Extending T^p automorphisms over $\mathbb{R}^{p+2}$ and realizing DE attractors

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Abstract

In this paper we consider the realization of DE attractors by self-diffeomorphisms of manifolds. For any expanding self-map $\phi : M \to M$ of a connected, closed p-dimensional manifold M, one can always realize a (p,q)-type attractor derived from ϕ by a compactly-supported self-diffeomorphsm of $\mathbb{R}^{p+q}$, as long as $q \geq p+1$. Thus lower codimensional realizations are more interesting, related to the knotting problem below the stable range. We show that for any expanding self-map ϕ of a standard smooth p-dimensional torus T^p, there is compactly-supported self-diffeomorphism of $\mathbb{R}^{p+2}$ realizing an attractor derived from ϕ. A key ingredient of the construction is to understand automorphisms of T^p which extend over $\mathbb{R}^{p+2}$ as a self-diffeomorphism via the standard unknotted embedding $\imath_p : T^p \hookrightarrow \mathbb{R}^{p+2}$. We show that these automorphisms form a subgroup $E_{\imath_p}$ of $\mathrm{Aut}(T^p)$ of index at most $2^p - 1$.

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1 Introduction

Hyperbolic attractors derived from expanding maps were introduced by Steve Smale in his celebrated paper [Sm] in the 1960s. Smale posed four families of basic sets for his Spectral Decomposition Theorem for the non-wandering set of self-diffeomorphisms of smooth manifolds: *Group 0* which are zero dimensional ones such as isolated points and the Smale horseshoe; *Group A* and *Group DA*, both of which are derived from Anosov maps; and *Group DE* which are attractors derived from expanding maps. While the first three families arise easily or automatically from self-diffeomorphisms of manifolds, it is less obvious whether and how attractors of Group DE could be realized via self-diffeomorphisms of manifolds. In this paper, we study the realization problem of DE attractors. We shall stay in the smooth category.

Definition 1.1. (1) Let M be an connected, closed p-dimensional smooth manifold. A smooth map $\phi : M \to M$ is said to be *expanding* if for some complete Riemannian metric on M, there exist constants $c > 0$, $\lambda > 1$ such that for any $x \in N$ and $v \in T_xM$, $\|\mathrm{d}\phi^m(v)\| \geq c\lambda^m \|v\|$, for every integer $m > 0$.

(2) Let M, ϕ be as above. We say ϕ *lifts* to a hyperbolic bundle embedding $e : E \hookrightarrow E$, if $\pi : E \to M$ is a compact unit (Euclidean) q-dimensional disk bundle over M, and e is a smooth self-embedding of E which descends to ϕ under π, and for some positive number $r < 1$, e sends every fiber q-disk to an embedded q-subdisk of radius r in the interior of the target fiber. Suppose there exists such a lift, then $\Lambda = \bigcap_{i \geq 0} e^i(E)$ is called a (p,q)*-type attractor* derived from the expanding map ϕ (w.r.t. e), or simply a *DE attractor*.

(3) Let M, ϕ, E, e be as above. Let X be a $(p+q)$-dimensional smooth manifold, and $f : X \to X$ be a self-diffeomorphism of X. If there is a smooth embedding $E \subset X$ such that $\Lambda = \bigcap_{i \geq 0} f^i(E)$, we say that f *realizes* the DE attractor Λ in manifolds.

Expanding maps of closed manifolds are topologically conjugate to expanding infranil endomorphisms of infranil manifolds ([Gr]). It is also known that any flat manifold admits an expanding map ([ES]). In his original paper, Smale only considered trivial $(p+1)$-disk bundle embeddings which lifts an expanding map ϕ, but it seems to be reasonable and necessary to allow twisted bundles here to ensure Theorem 1.2. For example, if M is an orientable closed non-spin flat manifold (e.g. [DSS]) while X is spin, it is impossible to realize any DE attractor of M on X if we restrict ourselves to trivial bundle lifts, simply because the normal bundle of any embedding of M into X is also non-spin hence nontrivial.

