\left(1 - \frac{b_1\alpha^2}{(\alpha+2\mathrm{i}\omega_0)^2}\right)\right] - k_{11}c_{21}\left(1 - \frac{b_2\alpha^2}{(\alpha+2\mathrm{i}\omega_0)^2}\right) \end{pmatrix}. \tag{58}$$

$$\mathcal{R}_0(z,\overline{z}) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} k_{11}z^2 + k_{12}z\overline{z} + k_{13}\overline{z}^2 + \cdots \\ k_{21}z^2 + k_{22}z\overline{z} + k_{23}\overline{z}^2 + \cdots \end{pmatrix}, \tag{51}$$

so that

$$\begin{aligned} g(z,\overline{z}) &= \overline{q}^{*^T}(0)\mathcal{R}_0(z,\overline{z}) = \overline{D}(1, \overline{B}_2)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= \overline{D}\left[\left(k_{11}z^2 + k_{12}z\overline{z} + k_{13}\overline{z}^2 + k_{14}z^2\overline{z} + \cdots\right)\right. \\ &\quad \left. + \overline{B}_2\left(k_{21}z^2 + k_{22}z\overline{z} + k_{23}\overline{z}^2 + k_{24}z^2\overline{z} + \cdots\right)\right]. \end{aligned} \tag{52}$$

By comparing with (38), we obtain

$$\begin{aligned} g_{20} &= 2\overline{D}(k_{11} + \overline{B}_2 k_{21}), \\ g_{02} &= \overline{D}(k_{12} + \overline{B}_2 k_{22}), \\ g_{11} &= 2\overline{D}(k_{13} + \overline{B}_2 k_{23}), \\ g_{21} &= 2\overline{D}(k_{14} + \overline{B}_2 k_{24}). \end{aligned} \tag{53}$$

Otherwise, substituting (51) in (3), and comparing with (41), we have

$$\begin{aligned} H(z, \overline{z}, 0) &= -\left(g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots\right)q(0) \\ &\quad -\left(\overline{g}_{20}\frac{\overline{z}^2}{2} + g_{11}z\overline{z} + \overline{g}_{02}\frac{z^2}{2} + \overline{g}_{21}\frac{\overline{z}^2 z}{2} + \cdots\right)\overline{q}(0) \\ &\quad + \begin{pmatrix} k_{11}z^2 + k_{12}z\overline{z} + k_{13}\overline{z}^2 + \cdots \\ k_{21}z^2 + k_{22}z\overline{z} + k_{23}\overline{z}^2 + \cdots \end{pmatrix}, \end{aligned} \tag{54}$$

$$\begin{cases} H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2(k_{11}, k_{21})^T, \\ H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + (k_{12}, k_{22})^T. \end{cases} \tag{55}$$

Eq. (43) can be rewritten as

$$\mathcal{A}(0)W_{20}(0) = 2\mathrm{i}\omega_0 W_{20}(0) - H_{20}(0). \tag{56}$$

By the definition of $\mathcal{A}(0)$, substituting (47) in (56) gives

$$(2\mathrm{i}\omega_0 I - \mathcal{A}(0))E_1 = 2(k_{11}, k_{21})^T, \tag{57}$$

and then we obtain

By the similar way, we can also get

$$E_2 = \begin{pmatrix} k_{12} - k_{12}c_{22}(1-b_2) - k_{22}c_{12}(1-b_1) \\ k_{22} - k_{22}c_{11}(1-b_1) - k_{12}c_{21}(1-b_2) \end{pmatrix}. \tag{59}$$

Thus, $W_{20}(\theta)$ and $W_{11}(\theta)$ can be obtained.

