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Dispersion relation of degenerated electron–positron plasma in an ultra-relativistic regime

S. Q. LIU, Y. L. LIAO, X. L. LIU, Q. S. XIAO and W. D. ZHANG

Department of Physics, NanChang University, JiangXi, NanChang 330031, People's Republic of China
(liao0705@163.com)

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Abstract. The dispersion relation for superluminal waves in degenerated and isotropic electron–positron plasmas is investigated. The dispersion equation of linear waves is derived from the relativistically correct form of the dielectric function and the Fermi distribution function. Analytical dispersion laws for the real part of the wave frequency are derived by applying the long-wavelength approximation and the short-wavelength approximation. Using the numerical simulation method, we obtain the full dispersion curve which cannot be given by an analytic method.

1. Introduction

Electron–positron plasmas have been widely studied theoretically owing to their relevance in astrophysical contexts such as pulsars, Solar flares, black holes, and the jet phenomena associated with active galactic nuclei (Hirotani and Iguchi 2000; Iwamoto and Takahapa 2002; Nishikawa et al. 2005). Electron–positron plasmas represent the larger class of plasmas with the same masses and electric charge magnitudes for the particle and the antiparticle, a class of plasmas that may offer plasma physical properties quite different from those of conventional electron–ion plasmas. In electron–positron plasmas, the theory of dispersion relations is expected to provide predictions for a variety of interesting phenomena unique to particle–antiparticle plasmas.

Recently, there has been a great deal of interest in studying the physics of highly relativistic electron–positron plasma (Gedalin et al. 1998; Laing and Diver 2005, 2006). Therefore, in this paper, we describe an ultra-relativistic dispersion relation in an unmagnetized, collisionless electron–positron plasma. Such highly relativistic plasmas can be found in the early universe. In the cosmological case, the plasma density is from 10^{33} to 10^{28} cm^{-3} (Tajima and Taniuti 1990). It has provided the decision of plasma parameters for numerical evaluation.

Collective effects in relativistically electron–positron plasmas are believed to play an important role. In view of the complexity of the charged-particle distributions at relativistic temperatures, the kinetic theory of a relativistically electron–positron plasma is expected to be considerably different from that of the usual electron–ion plasma (Tsintsadze and Shukla 1992).

In the present work, dispersion relations for linear waves in a relativistic, unmagnetized electron–positron plasma are derived. The derivation is restricted to a collisionless and homogeneous pair plasma in thermal equilibrium. Dispersion relations are fundamental for the understanding of the linear properties of homogeneous plasmas. To derive a dispersion relation in kinetic plasma theory, a distribution function for each particle species is introduced to describe the unperturbed plasma (Tjulin et al. 2000). Our derivation includes both the relativistically correct form of the Vlasov equation and the Fermi distribution function. We obtain the analytical results of longitudinal oscillations in these plasmas. The numerical solution of the full dispersion laws which cannot be given by an analytic method is obtained.

This paper is structured in the following way. In Sec. 2 we derive the longitudinal permittivity, starting from the ultra-relativistic Vlasov–Fermi system. The dispersion equation for longitudinal oscillations is obtained. In Sec. 3 we obtain the analytic dispersion laws, and in Sec. 4 we present numerical calculations. The results are summarized and discussed in Sec. 5.

2. Dispersion equation

Using kinetic theory, one must also take the particular choice of distribution function into account (Bergman and Eliasson 2001). Positrons and electrons will be taken to have the same equilibrium distribution function, which in this case is the counterpart of the Fermi distribution function,

$$f_\alpha^R(\mathbf{p}) = \frac{2}{h^3}\frac{1}{e^{(\varepsilon-\mu_\alpha)/k_{\mathrm{B}}T_\alpha}+1}, \tag{1}$$

where μ_α denotes the chemical potential, $\varepsilon = c\sqrt{m^2c^2+p^2}$ is the energy, $\mathbf{p}$ denotes the momentum of particles, c is the speed of light in vacuum, m is the mass of particles, T_α is the temperature of particles, k_{B} is the Boltzmann constant, h is the Planck's constant and the label α characterizes the electrons ($\alpha = \mathrm{e}$) and the positrons ($\alpha = \mathrm{p}$).

