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A Simple Method for Testing Variance Components in Unbalanced Nested Model

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In this article, we consider the unbalanced case of the three fold nested random effects model under partial balance. The distributions of unweighted sums of squares are obtained first. Using the method of generalized p value introduced in Tsui and Weerahandi (1989), a new method is proposed for hypothesis tests involving functions of variance components. To evaluate the sizes of the generalized p value, a simulation study is conducted. The results indicate that the proposed method performs well under all examined conditions.

Keywords Generalized p value; Unbalanced nested model; Unweighted sums of squares; Variance components.

Mathematics Subject Classification 62F03; 62J10.

1. Introduction

In many experimental situations, processes are divided into several stages, and estimators of the variance components of the stages are quite useful in identifying major sources of variation. Nested model are used for this purpose; see details in Sahai and Ojeda (2005).

For the balanced cases, several methods can be used for constructing confidence intervals for functions of variance components by sums of squares which are independently distributed as chi-squared distribution; see Burdick and Graybill (1992), Weeranhandi (1995), and Chiang (2001). However, it is impossible to select equal sample sizes for each stage in many cases. Since there may be some occasions, on which a subject does not appear for one test, so that the experiment design becomes unbalanced. But, in the unbalanced case, the sums of squares no longer have chi-squared distributions. Furthermore, they are not independent. Therefore, efforts in statistical inference for the variance of random effects seem to be incommensurable although the initial attempt can be traced back to Wald (1947). For testing whether a variance component is zero in a mixed linear model with two variance components, the test by Wald (1947) leads an exact F-test. Later, Seely

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and El-Bassiouni (1983) extended the test method in the mixed linear model with more than two variance components. Recently, likelihood-based method for linear mixed-effects models have been well studied in statistical literature, and REML procedures for testing variance components are implemented in well-developed statistical software packages such as SAS and Splus/R with wide applications in many scientific fields; see Scheipl et al. (2008) and Greven et al. (2008). According to Scheipl et al. (2008), the restricted likelihood ratio test (RLRT), which implemented in the R-package RLRSim (Scheipl, 2010), is better than other tests, including Wald test. And they recommend RLRT to test zero variance components.

Yates' (1934) unweighted sums of squares (USSs) have long been used to derive a test concerning the parameters of unbalanced random or mixed models. Several authors have investigated it and used it for constructing confidence intervals on functions of variance components; for example, Thomas and Hultiquist (1978), Burdick and Graybill (1985), Hernandez and Burdick (1993), and Khuri et al. (1998). However, to the best of our knowledge, there has not yet been much attempt to the problem on testing non zero variance components.

In many experimental situations, a partial balance is natural in many situations since the error stage usually corresponds to replication of the experiments, and hence achieving a balanced at this stage is more in the control of experimenter. Tietjen (1974) called this condition "stage uniformity." The model under consideration is the unbalanced three-fold nested model with unequal number of levels for the third random factor. In this article, we focus attention on constructing generalized p value for functions of variance components in this model. The generalized inference introduced by Tsui and Weerahandi (1989) and Weeranhandi (1993) appear to be appropriate for constructing confidence intervals, since it's built from an exact probability statement, and can provide procedures applicable to small samples. During the past few years, the idea of generalized inference have been used by many authors to obtain useful inference procedure in nonstandard problems; see, for instance, Weeranhandi (1995), Ananda (1995), Iyer et al. (2004), Mathew and Webb (2005), and Hannig et al. (2006). Recently, Burch (2007) proposed a generalized confidence interval approach for proportions of total variance in mixed linear models having more than two variance components.

The article is organized as follows. In Sec. 2, we present the USSs in three-fold nested model with unequal number of levels for the third random factor. Section 3 reviews some tests for zero variance component firstly, then provides the method to construct generalized p-value for function of variance components. In Sec. 4, a Monte–Carlo simulation study is applied to discussing the size of the resulted generalized p value. The results indicate that the proposed method performs well under all examined conditions. Section 5 contains summary and discussion.

2. The Three-Fold Nested Model

In many experimental situations, an investigator can control an experiment and achieve equal sample sizes for the error stage. Tietjen (1974) called this condition "last stage uniformity". The model under consideration is the unbalanced three-fold nested model with unequal number of levels for the third random factor, which is written as

$$Y_{k_1k_2k_3k_4} = \mu + \alpha_{k_1} + \beta_{k_1k_2} + \gamma_{k_1k_2k_3} + e_{k_1k_2k_3k_4}, \quad k_1 = 1, \ldots, a, \ k_2 = 1, \ldots, b,$$
$$k_3 = 1, \ldots, c_{k_1k_2}, \ k_4 = 1, \ldots, d, \tag{2.1}$$

where α_{k_1}, $\beta_{k_1k_2}$, $\gamma_{k_1k_2k_3}$, and $e_{k_1k_2k_3k_4}$ are mutually independent, normally distributed random variables with zero means and variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ and σ_4^2, respectively. Let

$$\begin{aligned}
Y_{k_1k_2} &= (Y_{k_1k_211}, \ldots, Y_{k_1k_21d}, \ldots, Y_{k_1k_2c_{k_1k_2}1} \ldots, Y_{k_1k_2c_{k_1k_2}d})', \\
Y &= (Y'_{11}, \ldots, Y'_{ab})', \quad \alpha = (\alpha_1, \ldots, \alpha_a)', \\
\beta &= (\beta_{11}, \ldots, \beta_{1b}, \ldots, \beta_{a1}, \ldots \beta_{ab})', \\
\gamma &= (\gamma_{111}, \ldots, \gamma_{11c_{11}}, \ldots, \gamma_{ab1}, \ldots \gamma_{abc_{ab}})'.
\end{aligned}$$