Thus, we get that each g_{ij} in (39) is determined by those parameters in system (4). Furthermore, we can compute

$$\begin{cases} c_1(0) = \dfrac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \dfrac{|g_{02}|^2}{3}\right) + \dfrac{g_{21}}{2}, \\ \mu_2 = -\dfrac{\Re\{c_1(0)\}}{\Re\{\lambda'(\alpha_0)\}}, \\ \beta_2 = 2\Re\{c_1(0)\}, \\ T_2 = -\dfrac{\Im\{c_1(0)\} + \mu_2\Im\{\lambda'(\alpha_0)\}}{\omega_0}. \end{cases} \tag{60}$$

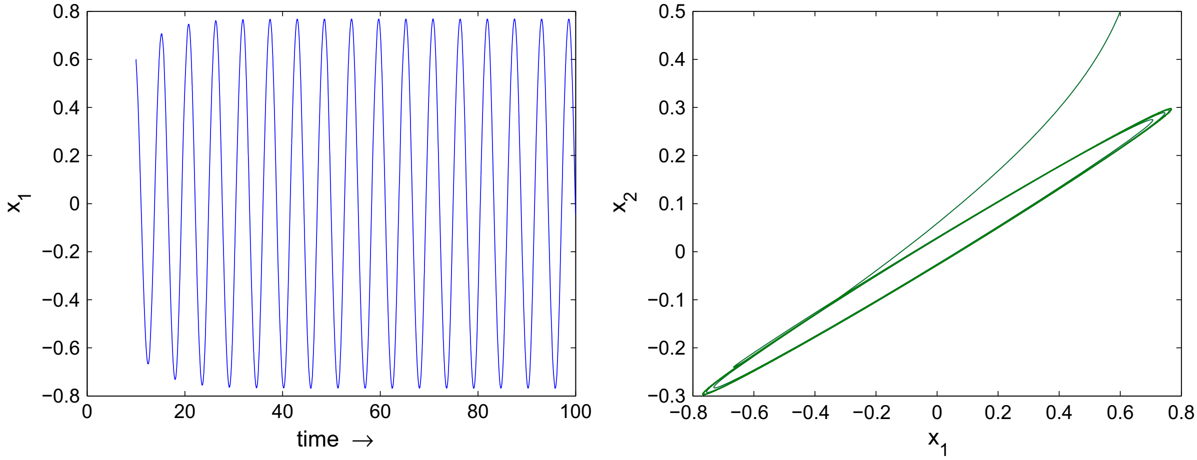

Fig. 1 Waveform plot and phase plot with $\alpha = 2$

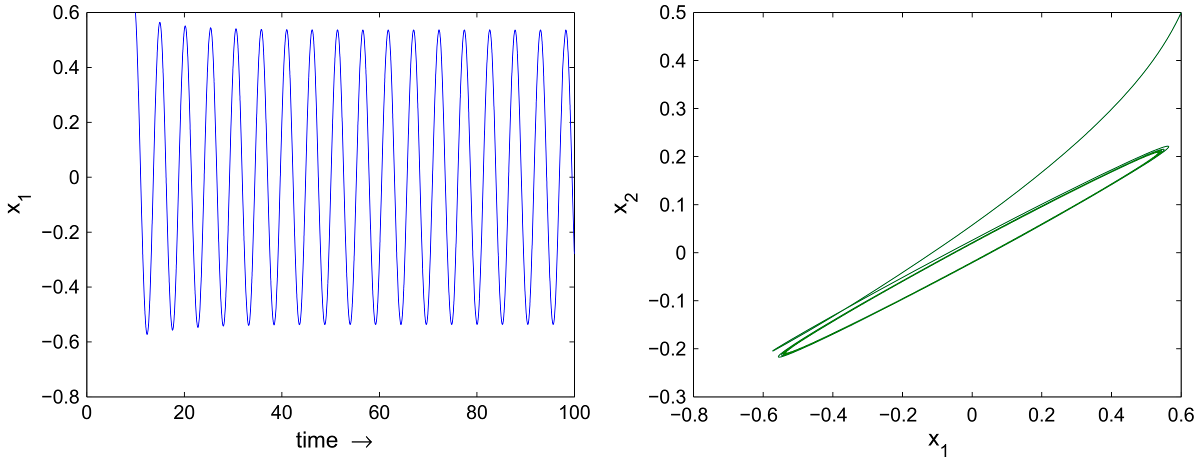

Fig. 2 Waveform plot and phase plot with $\alpha = 2.2$

By using theories given by Hassard et al. [18], we have the following main results in this section.

Theorem 3.1 *In (60), the following results hold:*

(i) The sign of μ_2 determines the direction of the Hopf bifurcation: If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcation periodic solutions exist for $\alpha > \alpha_0$ ($\alpha < \alpha_0$).