The normalization is chosen so the particle number density n_α is defined as

$$\int f_\alpha^R \, d\mathbf{p} = n_\alpha. \tag{2}$$

For the ultra-relativistic case, $\varepsilon = cp$ (Bergman and Eliasson 2001), the distribution function also satisfies the normalizing condition:

$$\int f(\mathbf{p})\, d\mathbf{p} = \int \frac{2}{h^3}\frac{4\pi(\varepsilon/c)^2}{e^{(\varepsilon-\mu)/k_{\mathrm{B}}T}+1}\frac{1}{c}\, d\varepsilon = n_\alpha. \tag{3}$$

Taking into account the condition $\mu \gg k_{\mathrm{B}}T$ (Pathria 2003), we can calculate the ε integral in (3):

$$\begin{aligned} I &= \int_0^\infty \frac{\varepsilon^2}{e^{(\varepsilon-\mu)/k_{\mathrm{B}}T}+1}\, d\varepsilon = \int_{-\mu/k_{\mathrm{B}}T}^0 k_{\mathrm{B}}T\frac{(k_{\mathrm{B}}Tx+\mu)^2}{e^x+1}\, dx \\ &= \frac{1}{3}\pi^2 k_{\mathrm{B}}^2T^2\mu^2 + \frac{1}{3}\mu^3. \end{aligned}$$

So we can obtain

$$\frac{8\pi}{(hc)^3}\left(\frac{1}{3}\pi^2k_{\mathrm{B}}^2T^2\mu^2 + \frac{1}{3}\mu^3\right) = n_\alpha. \tag{4}$$

According to (4), it is obvious that this equation tells us of the dependence of the chemical potential on the particle number density and the temperature of electrons and positrons of the system.

Let us consider an unmagnetized, collisionless plasma. Using kinetic theory to calculate the quantity ε_k^l, the longitudinal dielectric function can be obtained (Alexandrov et al. 1984):

$$\varepsilon_k^l = 1 + \sum_\alpha \frac{4\pi e^2}{\omega k^2} \int d\mathbf{p} \frac{\partial f_\alpha^R}{\partial \varepsilon} \frac{(\mathbf{k}\cdot\mathbf{v})^2}{\omega - \mathbf{k}\cdot\mathbf{v} + i\varepsilon}. \tag{5}$$

We now choose the z-axis along the direction of the wave vector $\mathbf{k}$, $\mathbf{k} = k\hat{\mathbf{z}}$, so that (5) becomes

$$\varepsilon_k^l = 1 + \sum_\alpha \frac{4\pi e^2}{\omega} \int d\mathbf{p} \frac{\mathrm{v}_z^2}{\omega - k\mathrm{v}_z + i\varepsilon} \frac{\partial f}{\partial \varepsilon}. \tag{6}$$

Introducing the reduced velocity, $\mathbf{u} = \mathbf{p}/m_\mathrm{e}c = \gamma\mathbf{v}/c$, so $\gamma = (1+u^2)^{1/2}$. Here $(\mathbf{u}, \theta, \varphi)$ represent the spherical coordinate system, so the dielectric function ε_k^l is given by

$$\varepsilon_k^l = 1 - \sum_\alpha \frac{8\pi^2 e^2 c}{\omega k}(m_\mathrm{e}c)^3 \int_0^\infty du \frac{u^3}{\gamma}\frac{\partial f}{\partial \varepsilon} \int_{-1}^{1} \frac{x^2\,dx}{x - (\gamma\bar{v}_\mathrm{p}/u) - i\varepsilon}, \tag{7}$$

where $\bar{v}_\mathrm{p} \equiv \omega/kc$.