Then the model (2.1) can be expressed in matrix form as

$$Y = X_0\mu + X_1\alpha + X_2\beta + X_3\gamma + X_4e, \tag{2.2}$$

where e is defined similarly to Y. Let $c_{..} = \sum_{k_1=1}^{a}\sum_{k_2=1}^{b} c_{k_1k_2}$, $c_{k_1\cdot} = \sum_{k_2=1}^{b} c_{k_1k_2}$, $n_1 = a$, $n_2 = ab$, $n_3 = c_{..}$, $n_4 = n_3d$, then $\alpha \sim N_{n_1}(0, \sigma_1^2 I_{n_1})$, $\beta \sim N_{n_2}(0, \sigma_2^2 I_{n_2})$, $\gamma \sim N_{n_3}(0, \sigma_3^2 I_{n_3})$, and $e \sim N_{n_4}(0, \sigma_4^2 I_{n_4})$. The design matrices X_0, X_1, X_2, X_3, X_4 are as follows:

$$\begin{aligned}
&X_0 = 1_{n_4}, \quad X_1 = diag(1_{c_{1\cdot}}, \ldots, 1_{c_{a\cdot}}) \otimes 1_d, \\
&X_2 = diag(1_{c_{11}}, \ldots, 1_{c_{ab}}) \otimes 1_d, \quad X_3 = I_{n_3} \otimes 1_d, \quad X_4 = I_{n_4}.
\end{aligned}$$

Lemma 2.1. *The properties of the design matrices are stated as follows.*

(1) $X_1 = X_2(I_a \otimes 1_b)$, $X_2'X_3 = diag(d1'_{c_{11}}, \ldots, d1'_{c_{ab}})$, $X_3'X_3 = dI_{n_3}$.
(2) $\mathscr{L}(X_0) \subset \mathscr{L}(X_1) \subset \mathscr{L}(X_2) \subset \mathscr{L}(X_3) \subset \mathscr{L}(X_4)$, *where* $\mathscr{L}(\cdot)$ *denotes range space (column space).*

Denote a projection matrix onto the column space of X_i by $P_i, i = 1, 2, 3, 4$. From Lemma 2.1, we see that

$$\begin{aligned}
&P_0 = \frac{1}{n_4}J_{n_4}, \quad P_1 = diag\left(\frac{1}{c_{1\cdot}}J_{c_{1\cdot}}, \ldots, \frac{1}{c_{a\cdot}}J_{c_{a\cdot}}\right) \otimes \frac{1}{d}J_d \\
&P_2 = diag\left(\frac{1}{c_{11}}J_{c_{11}}, \ldots, \frac{1}{c_{ab}}J_{c_{ab}}\right) \otimes \frac{1}{d}J_d, \quad P_3 = I_{c_{..}} \otimes \frac{1}{d}J_d, \quad P_4 = I_{n_4}.
\end{aligned} \tag{2.3}$$

Now, we employ the transformation $R = (R_1', R_2')'$ to model (2.2), where $R_1 = (X_2'X_2)^{-1}X_2'$, $R_2 = I - P_2$. By Lemma 2.1, we obtain

$$R_1Y = \mu 1_{ab} + (I_a \otimes 1_b)\alpha + \beta + R_1X_3\gamma + R_1e, \tag{2.4}$$

with

$$E(R_1Y) = \mu 1_{ab}, \quad Var(R_1Y) = \sigma_1^2(I_a \otimes J_b) + \sigma_2^2 I_{ab} + \left(\sum_{i=3}^{4}\frac{n_4}{n_i}\sigma_i^2\right)D,$$

and

$$R_2Y = (I_{n_4} - P_2)X_3\gamma + (I_{n_4} - P_2)e, \tag{2.5}$$

with

$$E(R_2Y) = 0, \quad Var(R_2Y) = \sum_{i=3}^{4} \sigma_i^2 (I_{n_4} - P_2) X_i X_i' (I_{n_4} - P_2),$$

where $D = diag\left(\frac{1}{c_{11}d}, \ldots, \frac{1}{c_{ab}d}\right)$. Let $SS_3 = Y'(P_3 - P_2)Y$, and $SS_4 = Y'(I_{n_4} - P_3)Y$, then $SS_{R_2Y} = Y'(I_{n_4} - P_2)Y = SS_3 + SS_4$.

Theorem 2.1.

(1) R_1Y, SS_3, SS_4 *are mutually independent.*
(2) $SS_i \sim \theta_i \chi^2_{n_i - n_{i-1}}$, $i = 3, 4$, *where* $\theta_i = \sum_{r=i}^{4} \frac{n_4}{n_r} \sigma_r^2$.