Proposition 1.2. *Suppose M is a connected, closed p-dimensional smooth manifold, $p \geq 1$, and $\phi : M \to M$ is an expanding map. For any $q \geq p+1$, there is a compactly-supported self-diffeomorphism of $\mathbb{R}^{p+q}$, which realizes a (p,q)-type attractor derived from ϕ.*

Remark 1.3. A self-diffeomorphism of a smooth manifold is said to be compactly-supported if it fixes every point outside a compact set. This clear implies that the result holds for any $(p+q)$-dimensional manifold X besides $\mathbb{R}^{p+q}$.

Proposition 1.2 suggests that the realization problem of DE attractors of codimensions below the 'stable range' is more interesting. This is basically because in lower codimensions, for an arbitrary embedding $E \subset X$ (with notations in Definition 1.1), if any, the cores of $e^i(E) \subset X$, for different $i \geq 0$ may be knotted in different ways. In this sense, the realization problem of DE attractors is essentially about the knotting problem of embeddings and satellite knot constructions. On the other hand, as any (p,q)-type attractor Λ derived from an expanding map $\phi : M \to M$ is homeomorphic to the inverse limit $\varprojlim (M, \phi) = \{(x_0, x_1, \cdots) \in \prod_{n=0}^{\infty} M \mid x_i = \phi(x_{i+1}),\ 0 \leq i < \infty\}$ (sometimes known as a p-dimensional *solenoid*), it cannot be embedded into any $(p+1)$-dimensional closed orientable manifold if M is also orientable ([JWZ]). Thus it becomes natural to wonder whether there are DE attractors realizable in orientable closed manifolds of codimension 2. When M is diffeomorphic to the flat p-dimensional torus $T^p = S^1 \times \cdots \times S^1$ (p copies), probably the simplest p-dimensional manifold admitting expanding maps, the study of unknotted embeddings of T^p into $\mathbb{R}^{p+2}$ allows us to give a positive answer:

Theorem 1.4. *For any expanding map $\phi : T^p \to T^p$, $p \geq 1$, there is a compactly-supported self-diffeomorphism of $\mathbb{R}^{p+2}$ which realizes a $(p,2)$-type attractor derived from ϕ.*

To get some idea of the proof of Theorem 1.4, consider the toy case when $p = 1$. An expanding map $\phi : S^1 \to S^1$ is nothing but a d-fold covering where $d > 1$. If $e : E \hookrightarrow E$ is a disk bundle embedding lifting ϕ, and $\jmath : E \hookrightarrow \mathbb{R}^3$ is an embedding, the core of $\jmath(E)$ is a knot $K \subset \mathbb{R}^3$, and the core of $\jmath \circ e(E)$ is a satelite knot K' with the companion K and the pattern a braid in the solid torus of winding number d. Note also that E must be a trivial bundle $S^1 \times D^2$ in this case. If there exists a self-diffeomorphism f of $\mathbb{R}^3$ so that $f \circ \jmath = \jmath \circ e$, K, K' should have homeomorphic exteriors, so K has to be trivial and the braid could be, for example, a $(d,1)$-cable (in the usual longitude-meridian notations of classical knot theory). This suggests in the general case we should also consider unknotted, framing untwisted embeddings $\jmath : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ (Definitions 3.4 and 4.3) and bundle embeddings $e : T^p \times D^2 \hookrightarrow T^p \times D^2$ respecting the framing.

The real issue when $p \geq 2$ is that $\jmath \circ e$ and $\jmath$ may still be non-isotopic even if they have isotopic images, no matter how we choose e and $\jmath$. This is essentially because there are self-diffeomorphisms of T^p that cannot be extended as a self-diffeomorphism of $\mathbb{R}^{p+2}$ via a given embedding $\imath : T^p \hookrightarrow \mathbb{R}^{p+2}$, due to certain spin obstructions, ([DLWY]). To overcome this difficulty, we say two embeddings $\jmath, \jmath' : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ have different types if they are isotopic up to image but not isotopic as embeddings, (Definitions 4.3, 3.6, cf. Remark 3.5). We will show that there are at most finitely many types arising in $\jmath \circ e^i$ for $i \geq 0$, so that some power of e extends as a (compactly-supported) self-diffeomorphism of $\mathbb{R}^{p+2}$ over some $\jmath \circ e^k$. The key ingredient is the following theorem, where $\mathrm{Aut}(T^p) \cong \mathrm{SL}(p,\mathbb{Z})$ is the group of automorphisms on T^p, and $E_\imath \leq \mathrm{Aut}(T^p)$ consists of elements which extend as compactly-supported self-diffeomorphisms of $\mathbb{R}^{p+2}$ via $\imath$, (cf. Section 3 and Definition 3.8):