(ii) The sign of β_2 determines the stability of the bifurcating periodic solution: The bifurcation periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$).

(iii) The sign of T_2 determines the period of the bifurcating periodic solutions: The period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

4 Numerical examples

In this section, some numerical results of system (4) are presented to support the analytic results given by Sects. 2 and 3. We consider system (3) and let $f(\cdot) = \tan h(\cdot)$. Then we have

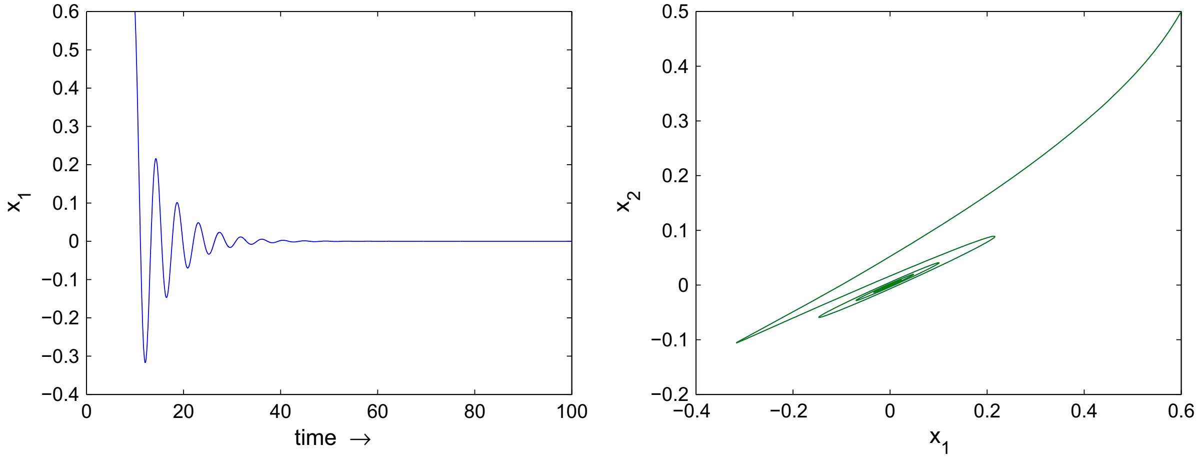

Fig. 3 Waveform plot and phase plot with $\alpha = 3$

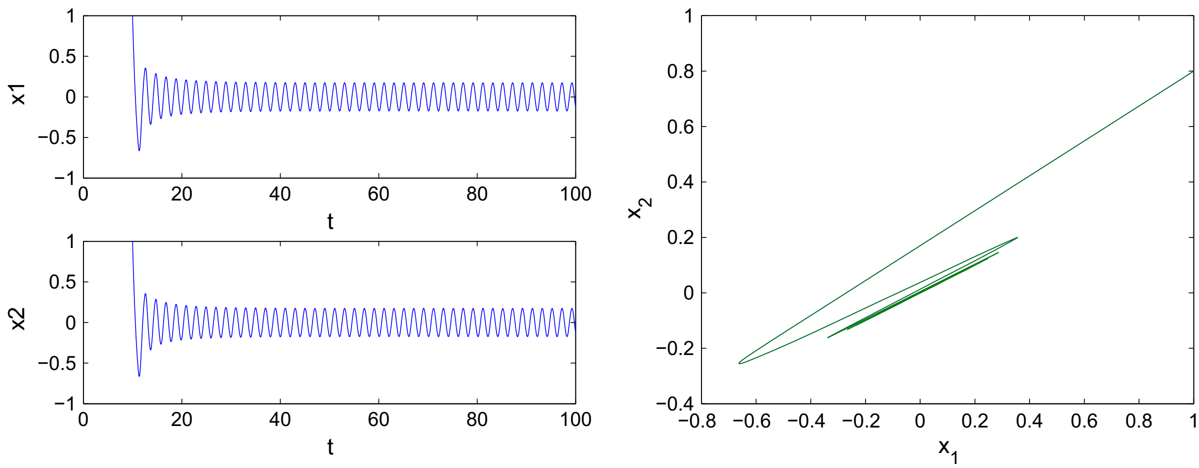

Fig. 4 Waveform plot and phase plot with $\alpha = 3$

$$\begin{cases} \dot{y}_1(t) = -y_1(t) + a_{11} \tan h \\ \qquad \left[y_1(t) - b_1 \int_{-\infty}^{0} F(-s) y_1(t+s) \mathrm{d}s - c_1 \right] \\ \qquad + a_{12} \tan h \left[y_2(t) - b_1 \int_{-\infty}^{0} F(-s) y_2(t+s) \mathrm{d}s - c_2 \right], \\ \dot{y}_2(t) = -y_2(t) + a_{21} \tan h \\ \qquad \left[y_1(t) - b_2 \int_{-\infty}^{0} F(-s) y_1(t+s) \mathrm{d}s - c_1 \right] \\ \qquad + a_{22} \tan h \left[y_2(t) - b_2 \int_{-\infty}^{0} F(-s) y_2(t+s) \mathrm{d}s - c_2 \right]. \end{cases} \tag{61}$$