Proof. First we show that $R_1Y, SS_i, i = 2, 3, 4$ are mutually independent. From Lemma 2.1, we see that $P_iP_j = P_{min\{i,j\}}, i \neq j, i, j = 1, 2, 3, 4$. Hence, $(P_i - P_{i-1})(P_j - P_{j-1}) = 0$, and $(P_i - P_{i-1})P_j = 0$ when $i > j$ $(i, j = 1, 2, 3, 4)$, which implies that R_1Y, SS_3, SS_4 are mutually independent.

From model (2.2), we see that

$$E(Y) = \mu 1_{n_4}, \quad Var\, Y = \sum_{i=1}^{4} \sigma_i^2 X_i X_i'.$$

By (2.3) and the above results, for $i = 3, 4$, we obtain that

$$\begin{aligned}
E(P_i - P_{i-1})Y &= 0, \\
Var(P_i - P_{i-1})Y &= (P_i - P_{i-1}) Var\, Y (P_i - P_{i-1}) \\
&= \sum_{r=1}^{4} \sigma_r^2 (P_i - P_{i-1}) X_r X_r' (P_i - P_{i-1}) \\
&= \sum_{r=i}^{4} \sigma_r^2 (P_i - P_{i-1}) \frac{n_4}{n_r} P_r (P_i - P_{i-1}) \\
&= \sum_{r=i}^{4} \frac{n_4}{n_r} \sigma_r^2 (P_i - P_{i-1}) \\
&= \theta_i (P_i - P_{i-1}).
\end{aligned}$$

That is,

$$(P_i - P_{i-1})Y \sim N(0, \theta_i (P_i - P_{i-1})).$$

We see that $P_i - P_{i-1}$ is idempotent, and $rank(P_i - P_{i-1}) = rank(X_i) - rank(X_{i-1}) = n_i - n_{i-1}$, this follows that $SS_i \sim \theta_i \chi^2_{n_i - n_{i-1}}$, $i = 3, 4$.

This completes the proof of Theorem 2.1.

Let $Q_i = (Q_{i1}', Q_{i2}')'$ $(i = 1, 2)$ be orthogonal matrices, where $Q_{11} = \frac{1}{\sqrt{a}} 1_a'$, $Q_{21} = \frac{1}{\sqrt{b}} 1_b'$, $Q_{12}'Q_{12} = I_a - \frac{1}{a} J_a$, and $Q_{22}'Q_{22} = I_b - \frac{1}{b} J_b$. Next, we employ the orthogonal transformation $H = (H_0', H_1', H_2')'$ to model (2.4), where $H_0 = Q_{11} \otimes Q_{21}$,

$H_1 = Q_{12} \otimes Q_{21}$, $H_2 = I_a \otimes Q_{22}$. Let $HR_1Y = Z = (Z_0', Z_1', Z_2')'$, where $Z_0 = H_0R_1Y, Z_1 = H_1R_1Y, Z_2 = H_2R_1Y$. It follows that

$$E(Z) = \left(\sqrt{ab}\mu, 0\ 0\right)',$$
$$Var(Z) = diag(b\sigma_1^2 + \sigma_2^2, (b\sigma_1^2 + \sigma_2^2)I_{a-1}, \sigma_2^2 I_{a(b-1)}) + \theta_3 HDH'. \tag{2.6}$$

If the data set is balanced, that is $c_{11} = c_{12} = \cdots = c_{ab} = c$, then $HDH' = \frac{1}{cd}I_{ab}$. But in unbalanced cases, it is no longer a diagonal matrix. Next, we get the approximation of HDH' with the form λI_{ab}, that is to say, we want to find the value of λ_0 such that $\lambda_0 = argmin_{\lambda\in R}\|HDH' - \lambda I_{ab}\| = argmin_{\lambda \in R}(tr(D - \lambda I_{ab})^2)^{1/2}$. Then it is easy to obtain the following lemma.

Lemma 2.2. *The best approximation of D with a diagonal matrix of the form λI_{ab} is achieved when $\lambda = \frac{1}{\tilde{c}d}$, where $\tilde{c} = \frac{ab}{\sum_{k_1=1}^{a}\sum_{k_2=1}^{b}\frac{1}{c_{k_1k_2}}}$ is the harmonic mean of the $c_{k_1k_2}$.*

Using Lemma 2.2 and (2.6),

$$Var(Z) \approx diag(b\sigma_1^2 + \sigma_2^2, (b\sigma_1^2 + \sigma_2^2)I_{a-1}, \sigma_2^2 I_{a(b-1)}) + \frac{1}{\tilde{c}d}\theta_3 I_{ab},$$

which follows that

$$Z_1 \sim (approx.)N_{n_1-1}\left(0, \frac{1}{\tilde{c}d}\theta_1 I_{n_1-1}\right),$$
$$Z_2 \sim (approx.)N_{n_2-n_1}\left(0, \frac{1}{\tilde{c}d}\theta_2 I_{n_2-n_1}\right),$$

where $\theta_1 = b\tilde{c}d\sigma_1^2 + \tilde{c}d\sigma_2^2 + d\sigma_3^2 + \sigma_4^2, \theta_2 = \tilde{c}d\sigma_2^2 + d\sigma_3^2 + \sigma_4^2$. Denote $SS_1 = \tilde{c}dZ_1'Z_1$, $SS_2 = \tilde{c}dZ_2'Z_2$, then

$$SS_1 \sim (approx.)\theta_1\chi^2_{n_1-1}, \quad SS_2 \sim (approx.)\theta_2\chi^2_{n_2-n_1}.$$

Such an approximation amounts to replacing $c_{k_1k_2}$ by their harmonic mean $\tilde{c}$. A similar approximation was used by Thomas and Hultiquist (1978) to obtain an approximate test for the between-groups variance component for the unbalanced random one-way model. The next lemma gives the relationship between the USSs.