Theorem 1.5. *For the standard unknotted embedding $\imath_p : T^p \hookrightarrow \mathbb{R}^{p+2}$, $(p \geq 1)$, the subgroup $E_{\imath_p}$ contains the stabilizer of some nontrivial element in $H^1(T^p;\mathbb{Z}_2)$ under the natural action of $\mathrm{Aut}(T^p)$. Hence the index of $E_{\imath_p}$ in $\mathrm{Aut}(T^p)$ is at most $2^p - 1$.*

Remark 1.6. In fact, the index of $E_{\imath_p}$ in $\mathrm{Aut}(T^p)$ is exactly $2^p - 1$, combined with the inequality in the other direction as shown in [DLWY]. Theorem 1.5 may be rephrased as there are at most $2^p - 1$ modular types (Definition 3.6) of unknotted embeddings (Corollary 3.9).

Our strategy to prove Theorem 1.4 is to construct a 'favorite' lifting e of a given expanding map ϕ of T^p (represented by a integral matrix of expanding eigenvalues in this case) using the Smith normal form of integral matrices, so that whenever an embedding $\jmath : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ is framing untwisted, the same holds for $\jmath \circ e$. Choosing $\jmath$ to be standardly unknotted and framing untwisted, we show $\jmath \circ e^i$ are all unknotted of modular types for $i \geq 0$. Unlike the toy case, this is much less obvious in higher dimensions, and the proof involves some technical manipulations of integral matrices. After all these are done, one can apply the finiteness result about modular types to derive Theorem 1.4.

In Section 2, we prove Proposition 1.2 using a classical unknotting theorem. In Section 3, we define unknotted embeddings $\imath : T^p \hookrightarrow \mathbb{R}^{p+2}$ and their types, and prove Theorem 1.5. In Section 4, we construct disk bundle embeddings lifting expanding maps, and realize the corresponding DE attractor by compactly-supported self-diffeomorphisms of $\mathbb{R}^{p+2}$, proving Theorem 1.4. We also define unknotted, framing untwisted embeddings $\jmath : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ in Section 4.

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2 Realizing DE attractors in large codimensions

In this section, we prove Proposition 1.2. This is a consequence of an unknotting theorem of Wen-Tsün Wu in the 1950's about smooth embeddings into Euclidean spaces of codimension right below the stable range, ([Wu], cf. also [Ha] for generalizations).

Proof of Proposition 1.2. When $p = 1$, M is diffeomorphic to S^1 and any expanding map $\phi : S^1 \to S^1$ is a topologically conjugate to a non-zero degree covering. The realization of $(1, 2)$-attractors, i.e. classical solenoids, is somewhat well-known, for example, as implicitly contained in [Bo] and [JNW]. We may assume $p \geq 2$ from now on.

Since $q \geq p + 1$, we may pick a Whitney embedding of $\imath : M \hookrightarrow \mathbb{R}^{p+q}$ of M into $\mathbb{R}^{p+q}$. By [Gr], we may assume the expanding map $\phi : M \to M$ is a covering map induced by an infranil endomorphism. Now ϕ induces an immersion $\imath \circ \phi : M \looparrowright \mathbb{R}^{p+q}$, which can be pertubed along normal directions into a smooth embedding $\hat{\imath} : M \hookrightarrow \mathbb{R}^{p+q}$ by a Whitney type argument, such that the image $\hat{\imath}(M)$ remains in the interior of a compact tubular neighborhood $\mathcal{N} \subset \mathbb{R}^{p+q}$ of $\imath(M)$.