Using the Plemelj formula (where $\wp$ denotes the principal value)

$$\frac{1}{z \pm i0} = \wp\frac{1}{z} \mp i\pi\delta(z), \tag{8}$$

the x integral in (7) becomes (Li 2004)

$$\begin{aligned} G &= \int_{-1}^{1} \frac{x^2\,dx}{x - (\gamma\bar{v}_\mathrm{p}/u) - i\varepsilon} \\ &= 2\frac{\gamma\bar{v}_\mathrm{p}}{u} - \frac{1}{2}\left(\frac{\gamma\bar{v}_\mathrm{p}}{u}\right)^2 \ln\left(\frac{\bar{v}_\mathrm{p} + u/\gamma}{\bar{v}_\mathrm{p} - u/\gamma}\right)^2 + i\pi\left(\frac{\gamma\bar{v}_\mathrm{p}}{u}\right)^2 \theta\left(1 - \frac{\gamma\bar{v}_\mathrm{p}}{u}\right), \end{aligned} \tag{9}$$

where $\theta(\xi)$ is the step function, indicating that $\theta(\xi) = 1$ at $\xi > 0$ and $\theta(\xi) = 0$ at $\xi < 0$.

Considering an ultra-relativistic electron–positron plasma $u/\gamma \sim 1$ (Chen et al. 2006), (9) then reduces to

$$G' = 2\bar{v}_\mathrm{p} - \frac{1}{2}(\bar{v}_\mathrm{p})^2 \ln\left(\frac{\bar{v}_\mathrm{p}+1}{\bar{v}_\mathrm{p}-1}\right)^2 + i\pi(\bar{v}_\mathrm{p})^2\theta(1-\bar{v}_\mathrm{p}). \tag{10}$$

Assuming an isotropic medium, we obtain the longitudinal dispersion relation (Carrington et al. 2004)

$$\varepsilon_k^l(\omega, k) = 0. \tag{11}$$

According to (9), we substitute (1) into (7). Therefore, we can obtain the dispersion equation of longitudinal modes:

$$\varepsilon_k^l = 1 - \sum_\alpha \frac{8\pi^2 e^2 c}{\omega k}(m_\mathrm{e}c)^3 \int_0^\infty du \frac{u^3}{\gamma} G \frac{\partial}{\partial \varepsilon}\left(\frac{2}{h^3}\frac{1}{e^{(\varepsilon-\mu_\alpha)/k_\mathrm{B}T_\alpha} + 1}\right). \tag{12}$$

3. Linear wave dispersion laws

We restrict our discussion here to superluminal waves ($\bar{v}_{\mathrm{p}} > 1$), so the imaginary parts of the dispersion relation disappear (Schlickeiser and Mause 1995); the case of subluminal waves ($\bar{v}_{\mathrm{p}} < 1$) is the subject for further research. We are now in a position to investigate the dispersion relations of the longitudinal modes.

Considering an ultra-relativistic electron–positron plasma $u/\gamma \sim 1$ and $\varepsilon = cp = mc^2 u$, the integral in (7) becomes

$$\begin{aligned}
&\int_0^\infty du \frac{u^3}{\gamma}\frac{\partial f}{\partial \varepsilon} \\
&\quad = \int_0^\infty u^2 \frac{\partial f}{\partial \varepsilon}\, du \\
&\quad = -\frac{2}{h^3 k_{\mathrm{B}} T}\int_0^\infty u^2 \frac{e^{(mc^2u-\mu)/k_{\mathrm{B}}T}}{(e^{(mc^2u-\mu)/k_{\mathrm{B}}T}+1)^2}\, du \\
&\quad = -\frac{2}{h^3}\frac{1}{(mc^2)^3}\left[\int_{-\mu/k_{\mathrm{B}}T}^\infty \frac{(k_{\mathrm{B}}Tx+\mu)^2}{e^x+1}\, dx - \int_{-\mu/k_{\mathrm{B}}T}^\infty \frac{(k_{\mathrm{B}}Tx+\mu)^2}{(e^x+1)^2}\, dx\right].
\end{aligned}$$