Lemma 2.3. *The USSs for α and β in model* (2.1) *are equal to SS_1 and SS_2, respectively. And SS_3 and SS_4 are the sums of squares of γ and e.*

Proof. Denote

$$\overline{Y}_{k_1k_2k_3} = \frac{1}{d}\sum_{k_4=1}^{d} Y_{k_1k_2k_3k_4}, \quad \overline{Y}_{k_1k_2} = \frac{1}{c_{k_1k_2}d}\sum_{k_3=1}^{c_{k_1k_2}}\sum_{k_4=1}^{d} Y_{k_1k_2k_3k_4},$$
$$\overline{Y}^*_{k_1\cdot} = \frac{1}{b}\sum_{k_2=1}^{b}\overline{Y}_{k_1k_2}, \quad \overline{Y}^*_{\cdot\cdot} = \frac{1}{a}\sum_{k_1=1}^{a}\overline{Y}^*_{k_1\cdot}$$

Since

$$R_1Y = diag\left(\frac{1}{c_{11}d}1'_{c_{11}d}, \ldots, \frac{1}{c_{ab}d}1'_{c_{ab}d}\right)Y = (\overline{Y}_{11}, \ldots, \overline{Y}_{ab})',$$

$$Y'P_2Y = Y'diag\left(\frac{1}{c_{11}d}J_{c_{11}d}, \ldots, \frac{1}{c_{ab}d}J_{c_{ab}d}\right)Y = \sum_{k_1=1}^{a}\sum_{k_2=1}^{b} c_{k_1k_2}d\overline{Y}^2_{k_1k_2},$$

$$Y'P_3Y = Y'\left(I_{n_3} \otimes \left(\frac{1}{d}J_d\right)\right)Y = \sum_{k_1=1}^{a}\sum_{k_2=1}^{b}\sum_{k_3=1}^{c_{k_1k_2}} d\overline{Y}^2_{k_1k_2k_3},$$

then we obtain that

$$\begin{aligned}
SS_1 &= \tilde{c}dY'R_1'H_1'H_1R_1Y \\
&= \tilde{c}d(\overline{Y}_{11}, \ldots, \overline{Y}_{ab})\left(\left(I_a - \frac{1}{a}J_a\right) \otimes \frac{1}{b}J_b\right)(\overline{Y}_{11}, \ldots, \overline{Y}_{ab})' \\
&= b\tilde{c}d\sum_{k_1=1}^{a}(\overline{Y}^*_{k_1\cdot} - \overline{Y}^*_{\cdot\cdot})^2, \\
SS_2 &= \tilde{c}dY'R_1'H_2'H_2R_1Y \\
&= \tilde{c}d(\overline{Y}_{11}, \ldots, \overline{Y}_{ab})\left(I_a \otimes \left(I_b - \frac{1}{b}J_b\right)\right)(\overline{Y}_{11}, \ldots, \overline{Y}_{ab})' \\
&= \tilde{c}d\sum_{k_1=1}^{a}\sum_{k_2=1}^{b}(\overline{Y}_{k_1k_2} - \overline{Y}^*_{k_1\cdot})^2, \\
SS_3 &= Y'(P_3 - P_2)Y = d\sum_{k_1=1}^{a}\sum_{k_2=1}^{b}\sum_{k_3=1}^{c_{k_1k_2}}(\overline{Y}_{k_1k_2k_3} - \overline{Y}_{k_1k_2})^2, \\
SS_4 &= Y'(I_{n_4} - P_3)Y = \sum_{k_1=1}^{a}\sum_{k_2=1}^{b}\sum_{k_3=1}^{c_{k_1k_2}}\sum_{k_4=1}^{d}(Y_{k_1k_2k_3k_4} - \overline{Y}_{k_1k_2k_3})^2.
\end{aligned}$$

The proof is completed.

Remark 2.1. On the basis of Lemmas 2.1 and 2.3, we conclude that

$$\begin{aligned}
SS_1 &\sim (approx.)(b\tilde{c}d\sigma_1^2 + \tilde{c}d\sigma_2^2 + d\sigma_3^2 + \sigma_4^2)\chi^2_{n_1-1}, \\
SS_2 &\sim (approx.)(\tilde{c}d\sigma_2^2 + d\sigma_3^2 + \sigma_4^2)\chi^2_{n_2-n_1}, \\
SS_3 &\sim (d\sigma_3^2 + \sigma_4^2)\chi^2_{n_3-n_2}, \\
SS_4 &\sim \sigma_4^2\chi^2_{n_4-n_3}. \qquad (2.7)
\end{aligned}$$

3. Testing Variance Component

3.1. *Testing for Zero Variance Component*

Under a regression model which has two variance components, Wald (1947) provided a method to construct the confidence interval on the ratio of variance to the error variance via the F-distribution. Seely and El-Bassiouni (1983) extended the Wald's procedure to test whether the variance is zero in the linear mixed model which has several variance components. Suppose the variance component of interest in the three-fold nested model (2.2) is σ_1^2, and the hypothesis problem to be tested is

$$H_0 : \sigma_1^2 = 0 \quad \text{versus} \quad H_1 : \sigma_1^2 > 0.$$

Now we give the Wald test for above testing problem.