The result in [Wu] says that for any connected, closed p-dimensional smooth manifold M ($p \geq 2$), any two smooth embeddings of M into $\mathbb{R}^{p+q}$ are smoothly isotopic to each other. Note the connectedness is an indispensable assumption here. Applying this unknotting theorem, there is a smooth isotopy $F : M \times [0, 1] \to \mathbb{R}^{p+q}$, such that $F|_{M\times\{0\}} = \imath$, and $F|_{M\times\{1\}} = \hat{\imath}$. By the isotopy extension theorem (cf. [Hi, Theorem 1.3]), F extends as a diffeotopy $F : \mathbb{R}^{p+q} \times [0, 1] \to \mathbb{R}^{p+q}$ with compact support. Denote $f = F|_{\mathbb{R}^{p+q}\times\{1\}}$. By shrinking the radius of $f(\mathcal{N})$ by a diffeotopy of $\mathbb{R}^{p+q}$ supported near $f(\mathcal{N})$ if necessary, we may assume $f(\mathcal{N})$ is contained in the interior of $\mathcal{N}$. Identify $\mathcal{N}$ as the unit disk bundle of the normal bundle $N_{\mathbb{R}^{p+q}}(\imath(M))$ of $\imath(M)$ (w.r.t. any Euclidean fiber metric) via a diffeomorphism. By a standard differential topology argument, we may further assume f is diffeotoped supported near $f(\mathcal{N})$ so that it maps every fiber disk into a fiber disk of $\mathcal{N}$, and that it satisfies Definition 1.1 (2). Note $f : \mathbb{R}^{p+q} \to \mathbb{R}^{p+q}$ is also compactly supported by the construction.

Now $\mathcal{N}$ may be regarded as a q-disk bundle over M, and $f|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ may be regarded as a hyperbolic bundle embedding lifted from ϕ, via the natural identification. Then clearly f realizes a (p, q)-type attractor derived from ϕ via the inclusion $\mathcal{N} \subset \mathbb{R}^{p+q}$. $\square$

It is remarkable at this point that while Proposition 1.2 allows one to realize DE attractors in fairly arbitrary manifolds of sufficiently large dimensions, it is still a 'local' realization anyways. In contrast, for example, if a self-diffeomorphism f of a closed, orientable n-dimensional smooth manifold X satisfies that the non-wandering set $\Omega(f)$ is a union of finitely many DE attractors and repellers (i.e. attractors of f^{-1}), then X must be a rational homology sphere, and each DE attractor/repeller must have codimension 2, namely of type $(n-2, 2)$, ([DPWY], cf. [JNW] for an example of $n = 3$).

3 Unknotted T^p in $\mathbb{R}^{p+2}$ and extendable automorphisms

In this section, we introduce and study unknotted embeddings of T^p into $\mathbb{R}^{p+2}$, and prove Theorem 1.5.

Regard T^p as the standard p-dimensional torus obtained by quotienting $\mathbb{R}^p$ by its integral lattice. The natural action of $\mathrm{SL}(p, \mathbb{Z})$ on $\mathbb{R}^p$ descends to an action on T^p, so there is a subgroup $\mathrm{Aut}(T^p)$ of the orientation-preserving diffeomorphism group $\mathrm{Diff}_+(T^p)$ consisting of the transformations induced by the action. The elements of $\mathrm{Aut}(T^p)$ will be refered as *automorphisms* of T^p. After choosing a product structure of $T^p \cong S^1_1 \times \cdots \times S^1_p$, one may naturally identify $\mathrm{Aut}(T^p)$ with $\mathrm{SL}(p, \mathbb{Z})$.

We start by investigating some important aspects of unknotted embeddings. It is reasonable to expect that such embeddings are fairly simple and symmetric, largely agreeing with our low-dimension intuition. We will parametrize S^1 and D^2 as the unit circle and the compact unit disk of $\mathbb{C}$, respectively. The real and imaginary part of $z \in \mathbb{C}$ are often written as z_x, z_y. The standard basis of $\mathbb{R}^n$ is $(\vec{\varepsilon}_1, \cdots, \vec{\varepsilon}_n)$, and the m-subspace spanned by $(\vec{\varepsilon}_{i_1}, \cdots, \vec{\varepsilon}_{i_m})$ will be written as $\mathbb{R}^m_{i_1,\cdots,i_m}$. Note there is a natural inclusion of $\mathbb{R}^n = \mathbb{R}^n_{1,\cdots,n}$ into $\mathbb{R}^{n+1}$.