If we define the integrals

$$\begin{aligned}
A &= \int_{-\mu/k_{\mathrm{B}}T}^\infty \frac{(k_{\mathrm{B}}Tx+\mu)^2}{e^x+1}\, dx, \\
B &= -\int_{-\mu/k_{\mathrm{B}}T}^\infty \frac{(k_{\mathrm{B}}Tx+\mu)^2}{(e^x+1)^2}\, dx \\
&= \int_{-\mu/k_{\mathrm{B}}T}^\infty e^{-x}(k_{\mathrm{B}}Tx+\mu)^2\, dx - \int_{-\mu/k_{\mathrm{B}}T}^\infty \frac{e^{-x}(k_{\mathrm{B}}Tx+\mu)^2}{e^{-x}+1}\, dx \\
&\quad - 2k_{\mathrm{B}}T\int_{-\mu/k_{\mathrm{B}}T}^\infty e^{-x}(k_{\mathrm{B}}Tx+\mu)\, dx + 2k_{\mathrm{B}}T\int_{-\mu/k_{\mathrm{B}}T}^\infty \frac{e^{-x}(k_{\mathrm{B}}Tx+\mu)}{e^{-x}+1}, \\
C &= \int_{-\mu/k_{\mathrm{B}}T}^\infty e^{-x}(k_{\mathrm{B}}Tx+\mu)^2\, dx \\
&= 2k_{\mathrm{B}}T\int_{-\mu/k_{\mathrm{B}}T}^\infty e^{-x}(k_{\mathrm{B}}Tx+\mu)\, dx, \\
D &= -\int_{-\mu/k_{\mathrm{B}}T}^\infty \frac{e^{-x}(k_{\mathrm{B}}Tx+\mu)^2}{e^{-x}+1}\, dx = -A, \\
E &= -2k_{\mathrm{B}}T\int_{-\mu/k_{\mathrm{B}}T}^\infty e^{-x}(k_{\mathrm{B}}Tx+\mu)\, dx = -C, \\
F &= 2k_{\mathrm{B}}T\int_{-\mu/k_{\mathrm{B}}T}^\infty \frac{e^{-x}(k_{\mathrm{B}}Tx+\mu)}{e^{-x}+1}\, dx \\
&= \mu^2 + \frac{1}{3}\pi^2 k_{\mathrm{B}}^2 T^2,
\end{aligned}$$

then we obtain

$$\int_0^\infty du \frac{u^3}{\gamma}\frac{\partial f}{\partial \varepsilon} = -\frac{2}{h^3}\frac{1}{(mc^2)^3}\left(\mu^2 + \frac{1}{3}\pi^2 k_{\mathrm{B}}^2 T^2\right). \tag{13}$$

In the ultra-relativistic approximation, taking into account (11)–(13), the longitudinal dispersion equation becomes

$$1 + \sum_\alpha \frac{16\pi^2 e^2}{\omega k}\frac{1}{c^2 h^3}\left(\mu_\alpha^2 + \frac{1}{3}k_{\mathrm{B}}^2 T_\alpha^2 \pi^2\right)\left[2\bar{v}_{\mathrm{p}} - \frac{1}{2}\bar{v}_{\mathrm{p}}^2 \ln\left(\frac{\bar{v}_{\mathrm{p}}+1}{\bar{v}_{\mathrm{p}}-1}\right)^2\right] = 0. \tag{14}$$

From this general relation we will consider two limiting cases: the long wave and the short wave. We consider each case in turn. As a first check of the general longitudinal dispersion relation (14), we consider the limiting case of the long wave for which $\bar{v}_{\mathrm{p}} > 1$. The longitudinal dispersion relation can be written as

$$\sum_\alpha \frac{k^2 c^3 h^3}{32\pi^2 e^2 (\mu_\alpha^2 + \frac{1}{3}k_{\mathrm{B}}^2 T_\alpha^2 \pi^2)} = \frac{1}{3\bar{v}_{\mathrm{p}}^2} + \frac{1}{5\bar{v}_{\mathrm{p}}^4}. \tag{15}$$