Denote $X = X_0, B = X_1, C = (X_2, X_3)$ for simplicity. Then the model (2.2) is equivalent to

$$Y = X\mu + B\alpha + C(\beta', \gamma')' + X_4 e.$$

Let L be an $k \times n_4$ matrix satisfying $LL' = I_k$, $L'L = P_{(X,B,C)} - P_{(X,C)}$, and F be an $f \times n_4$ matrix satisfying $FF' = I_f$, $F'F = I_{n_4} - P_{(X,B,C)}$, where $k = rank(P_{(X,B,C)} - P_{(X,C)}) = rank(X, B, C) - rank(X, C)$ and $f = rank(I_{n_4} - P_{(X,B,C)}) = n_4 - rank(X, B, C)$. We can obtain that

$$\begin{aligned} LY &= LB\alpha + Le \sim N_k(0, \sigma_1^2 LBB'L' + \sigma_4^2 I_k), \\ FY &= Fe \sim N_f(0, \sigma_4^2 I_f). \end{aligned}$$

Consequently, there exists

$$T = \frac{Y'L'(I_k + \sigma_1^2 LBB'L'/\sigma_4^2)^{-1} LY/k}{Y'F'FY/f} \sim F_{k,f},$$

where $F_{k,f}$ denotes F distribution with k and f degrees of freedom. Note that the matrix $LBB'L'$ is a positive definite matrix, T is a decreasing function of σ_1^2.

When the null hypothesis is true, that is $\sigma_1^2 = 0$, $T(0)$ is the test statistic. Then the corresponding p value is calculated as

$$p = Pr(T \geq T(0)) = F_{k,f}(T(0)).$$

Recently, likelihood-based method for linear mixed-effects models have been well studied in statistical literature, and REML procedures for testing variance components are implemented in well-developed statistical software packages such as SAS and Splus/R with wide applications in many scientific fields. Recently, Greven et al. (2008) proposed two approximation methods for the sample distribution of the likelihood ratio statistics. Although the asymptotic distribution of the restricted likelihood ratio (RLRT) test is derived for a single variance component for the case of linear mixed-effects models, the extension to multiple variance components remains quite challenging. Scheipl et al. (2008) compared a variety of tests for a zero random effect variance with respect to their power and their adherence to the nominal level in a broad range of settings. Their simulation study showed that RLRT test which implemented in the R-package RLRSim (Scheipl, 2010), as implemented in the R-package RLRSim is better than Wald test. And they recommend RLRT to test zero variance components.

In the next subsection, we will develop a nonzero-variance component test and a corresponding efficient implementation procedure.

3.2. Testing for Non Zero Variance Component

Let $\xi = h(\sigma_1^2, \ldots, \sigma_4^2)$ be the function of variance components. It is often desired to consider the following hypotheses for ξ,

$$\begin{aligned} &H_0^I : \xi = \xi_0 \quad versus \quad H_1 : \xi \neq \xi_0, \\ &H_0^{II} : \xi \leq \xi_0 \quad versus \quad H_1 : \xi > \xi_0, \\ &H_0^{III} : \xi \geq \xi_0 \quad versus \quad H_1 : \xi < \xi_0. \end{aligned} \tag{3.1}$$

Now the USSs SS_i, $i = 1, \ldots, 4$ are applied to conduct above hypothesis tests.

Tsui and Weerahandi (1989) and Weeranhandi (1993, 1995) introduced the concept of generalized inference. Assume an observable random vector X has a probability distribution $P_\eta(\cdot)$, where $\eta = (\theta, \delta)$ is an unknown vector in parameter space Ω. Suppose that $\theta = \theta(\eta)$ is the parameter of interest, and δ is the nuisance parameter. Let x is the observed value of X. $T = T(X; x, \eta)$ is a generalized test variable if it satisfies the following properties:

(a) The observed value $t = T(x; x, \eta)$ does not depend on the nuisance parameter δ.
(b) T has a probability distribution free of unknown parameters.
(c) For fixed x and δ, $Pr\{T(X; x, \eta) \geq t \mid \theta\}$ is either nondecreasing or nonincreasing in θ for any given t.