Example 1 Choosing $a_{11} = 1.14, a_{12} = 1.75, a_{21} = 0.5, a_{22} = 0.5, b_1 = 1.55, b_2 = 0.1$, we compute $\alpha_0 = 2.229$. According to Theorems 2.2, a Hopf bifurcation occurs at the equilibrium $(0, 0)$ when α passes through $\alpha_0 = 2.229$. Furthermore, a periodic solution exists when $\alpha \in (0, 2.229)$ (Figs. 1, 2) and system (61) is asymptotically stable when $\alpha > 2.229$ (Fig. 3).

Example 2 Choosing $a_{11} = a_{22} = 7/16,\ a_{12} = 7/8,\ a_{21} = 7/32,\ b_1 = 8,\ b_2 = 8$ to make **(H4)** and **(H5)** be satisfied. We compute $\alpha_0 = 3.357$. According to Corollary 2.1, a Hopf bifurcation occurs at the equilibrium $(0, 0)$ as α passes through $\alpha_0 = 3.357$. Meanwhile, a periodic solution exists if $\alpha \in (0, 3.357)$ (Fig. 4). System (61) is asymptotically stable if $\alpha > 3.357$ (Fig. 5).

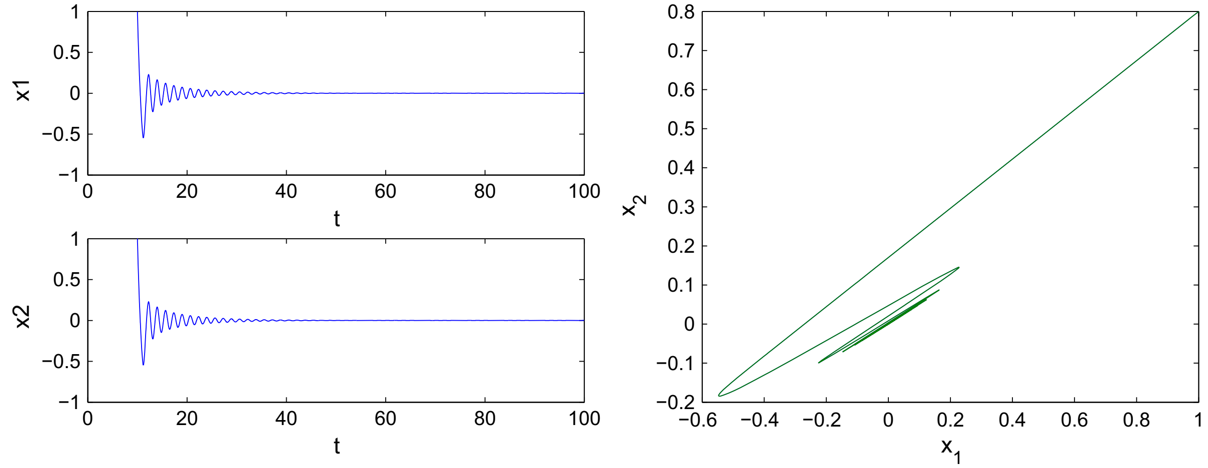

Fig. 5 Waveform plot and phase plot with $\alpha = 4$

5 Conclusions

A delay neural network model with self-connections and strong kernel has been studied in this paper. It shows that a Hopf bifurcation occurs when the bifurcation parameter passes through a critical value. The stability and direction of the bifurcating periodic orbits have also been analyzed by using the theory of center manifold and the normal form method.