Example 3.1 (The standard model)**.** Let $\imath_0 : \mathrm{pt} = T^0 \hookrightarrow \mathbb{R}^2$ be the map from the single point to the origin of $\mathbb{R}^2$ by convention. Inductively suppose $\imath_{p-1}$ has been constructed ($p \geq 1$) such that $\imath_{p-1}(T^{p-1}) \subset \mathrm{Int}(D^p) \subset \mathbb{R}^p_{2,\cdots,p+1}$. Denote the rotation of $\mathbb{R}^{p+2}$ on the subspace $\mathbb{R}^2_{2,p+2}$ of angle $\arg(u)$ as $\rho_p(u) \in \mathrm{SO}(p+2)$, for any $u \in S^1$. We define $\imath_p : T^p = T^{p-1} \times S^1_p$ as:

$$\imath_p(v, u) = \rho_p(u)(\frac{1}{2} \cdot \vec{\varepsilon}_2 + \frac{1}{4} \cdot \imath_{p-1}(v)).$$

This explicitly describes an embedding of $T^p = S^1_1 \times \cdots \times S^1_p$ into $\mathbb{R}^{p+1}_{2,\cdots,p+2}$. In Figure 1, the images of $\imath_{p-1}$ and $\imath_p$ are schematically presented on the left and the right respectively. One may imagine $\vec{\varepsilon}_1$ points perpendicularly outward the page. Observe that the image of T^p is invariant under $\rho_p(u)$.

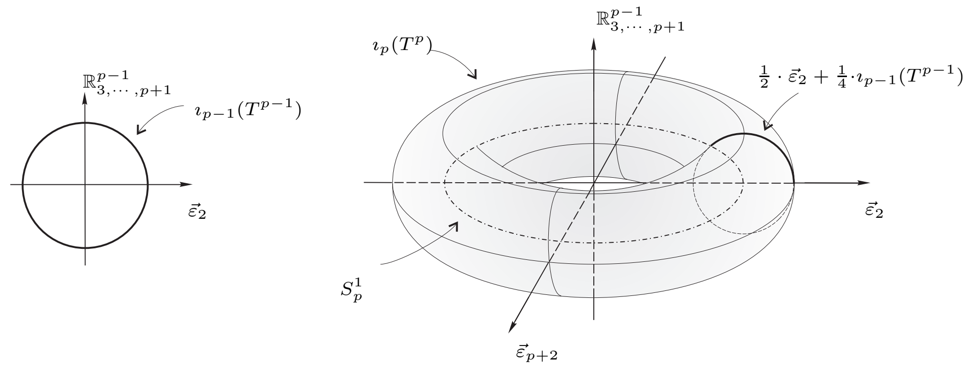

Figure 1: The standard model.

The basic feature of the standard model is that it is highly 'compressible', in the sense of the following lemma. Let $B^4 \subset \mathbb{R}^4$ be the compact 4-dimensional disk centered at the origin with radius 2.

Lemma 3.2. *In the standard model for $p \geq 2$, for each $i = 2, \cdots, p$, the embedding $\imath_p : T^p = S^1_1 \times \cdots \times S^1_p \hookrightarrow \mathbb{R}^{p+2}$ extends as an embedding:*

$$k_{1i} : B^4 \times (S^1_2 \times \cdots \times \hat{S}^1_i \times \cdots \times S^1_p) \hookrightarrow \mathbb{R}^{p+2},$$

where $S^1_1 \times \cdots \times S^1_p \subset B^4 \times (S^1_2 \times \cdots \times \hat{S}^1_i \times \cdots \times S^1_p)$ is the standard embedding (i.e. inclusion) for $S^1_1 \times S^1_i \subset B^4$, and the identity on other factors.

Proof. To see the idea consider a standard $T^2 = S^1_1 \times S^1_2$ in $\mathbb{R}^4$. Make a solid torus $D^2_1 \times S^1_2$ by filling up the S^1_1 factor, and attach a semi-sphere in $\mathbb{R}^4$ along the core of that solid torus. The result is a 'hat' whose regular neighborhood is diffeomorphic to D^4. When $i > 2$, one only cares about S^1_1 and S^1_i.