Lemma 3.1. *Denote* $\eta = (\mu, \sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2)$, $T_{\sigma_1^2}(Y, y, \eta) = \frac{1}{b\tilde{c}d}\left(\frac{ss_1}{SS_1}\theta_1 - \frac{ss_2}{SS_2}\theta_2\right)$, $T_{\sigma_2^2}(Y, y, \eta) = \frac{1}{\tilde{c}d}\left(\frac{ss_2}{SS_2}\theta_2 - \frac{ss_3}{SS_3}\theta_3\right)$, $T_{\sigma_3^2}(Y, y, \eta) = \frac{1}{d}\left(\frac{ss_3}{SS_3}\theta_3 - \frac{ss_4}{SS_4}\theta_4\right)$, *and* $T_{\sigma_4^2}(Y, y, \eta) = \frac{ss_4}{SS_4}\theta_4 = \frac{ss_4}{\chi^2_{n_4-n_3}}$, *where* ss_i *is the observed value of* SS_i, $i = 1, 2, 3, 4$. *Then an approximate generalized test variable for* ξ *can be given as*

$$T_\xi(Y, y, \eta) = h(T_{\sigma_1^2}, \ldots, T_{\sigma_4^2}). \tag{3.2}$$

Proof. Since $T_{\sigma_i^2}(y, y, \eta) = \sigma_i^2$, $i = 1, \ldots, 4$, $T_\xi(y, y, \eta) = \xi$. On the other hand, by (2.7), we have

$$\begin{aligned} T_{\sigma_1^2} &= \frac{1}{b\tilde{c}d}\left(\frac{ss_1}{SS_1}\theta_1 - \frac{ss_2}{SS_2}\theta_2\right) \sim (approx.)\frac{1}{b\tilde{c}d}\left(\frac{ss_1}{\chi^2_{n_1-1}} - \frac{ss_2}{\chi^2_{n_2-n_1}}\right), \\ T_{\sigma_2^2} &= \frac{1}{\tilde{c}d}\left(\frac{ss_2}{SS_2}\theta_2 - \frac{ss_3}{SS_3}\theta_3\right) \sim (approx.)\frac{1}{\tilde{c}d}\left(\frac{ss_2}{\chi^2_{n_2-n_1}} - \frac{ss_3}{\chi^2_{n_3-n_2}}\right), \\ T_{\sigma_3^2} &= \frac{1}{d}\left(\frac{ss_3}{SS_3}\theta_3 - \frac{ss_4}{SS_4}\theta_4\right) \sim \frac{1}{d}\left(\frac{ss_3}{\chi^2_{n_3-n_2}} - \frac{ss_4}{\chi^2_{n_4-n_3}}\right), \\ T_{\sigma_4^2} &= \frac{ss_4}{SS_4}\theta_4 \sim \frac{ss_4}{\chi^2_{n_4-n_3}}, \end{aligned}$$

Then the approximate distribution of $T_\xi(Y, y, \eta)$ does not depend on any parametric. Furthermore, we have

$$Pr(T(Y; y, \eta) \geq T(y; y, \eta)) = Pr(h(T_{\sigma_1^2}, \ldots, T_{\sigma_4^2}) \geq \xi),$$

which is non increasing in ξ. Hence, $T(X; x, \eta)$ is the approximate generalized test variable.

Following from Lemma 3.1, the generalized p values for the problems (3.1) are

$$\begin{aligned} H_0^I &: p = 2 \cdot \min\{Pr(T_\xi(Y, y, \eta) \le \xi_0), Pr(T_0(Y, y, \eta) \ge \xi_0)\}, \\ H_0^{II} &: p = Pr(T_\xi(Y, y, \eta) \le \xi_0), \\ H_0^{III} &: p = Pr(T_\xi(Y, y, \eta) \ge \xi_0). \end{aligned} \tag{3.3}$$

4. Simulation Study

To appraise the accuracies of the generalized test provided in Sec. 3, a Monte Carlo simulation is used to estimate its size. The magnitude of variance components and the intraclass correlation coefficient, which are significant in identifying and controlling major sources of variation, are of particular interest in many applications. Without loss of generality, define $\xi_1 = \sigma_1^2/\sum_{i=1}^4 \sigma_i^2$ and $\xi_2 = \sigma_1^2$. Then the two kinds of hypothesis problems are

$$\begin{aligned} H_0^1 &: \xi_1 = c_{01} \leftrightarrow H_1^1 : \xi_1 \ne c_{01}, \\ H_0^2 &: \xi_1 \ge c_{01} \leftrightarrow H_1^2 : \xi_1 < c_{01}, \\ H_0^3 &: \xi_1 \le c_{01} \leftrightarrow H_1^3 : \xi_1 > c_{01}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} H_0^1 &: \xi_2 = c_{02} \leftrightarrow H_1^1 : \xi_2 \ne c_{02}, \\ H_0^2 &: \xi_2 \ge c_{02} \leftrightarrow H_1^2 : \xi_2 < c_{02}, \\ H_0^3 &: \xi_2 \le c_{02} \leftrightarrow H_1^3 : \xi_2 > c_{02}. \end{aligned} \tag{4.2}$$