Because distributed delay can become discrete delay when the delay kernel is a delta function at a certain time, a neural network model with distributed delay is more general than that with discrete delay. For this reason a neural network model with distributed delay for strong kernel is studied in this paper. When choosing the kernel as a strong kernel with $F(s) = \alpha^2 s e^{-\alpha s}$, we can obtain the dynamical behaviors of the model by explicit algorithms. This helps understand the complexities of neural network systems.

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Appendix

Calculation for Eq. (11)

From (10), we have

$$\omega_0^2 = \frac{m_3 + \sqrt{m_3^2 - 4m_1m_5}}{2m_1}, \tag{62}$$

$$\begin{aligned}&\left(m_3^3 - m_1m_2m_3^2 + m_1^2m_3m_4 - m_1m_3m_5\right)\\ &\quad + \sqrt{m_3^2 - 4m_1m_5}\left(m_3^2 - m_1m_2m_3 + m_1^2m_4 - m_1m_5\right)\\ &\quad = 2m_1^3m_6 + m_1m_3m_5 - 2m_1^2m_2m_5,\end{aligned} \tag{63}$$

$$\begin{aligned}&4m_1^2\Big(m_1^3m_6^2 + m_1m_2^2m_5^2 + 3m_1m_3m_5m_6\\ &\quad - 2m_1^2m_2m_5m_6 - m_3^3m_6 + m_1m_2m_3^2m_6\\ &\quad - m_1^2m_3m_4m_6 - m_2m_3m_5^2 + m_3^2m_4m_5 + m_1^2m_4^2m_5\\ &\quad + m_5^3 - m_1m_2m_3m_4m_5 - 2m_1m_4m_5^2\Big) = 0,\end{aligned} \tag{64}$$

that is equivalent to $D_5(\alpha) = 0$.

Calculation for Eq. (12)

From (6), let

$$\begin{aligned}F &= \lambda^6 + m_1(\alpha)\lambda^5 + m_2(\alpha)\lambda^4 + m_3(\alpha)\lambda^3\\ &\quad + m_4(\alpha)\lambda^2 + m_5(\alpha)\lambda + m_6(\alpha) = 0.\end{aligned} \tag{65}$$

Since $\frac{dF}{d\alpha} = \frac{dF}{d\lambda} \cdot \frac{d\lambda}{d\alpha}$, so

$$\begin{aligned}\frac{d}{d\alpha}[Re\lambda]|_{\alpha_0} &= Re\left[\frac{d\lambda}{d\alpha}\right]_{\alpha_0}\\ &= Re\left[\frac{dF}{d\lambda}\right]^{-1} \cdot \frac{d}{d\alpha}[Re(F)]|_{\alpha_0}.\end{aligned} \tag{66}$$

For $\frac{\mathrm{d}}{\mathrm{d}\alpha}[Re(F)]$

$$\begin{aligned} F &= -\omega_0^6 + \omega_0^5 \mathrm{i} m_1(\alpha) + \omega_0^4 m_2(\alpha) - \omega_0^3 \mathrm{i} m_3(\alpha) \\ &\quad - \omega_0^2 m_4(\alpha) + \omega_0 \mathrm{i} m_5(\alpha) + m_6(\alpha), \end{aligned} \tag{67}$$