For $\bar{v}_{\mathrm{p}} \gg 1$, we can obtain the first-order approximate solution of (15):

$$\bar{v}_{\mathrm{p}}^2 = \frac{1}{3}\sum_\alpha \frac{32\pi^2 e^2}{k^2 c^3 h^3}\left(\mu_\alpha^2 + \frac{1}{3}k_{\mathrm{B}}^2 T_\alpha^2 \pi^2\right). \tag{16}$$

Substituting (16) into (15), we obtain

$$\omega^2 = \frac{1}{3}\sum_\alpha \frac{32\pi^2 e^2}{h^3 c}\left(\mu_\alpha^2 + \frac{1}{3}k_{\mathrm{B}}^2 T_\alpha^2 \pi^2\right) + \frac{3}{5}k^2 c^2. \tag{17}$$

For $\bar{v}_{\mathrm{p}} > 1$, and $\bar{v}_{\mathrm{p}} \sim 1$, the longitudinal dispersion equation of short wavelength becomes

$$1 + \sum_\alpha \frac{16\pi^2 e^2}{k^2}\frac{1}{c^3 h^3}\left(\mu_\alpha^2 + \frac{1}{3}k_{\mathrm{B}}^2 T_\alpha^2 \pi^2\right)\left[2 - \ln\left(\frac{\bar{v}_{\mathrm{p}}+1}{\bar{v}_{\mathrm{p}}-1}\right)\right] \approx 0, \tag{18}$$

or

$$\bar{v}_{\mathrm{p}} \approx 1 + 2\exp\left(-2 - \sum_\alpha \frac{k^2 h^3 c^3}{16\pi^2 e^2}\frac{1}{\mu_\alpha^2 + \frac{1}{3}k_{\mathrm{B}}^2 T_\alpha^2 \pi^2}\right). \tag{19}$$

Hence, the dispersion relation can be expressed as

$$\omega = kc\left[1 + 2\exp\left(-2 - \sum_\alpha \frac{k^2 h^3 c^3}{16\pi^2 e^2}\frac{1}{\mu_\alpha^2 + \frac{1}{3}k_{\mathrm{B}}^2 T_\alpha^2 \pi^2}\right)\right]. \tag{20}$$

The analytic solutions of the longitudinal dispersion relation are obtained. In comparison with the analytic dispersion laws in classical plasmas, it can be seen that the dispersion curves in the electron–positron plasma and the electron–ion plasma have analogous phenomena (Mikhailovskii 1979), which is that separate parts of this dispersion curve are described by different analytic formulas depending on the temperature and the chemical potential. From the analytical solutions, we have shown that the dispersion curve is discontinuous in the case of an ultra-relativistic plasma. Using the numerical simulation method we will study the full dispersion curves in the following section.

4. Numerical calculation

In this section we will numerically evaluate the wavenumber dependence of the frequency from the dispersion relation calculated in the previous section. We first introduce the dimensionless variables

$$\Omega = \frac{\omega}{\omega_{\rm pe}}, \quad K = \frac{kc}{\omega_{\rm pe}}, \quad \bar{v}_{\rm p} = \frac{\Omega}{K}, \quad x = \frac{mc^2 u - \mu}{k_{\rm B}T}, \quad y = \frac{e^2 c^2 (mc)^3}{k_{\rm B}T\omega_{\rm pe}^2 h^3}, \tag{21}$$

where $\omega_{\rm pe} = (4\pi n e^2/m_{\rm e})^{1/2}$ is the electron plasma frequency.