Following from Lemma 3.1, we can obtain the generalized pivotal quantities of $T_{\xi_i}, i = 1, 2$ are

$$\begin{aligned} T_{\xi_1} &= T_{\sigma_1^2} \Big/ \sum_{i=1}^4 T_{\sigma_i^2}, \\ T_{\xi_2} &= T_{\sigma_1^2}, \end{aligned} \tag{4.3}$$

with their observed values $t_{\xi_i} = c_{0i}, i = 1, 2$. It is obvious that they are stochastically monotonous in their respective parameter of interest and could be applied as the generalized test variables for the relevant hypothesis problems. Thus, for any given $c_{0i}, i = 1, 2$, the generalized p values for the two groups of testing problems (4.1) and (4.2) are

$$\begin{aligned} H_0^1 &: p = 2 \cdot \min\{Pr(T_{\xi_i} \le c_{0i}), Pr(T_{\xi_i} \ge c_{0i})\}, \quad i = 1, 2, \\ H_0^2 &: p = Pr(T_{\xi_i} \le c_{0i}), \quad i = 1, 2, \\ H_0^3 &: p = Pr(T_{\xi_i} \ge c_{0i}), \quad i = 1, 2. \end{aligned} \tag{4.4}$$

By using the Monte Carlo method, the Type I error rates are simulated numerically under various parameter configurations. Table 1 lists the eight designs

Table 1
Design configurations

Design	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
a	2	2	3	3	3	3	5	5
b	2	2	3	3	6	6	3	3
$c_{k_1k_2}$	$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$	$\begin{bmatrix} 3 & 15 \\ 4 & 20 \end{bmatrix}$	$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 5 & 8 \\ 4 & 8 & 10 \end{bmatrix}$	$\begin{bmatrix} 3 & 10 & 20 \\ 2 & 12 & 25 \\ 4 & 20 & 40 \end{bmatrix}$	$\begin{bmatrix} 2 & 4 & 6 & 8 & 10 & 12 \\ 3 & 5 & 8 & 10 & 15 & 20 \\ 4 & 8 & 15 & 20 & 25 & 30 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 15 & 15 & 30 & 30 \\ 35 & 35 & 35 & 5 & 5 & 5 \\ 20 & 20 & 2 & 2 & 40 & 40 \end{bmatrix}$	$\begin{bmatrix} 5 & 5 & 5 \\ 3 & 8 & 15 \\ 16 & 9 & 4 \\ 8 & 15 & 3 \\ 15 & 4 & 9 \end{bmatrix}$	$\begin{bmatrix} 3 & 12 & 30 \\ 5 & 20 & 40 \\ 2 & 50 & 2 \\ 30 & 3 & 12 \\ 40 & 2 & 20 \end{bmatrix}$
d	3	3	5	5	3	3	4	4

with different values of $a, b, c_{k_1k_2}$, and d, which set the values of sample size n_4 ranging from 42–1084.

Several values of ξ_1 and ξ_2 are selected for extensive simulation study. For the hypothesis test concerned ξ_1, we set $\sigma_2^2 = \sigma_3^2 = 1$ and $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 = 10$ with σ_4^2 varying to keep the sum of variance components equal to 10. In the testing procedure for ξ_2, we set $\sigma_2^2 = \sigma_3^2 = \sigma_4^2 = 1$. The algorithm of generalized p value (GP) method for evaluating the type I error rates for certain testing problem based on $T_{\xi_k}, k = 1, 2$ is as followed:

Algorithm

For $l = 1$ to N

Generate a random sample of y from certain designed model.

Compute n_i, $\tilde{c}$ and $SS_i, i = 1, 2, 3, 4$ defined in Sec. 2.

Produce $\chi_1^2 \sim \chi^2(n_1 - 1)$ and $\chi_i^2 \sim \chi^2(n_i - n_{i-1}), i = 2, 3, 4$, and compute T_{ξ_k} using (3.2) and (4.3).

Repeat the above step for M times, and obtain $t_{\xi_k}^j, j = 1, \ldots, M$.

For testing H_0^1, denote p_l as $2 \cdot \min\{proportion\{t_{\xi_k}^j \le c_{0k}, j = 1, \ldots, M\}, proportion\{t_{\xi_k}^j \ge c_{0k}, j = 1, \ldots, M\}\}$; for testing H_0^2, denote p_l as $proportion\{t_{\xi_k}^j \le c_{0k}, j = 1, \ldots, M\}$; for testing H_0^3, denote p_l as $proportion\{t_{\xi_k}^j \ge c_{0k}, j = 1, \ldots, M\}$.

End l loop.

Then the simulated Type I error rates of certain testing problem is the proportion of $\{p_l < \alpha, l = 1, \ldots, N\}$, where α is the given significance level. In the simulation procedure, we set $M = 50{,}000$ and $N = 5{,}000$.

Firstly, we consider the testing problem

$$H_0 : \sigma_1^2 = 0 \quad versus \quad H_1 : \sigma_1^2 > 0,$$

that is $c_{0i} = 0$ in H_0^3. For the given test level 0.05, we compare the size of the proposed GP test with the restricted likelihood ratio test (RLRT), which is conducted by R-package RLRSim for the test problem. The sizes are reported in Table 2. It is evident that the performance of the GP test is as good as the RLRT test.

Under the nominal level 0.05, Tables 3 and 4 provide the simulated results for testing hypothesis in (4.1) and (4.2), respectively. We observe from the table values that the sizes of H_0^1 nominal level, and they are in general very close to the nominal level 5%. And the tests of H_0^2 and H_0^3 are also generally close to the vary little, although the sizes of H_0^2 are smaller, and the sizes of H_0^3 are larger in some cases. Thus, the resulted generalized test has a good frequency property.