$$\begin{aligned} Re(F) &= -\omega_0^6 + \omega_0^4 m_2(\alpha) - \omega_0^2 m_4(\alpha) + m_6(\alpha) \\ &= \frac{1}{2m_1^3(\alpha)} \Big\{\big[2m_1^3(\alpha)m_6(\alpha) + 3m_1(\alpha)m_3(\alpha)m_5(\alpha) \\ &\quad - 2m_1^2(\alpha)m_2(\alpha)m_5(\alpha) - m_3^3(\alpha) \\ &\quad + m_1(\alpha)m_2(\alpha)m_3^2(\alpha) - m_1^2(\alpha)m_3(\alpha)m_4(\alpha)\big] \\ &\quad - \sqrt{m_3^2(\alpha) - 4m_1(\alpha)m_5(\alpha)} \\ &\quad \times \big[m_3^2(\alpha) - m_1(\alpha)m_2(\alpha)m_3(\alpha) \\ &\quad + m_1^2(\alpha)m_4(\alpha) - m_1(\alpha)m_5(\alpha)\big]\Big\} \\ &= \frac{1}{2m_1^3(\alpha)} \Big\{\big[2m_1^3(\alpha)m_6(\alpha) + 3m_1(\alpha)m_3(\alpha)m_5(\alpha) \\ &\quad - 2m_1^2(\alpha)m_2(\alpha)m_5(\alpha) - m_3^3(\alpha) \\ &\quad + m_1(\alpha)m_2(\alpha)m_3^2(\alpha) - m_1^2(\alpha)m_3(\alpha)m_4(\alpha)\big] \\ &\quad - \sqrt{m_3^2(\alpha) - 4m_1(\alpha)m_5(\alpha)} \\ &\quad \times \big[m_3^2(\alpha) - m_1(\alpha)m_2(\alpha)m_3(\alpha) \\ &\quad + m_1^2(\alpha)m_4(\alpha) - m_1(\alpha)m_5(\alpha)\big]\Big\}^{-1} \\ &\quad \times \Big\{\big[2m_1^3(\alpha)m_6(\alpha) + 3m_1(\alpha)m_3(\alpha)m_5(\alpha) \\ &\quad - 2m_1^2(\alpha)m_2(\alpha)m_5(\alpha) - m_3^3(\alpha) \\ &\quad + m_1(\alpha)m_2(\alpha)m_3^2(\alpha) - m_1^2(\alpha)m_3(\alpha)m_4(\alpha)\big]^2 \\ &\quad - \big[m_3^2(\alpha) - 4m_1(\alpha)m_5(\alpha)\big] \\ &\quad \times \big[m_3^2(\alpha) - m_1(\alpha)m_2(\alpha)m_3(\alpha) \\ &\quad + m_1^2(\alpha)m_4(\alpha) - m_1(\alpha)m_5(\alpha)\big]^2\Big\} \\ &= \frac{1}{2m_1^3(\alpha)} \cdot [M(\alpha)]^{-1} \cdot \big[-4m_1^3(\alpha)D_5(\alpha)\big] \\ &= -2[M(\alpha)]^{-1} \cdot D_5(\alpha), \end{aligned} \tag{68}$$

where

$$\begin{aligned} M(\alpha_0) &= \big(2 - m_1^3 m_6 + 3m_1m_3m_5 - 2m_1^2m_2m_5 - m_3^3 \\ &\quad + m_1m_2m_3^2 - m_1^2m_3m_4\big) + \sqrt{m_3^2 - 4m_1m_5} \\ &\quad \times \big(m_3^2 - m_1m_2m_3 + m_1^2m_4 - m_1m_5\big). \end{aligned} \tag{69}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\alpha}[ReF]|_{\alpha_0} &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big[-2[M(\alpha)]^{-1} \cdot D_5(\alpha)\Big] \\ &= -2\left\{\frac{\mathrm{d}}{\mathrm{d}\alpha}[M(\alpha)]^{-1} \cdot D_5(\alpha)\Big|_{\alpha_0} \right. \\ &\quad \left. + [M(\alpha)]^{-1} \cdot \frac{\mathrm{d}D_5(\alpha)}{\mathrm{d}\alpha}\Big|_{\alpha_0}\right\} \\ &= -\frac{2}{M(\alpha_0)} \cdot \frac{\mathrm{d}D_5(\alpha)}{\mathrm{d}\alpha}\Big|_{\alpha_0}. \end{aligned} \tag{70}$$