The construction is as follows. We first extend $\imath_1$ as $j_1 : D^2_1 \hookrightarrow \mathbb{C} \cong \mathbb{R}^2_{2,3}$ in an obious fashion. Inductively, suppose $j_{s-1} : D^2_1 \times S^1_2 \times \cdots \times S^1_{s-1} \hookrightarrow \mathrm{Int}(D^s) \subset \mathbb{R}^s_{2,\cdots,s+1}$ has been constructed. Then define $j_s : (D^2_1 \times S^1_2 \times \cdots \times S^1_{s-1}) \times S^1_s \hookrightarrow \mathrm{Int}(D^{s+1}) \subset \mathbb{R}^{s+1}_{2,\cdots,s+2}$ as:

$$j_s(v, u) = \rho_s(u)(\frac{1}{2} \cdot \vec{\varepsilon}_2 + \frac{1}{4} \cdot j_{s-1}(v)).$$

After $i-1$ steps we obtain $j_i : D^2_1 \times S^1_2 \times \cdots \times S^1_i \hookrightarrow \mathrm{Int}(D^{i+1}) \subset \mathbb{R}^{i+1}_{2,\cdots,i+2}$. Now let $\zeta_s(re^{i\theta})$ $(0 \leq r \leq 1)$ be the rotation of $\mathbb{R}^{s+2}$ of angle $\arccos(r)$, on the subspace spanned by $\rho_s(e^{i\theta})(\vec{\varepsilon}_2)$ and $\vec{\varepsilon}_1$ (from the former toward the latter). We may further define $k_{1i,i} : (D_1^2 \times S_2^1 \times \cdots \times S_{i-1}^1) \times D_i^2 \hookrightarrow \mathrm{Int}(D^{i+2}) \subset \mathbb{R}^{i+2}$ as, for example,

$$k_{1i,i}(v, re^{i\theta}) = \zeta_i(\frac{2r}{1+r^2}e^{i\theta})(j_i(v, e^{i\theta})).$$

Then repeat the standard construction, namely, let $k_{1i,s}(\vec{x}, u) = \rho_s(u)(\frac{1}{2} \cdot \vec{\varepsilon}_2 + \frac{1}{4} \cdot k_{1i,s-1}(\vec{x}))$ for $i < s \le p$. In the end we obtain:

$$k_{1i} = k_{1i,p} : D_1^2 \times D_i^2 \times (S_2^1 \times \cdots \hat{S}_i^1 \times \cdots \times S_p^1) \hookrightarrow \mathbb{R}^{p+2}.$$

From the construction, we see that k_{1i} can be extended a bit as an embedding:

$$k_{1i} : B^4 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1) \hookrightarrow \mathbb{R}^{p+2}.$$

□

Corollary 3.3. *For each $i = 1, \cdots, p$, $\imath_p$ also extends as an embedding:*

$$k_i : D^3 \times (S_1^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1) \hookrightarrow \mathbb{R}^{p+2},$$

where $S_1^1 \times \cdots \times S_p^1 \subset D^3 \times (S_1^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1)$ is an unknotted embedding for S_i^1 in $\mathrm{Int}(D^3)$, and the identity on other factors.

Proof. Clearly the inclusion $S_1^1 \times S_i^1 \subset B^4 \subset \mathbb{C} \times \mathbb{C}$ in Lemma 3.2 extends as an embedding $S^1 \times D^3 \cong (S_1^1 \times D^1) \times B^2 \subset B^4 \subset \mathbb{C} \times \mathbb{C}$ where $S_1^1 \times D^1$ is a tubular neighborhood of $S_1^1 \subset \mathbb{C}$ and B^2 is the 2-dimensional disk centered at the origin with radius $\frac{3}{2}$. Now k_i may be defined as k_{1i} composed with the latter embedding. □

We introduce the notion of unknotted embeddings and their types.

Definition 3.4. A smooth embedding $\imath : T^p \hookrightarrow \mathbb{R}^{p+2}$ is called *unknotted* if there is a compactly-supported self-diffeomorphism $g : \mathbb{R}^{p+2} \to \mathbb{R}^{p+2}$ of $\mathbb{R}^{p+2}$ such that $\imath$ and $g \circ \imath_p$ have the same image, i.e. $\imath(T^p) = g \circ \imath_p(T^p)$.