Positrons and electrons will be taken to have the same temperature. So the dimensionless longitudinal dispersion equation can be expressed as

$$\varepsilon_k^l = 1 + \frac{32\pi^2 y}{\Omega K}\int_0^\infty u^2 \frac{e^x}{(e^x+1)^2} G' \, du = 0. \tag{22}$$

Based on the analytical results discussed in the preceding section, we have chosen to solve the longitudinal dispersion equation (22) numerically for the ultra-relativistic approximation. The full dispersion equation is transformed so that it is well suited for numerical evaluation in the temperature range where an ultra-relativistic treatment is needed. Based on the discontinuity of the analytical dispersion curves we have chosen the calculation range, since we hope that the numerical dispersion curve can connect the two analytical dispersion curves.

In the previous section, we have shown that the analytical dispersion curves have a discontinuity in the range $1 < \bar{v}_{\rm p} < 10$. So our work is in calculating the dispersion relation which cannot be obtained by an analytic method.

The dynamics of electrons and the positrons with large number density ($\sim 10^{30}$ cm^{-3}) is relativistic, and the Fermi temperature of an electron–positron plasma is of the order $10^{10} K$ (K here is the unit of energy) in a state of complete degeneracy (Pathria 2003). So we consider these parameter values for an electron–positron plasma such that the plasma is simultaneously degenerate and relativistic. Then, the numerical calculation is completed by using these plasma parameters.

In Fig. 1 we show the analytical dispersion curves from (17) and (20) and the dispersion curve calculated by the numerical method. It can be seen that the dispersion curves are continuous. The long and the short waves are joined by the numerical dispersion wave. We have thus established the important result that we can obtain the full dispersion curve from the numerical method and the analytical method.

5. Summary and discussion

In the present paper, we have investigated the ultra-relativistic dispersion relation for superluminal waves in an unmagnetized and collisionless electron–positron plasma. The dispersion relations describe the linearized response of the system to the initial perturbation and thus define all existing linear plasma modes in the system. The dispersion function of longitudinal waves is obtained when we consider the Fermi distribution function.

The analytical results for longitudinal waves in an isotropic plasma are presented. Separate parts of this dispersion curve are described by different analytic formulas depending on the temperature and the chemical potential. The results in an electron–positron plasma and the results in an electron–ion plasma are similar with respect to the dispersion curve. Analytical dispersion laws for the real part of the

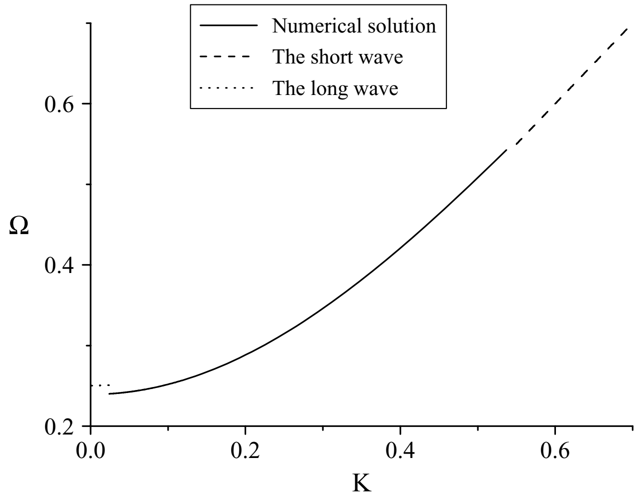

Figure 1. Numerical solution and analytical solutions of the longitudinal dispersion relation in an ultra-relativistic case.

wave frequency have been derived by applying the long wavelength approximation and the short wavelength approximation. From the analytical solutions, we have shown that dispersion curves have a discontinuity in the case of an ultra-relativistic plasma.

We have studied the full dispersion curve by using a numerical simulation method. The long and short waves (the analytical dispersion curves) were joined by a numerical dispersion wave. Therefore, we have established that the continuous dispersion curve of a degenerated electron–positron plasma in an ultra-relativistic regime can be obtained by jointly using a numerical method and an analytical method.

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