Table 2
Simulated Type I error rates for RLRT and GP test

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
RLRT	0.0455	0.055	0.05	0.048	0.039	0.042	0.0385	0.041
GP	0.047	0.05	0.05	0.0525	0.0435	0.0485	0.045	0.055

Table 3
Simulated Type I error rates for test on ξ_1

		Design							
ξ_1		D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
0.1	H_0^1	0.0384	0.0362	0.047	0.0442	0.044	0.0492	0.0512	0.0476
	H_0^2	0.024	0.0232	0.0352	0.0376	0.0496	0.0454	0.0432	0.0408
	H_0^3	0.0598	0.0556	0.0572	0.0522	0.048	0.0518	0.0594	0.0496
0.2	H_0^1	0.0354	0.0376	0.0434	0.0414	0464	0.047	0.0518	0.044
	H_0^2	0.0184	0.0226	0.0354	0.0378	0.0466	0.0464	0.0416	0.0414
	H_0^3	0.061	0.0632	0.0586	0.0534	0.0468	0.0516	0.0554	0.0508
0.4	H_0^1	0.0368	0.0374	0.0468	0.0402	0.0482	0.047	0.0482	0.0482
	H_0^2	0.0188	0.0222	0.0358	0.0376	0.0474	0.0464	0.041	0.0398
	H_0^3	0.0612	0.0586	0.0584	0.0514	0.0486	0.0484	0.0544	0.0532
0.6	H_0^1	0.0364	0.033	0.043	0.0416	0.048	0.0488	0.0484	0.0478
	H_0^2	0.0208	0.0214	0.0352	0.0366	0.0472	0.0474	0.041	0.04
	H_0^3	0.0586	0.054	0.0544	0.0506	0.0496	0.0486	0.0564	0.054
0.7	H_0^1	0.0396	0.0356	0.0446	0.0448	0.0486	0.0506	0.0456	0.0478
	H_0^2	0.0236	0.0232	0.0356	0.0368	0.0464	0.0476	0.0408	0.0384
	H_0^3	0.06	0.055	0.0562	0.0504	0.0504	0.049	0.0582	0.0534
0.8	H_0^1	0.0428	0.0376	0.0462	0.0484	0.0476	0.0496	0.0432	0.0482
	H_0^2	0.0428	0.0376	0.04	0.0396	0.0452	0.0472	0.0432	0.0482
	H_0^3	0.0574	0.0546	0.0528	0.0528	0.05	0.0492	0.057	0.0576

Table 4
Simulated Type I error rates for test on ξ_2

		Design							
ξ_2		D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
1	H_0^1	0.0372	0.036	0.0454	0.0438	0.047	0.0456	0.0512	0.0444
	H_0^2	0.0232	0.0226	0.0392	0.0374	0.0486	0.0472	0.0448	0.041
	H_0^3	0.0534	0.0558	0.0592	0.053	0.047	0.0516	0.057	0.05
2	H_0^1	0.0312	0.0366	0.0494	0.043	0.0466	0.0482	0.0498	0.0462
	H_0^2	0.0216	0.0234	0.039	0.0386	0.0488	0.0476	0.0448	0.0434
	H_0^3	0.0486	0.0574	0.0578	0.0542	0.0498	0.0506	0.054	0.0516
4	H_0^1	0.0352	0.0364	0.0496	0.046	0.0482	0.0524	0.0506	0.0522
	H_0^2	0.0242	0.0266	0.0412	0.0446	0.0518	0.0498	0.048	0.0454
	H_0^3	0.0528	0.0542	0.0546	0.0484	0.049	0.0498	0.054	0.0516
6	H_0^1	0.0378	0.0346	0.0496	0.047	0.0496	0.0516	0.0494	0.0532
	H_0^2	0.0256	0.0286	0.0456	0.0462	0.0508	0.0508	0.0486	0.0482
	H_0^3	0.0544	00506	0.0538	0.0498	0.0498	0.0484	0.0516	0.0526
8	H_0^1	0.0398	0.0362	0.0482	0.0482	0.05	0.0506	0.0516	0.0538
	H_0^2	0.0268	0.031	0.048	0.0472	0.0508	0.0516	0.05	0.0494
	H_0^3	0.052	00504	0.0528	0.048	0.0488	0.0482	0.0536	0.0512
10	H_0^1	0.0406	0.0352	0.0454	0.0462	0.0476	0.0492	0.0504	0.0504
	H_0^2	0.029	0.0316	0.0432	0.044	0.0462	0.0464	0.0444	0.0428
	H_0^3	0.0538	0.0512	0.0506	0.0506	0.0486	0.0508	0.0498	0.0518

5. Summary and Discussion

Testing for functions of variance components in three-fold nested mode is an important problem in statistics which needs to be addressed frequently in practice. Although testing zero variance component could be done by ML and REML test, one needs to consider the testing problem for non zero variance components. However, to the best of our knowledge, there does not exist an approach for the problem. This article aims to fill this gap.

In this article, we reconsider the Wald test and propose an generalized p value for functions of variance components in three-fold nested model with unequal number of levels for the third random factor. The proposed generalized approach can easily provide p values by using a few straightforward simulation steps. The simulation studies indicate that the Type I errors of the proposed generalized p value approach are generally satisfactory.

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