Remark 3.5. Since the orientation-preserving diffeomorphism group $\mathrm{Diff}_+(\mathbb{R}^n)$ deformation-retracts to $\mathrm{SO}(n)$, and hence that $\pi_0\mathrm{Diff}_+(\mathbb{R}^n)$ is trivial ([St]), clearly our definition of unknottedness agrees with the more common notion that $\imath(T^p)$ and $\imath_p(T^p)$ are equivalently knotted is if there is a diffeotopy of $\mathbb{R}^{p+2}$ taking $\imath(T^p)$ to $\imath_p(T^p)$.

Definition 3.6. Two unknotted embeddings $\imath_0, \imath_1 : T^p \hookrightarrow \mathbb{R}^{p+2}$ are called of the same *type* if they are the same up to a self-diffeomorphism of $\mathbb{R}^{p+2}$, namely there is a diffeomorphism $h : \mathbb{R}^{p+2} \to \mathbb{R}^{p+2}$ such that $h \circ \imath_0 = \imath_1$. This is an equivalence relation, and the equivalent classes are called *types*. The type of $\imath_p$ is called the *standard* type. For any $\tau \in \mathrm{Aut}(T^p)$, τ defines a *modular transformation* on types, namely $[\imath] \mapsto [\imath \circ \tau]$. A *modular* type is obtained by a modular transformation of the standard type.

Lemma 3.7. *For any unknotted embedding and any type, there is an unknotted embedding with the same image and of that type.*

Proof. Let $\imath_0$ be the embedding, and $[\imath_1]$ be the type. By Definition 3.6, there is some $\mathbb{R}^{p+2}$-self-diffeomorphism h_1 such that $h_1 \circ \imath_0(T^p) = \imath_1(T^p)$. Let $\tau = \imath_1^{-1} \circ h_1 \circ \imath_0 : T^p \to T^p$, then $h_1^{-1} \circ \imath_1 = \imath_0 \circ \tau^{-1}$. Thus $\imath_0 \circ \tau^{-1}$ has the same image as $\imath_0$, and the same type as $[\imath_1]$. □

Related is the notion of extendable automorphisms.

Definition 3.8. Let $\imath : T^p \hookrightarrow \mathbb{R}^{p+2}$ be a smooth embedding. An automorphism $\tau \in \mathrm{Aut}(T^p)$ is said to be *extendable* over $\imath$ if there is a compactly-supported self-diffeomorphism of $\mathbb{R}^{p+2}$ which commutes with τ via $\imath$. The subgroup of $\mathrm{Aut}(T^p)$ consisting of extendable automorphisms will be denoted as $E_\imath$.

Note $E_{\imath\circ\tau} = \tau^{-1} E_\imath \tau$ for any smooth embedding $\imath$ and any $\tau \in \mathrm{Aut}(T^p)$. It is also clear that modular-type embeddings are in natural one-to-one correspondence with the right cosets of $E_{\imath_p}$ in $\mathrm{Aut}(T^p)$, so Theorem 1.5 may be rephrased as:

Corollary 3.9. *For $p \geq 1$, there are at most $2^p - 1$ modular types of unknotted embeddings $\imath : T^p \hookrightarrow \mathbb{R}^{p+2}$.*

In the rest of this section, we prove Theorem 1.5.

Fix a product structure $T^p = S^1_1 \times \cdots \times S^1_p$ as in Example 3.1, then $\mathrm{Aut}(T^p)$ is identified with $\mathrm{SL}(p, \mathbb{Z})$ $(p \geq 2)$. Denote:

$$R_{ij} = I + E_{ij}, \quad Q_{ij} = R_{ij}^{-1} R_{ji} R_{ij}^{-1},$$

where $i \neq j$, and I is the identity matrix and E_{ij} has 1 for the (i, j)-entry and all other entries 0. Note that R_{ij} is the full Dehn twist on the sub-torus $S^1_i \times S^1_j$ along S^1_i, and Q_{ij} trades the two factors of $S^1_i \times S^1_j$.

When $p = 2$, there are two basic extendable automorphisms for the embedding $T^2 = S^1_1 \times S^1_2 \subset \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$.