Calculation for $Re[\mathrm{d}F/\mathrm{d}\lambda]^{-1}|_{\alpha_0}$

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}\lambda}|_{\alpha_0} &= 6\lambda^5 + 5m_1\lambda^4 + 4m_2\lambda^3 + 3m_3\lambda^2 + 2m_4\lambda + m_5 \\ &= 6\omega_0^5\mathrm{i} + 5m_1\omega_0^4 - 4m_2\omega_0^3\mathrm{i} - 3m_3\omega_0^2 + 2m_4\omega_0\mathrm{i} + m_5, \end{aligned} \tag{71}$$

$$Re\left[\frac{\mathrm{d}F}{\mathrm{d}\lambda}\right]^{-1}|_{\alpha_0} \tag{72}$$

$$\begin{aligned} &= Re\left[\frac{1}{(5m_1\omega_0^4 - 3m_3\omega_0^2 + m_5) + (6\omega_0^5 - 4m_2\omega_0^3 + 2m_4\omega_0)\,\mathrm{i}}\right] \\ &= \frac{5m_1\omega_0^4 - 3m_3\omega_0^2 + m_5}{(5m_1\omega_0^4 - 3m_3\omega_0^2 + m_5)^2 + (6\omega_0^5 - 4m_2\omega_0^3 + 2m_4\omega_0)^2} \\ &= \frac{5m_1\omega_0^4 - 3m_3\omega_0^2 + m_5}{N(\alpha_0)} \\ &= \frac{m_3\left(m_3 + \sqrt{m_3^2 - 4m_1m_5}\right) - 4m_1m_5}{m_1N(\alpha_0)}, \end{aligned} \tag{73}$$

where

$$\begin{cases} N(\alpha_0) = \left(5m_1\omega_0^4 - 3m_3\omega_0^2 + m_5\right)^2 \\ \qquad\qquad + \left(6\omega_0^5 - 4m_2\omega_0^3 + 2m_4\omega_0\right)^2, \\ \omega_0^2 = \dfrac{m_3 + \sqrt{m_3^2 - 4m_1m_5}}{2m_1}. \end{cases} \tag{74}$$

Thus, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\alpha}[Re\lambda]|_{\alpha_0} &= -\frac{2\left[m_3\left(m_3 + \sqrt{m_3^2 - 4m_1m_5}\right) - 4m_1m_5\right]}{m_1} \\ &\quad \cdot M(\alpha_0) \cdot N(\alpha_0) \cdot \frac{\mathrm{d}D_5(\alpha)}{\mathrm{d}\alpha}|_{\alpha_0}. \end{aligned}$$

Calculation for Eq. 14

From

$$\mathrm{Det}\begin{pmatrix} \lambda+1-c_{11}\left[1-b_1\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] & -c_{12}\left[1-b_1\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] \\ -c_{21}\left[1-b_2\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] & \lambda+1-c_{22}\left[1-b_2\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] \end{pmatrix} = 0, \tag{75}$$

we have

$$\left\{\lambda + 1 - c_{11}\left[1 - b_1\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right]\right\} \times \left\{\lambda + 1 - c_{22}\left[1 - b_2\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right]\right\} - \left\{c_{21}\left[1 - b_2\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right]\right\} \times \left\{c_{12}\left[1 - b_1\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right]\right\} = 0,$$

that is,

$$(\lambda+1)^2 - c_{11}(\lambda+1)\left[1 - b_1(\frac{\alpha}{\alpha+\lambda})^2\right] - c_{22}(\lambda+1)\left[1 - b_2(\frac{\alpha}{\alpha+\lambda})^2\right] + (c_{11}c_{22} - c_{21}c_{12})\left[1 - b_2(\frac{\alpha}{\alpha+\lambda})^2\right] \times \left[1 - b_1\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] = 0.$$

If $c_{11}c_{22} = c_{12}c_{21}$, it becomes

$$(\lambda+1)\left[\lambda + (1 - c_{11} - c_{22}) + (c_{11}b_1 + c_{22}b_2)\left(\frac{\alpha}{\alpha+\lambda}\right)^2\right] = 0,$$

that is,

$$(\lambda+1)(\lambda+\alpha)^2\left[\lambda^3 + m_1(\alpha)\lambda^2 + m_2(\alpha)\lambda + m_3(\alpha)\right] = 0, \tag{76}$$

where

$$\begin{cases} m_1(\alpha) = 2\alpha + (1 - c_{11} - c_{22}), \\ m_2(\alpha) = \alpha[\alpha + 2(1 - c_{11} - c_{22})], \\ m_3(\alpha) = [(1 - c_{11} - c_{22}) + (c_{11}b_1 + c_{22}b_2)]\,\alpha^2. \end{cases} \tag{77}$$

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