Lemma 3.10. *For the standard embedding $T^2 = S^1_1 \times S^1_2 \subset B^4 \subset \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$, the following automorphisms can be extended as self-diffeomorphisms of B^4 supported in the interior, (i.e. fixing an open neighborhood of ∂B^4):*

(1) the twice full Dehn twist along each factor circle;

(2) trading two factors with their orientations preserved.

Proof. (1) It suffices to prove for the first factor. Consider $S^1_1 \times S^1_2 \subset (S^1_1 \times D^1) \times D^2 = S^1_1 \times D^3 \subset \mathbb{R}^4$, where $S^1_1 \times D^1$ is a tubular neighborhood of S^1_1 in the first $\mathbb{C}$, and D^2 is the disk bounded by S^1_2 in the second $\mathbb{C}$, $S^1_1 \times D^3$ is a tubular neighborhood of S^1_1 in $\mathbb{R}^4$, $\partial(S^1_1 \times D^3) = S^1_1 \times S^2$, and $* \times S^1_2$ is the equator of $* \times S^2$.

The Dehn 2-twist $\tau : S^1_1 \times S^1_2 \to S^1_1 \times S^1_2$ is $(x, y) \mapsto (x, x^2 y)$. The map $x \mapsto x^2$, considered as a map from S^1 to $\mathrm{SO}(2)$, is of degree 2. Thus the map $x \mapsto x^2$, considered as a map $g : S^1 \to \mathrm{SO}(3)$, is homotopic to a constant map since $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$. We may extend the map $\tau : S^1_1 \times S^1_2 \to S^1_1 \times S^1_2$ to a map $\tilde{\tau} : S^1_1 \times S^2 \to S^1_1 \times S^2$, defined by $\tau(x, y) = (x, g(x)y)$. Since $g : S^1 \to \mathrm{SO}(3)$ is homotopic to a constant map, $\tilde{\tau}$ is diffeotopic to the identity. By the isotopy extension theorem (cf. [Hi, Theorem 1.3]), $\tilde{\tau}$ can be extended to a self-diffeomorphism of B^4 supported in the interior.

(2) First extend $T^2 = S^1_1 \times S^1_2 \to S^1_1 \times S^1_2$ as $f : \mathbb{R}^4 = \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, by $(z, w) \mapsto (\bar{w}, z)$. f is an orientation-reversing diffeomorphism. To adjust to get an orientation-preserving one, pick a self-diffeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$, supported in the interior of $B^4 \subset \mathbb{R}^4$, such that $h(T^2)$ lies on the subspace $\mathbb{R}^3_{2,3,4}$. Let $r_1 : \mathbb{R}^4 \to \mathbb{R}^4$ be the reflection with respect to $\mathbb{R}^3_{2,3,4}$, then r_1 is orientation-reversing, and is the identity restricted to $h(T^2)$. Now $f_1 = (h^{-1} r_1 h) \circ f$ is orientation preserving, extending the described automorphism on T^2. Furthermore, since $f_1(z, w) = (-w, z)$ when $|w|^2 + |z|^2 > 4 - \varepsilon$, where $\varepsilon > 0$ is sufficiently small, we may adjust f_1 near the boundary of B^4 to get an f_2 which is supported in the interior of B^4. ☐

From this observation we have the following lemma for $p \geq 2$.

Lemma 3.11. *$R^2_{1i}, Q_{1i} \in E_{\imath_p}$, for $1 < i \leq p$.*

Proof. Let τ be either R^2_{1i} or Q_{1i}. From Lemma 3.10, $\tau|_{S^1_1 \times S^1_i} : S^1_1 \times S^1_i \to S^1_1 \times S^1_i$ extends as $\bar{\tau} : B^4 \to B^4$ which is the identity near the boundary. Therefore by Lemma 3.2,

$$\bar{\tau} \times \mathrm{id} : B^4 \times (S^1_2 \times \cdots \times \hat{S}^1_i \times \cdots \times S^1_p) \to B^4 \times (S^1_2 \times \cdots \times \hat{S}^1_i \times \cdots \times S^1_p)$$

induces a self-diffeomorphism of $k_{1i}(B^4 \times (S^1_2 \times \cdots \times \hat{S}^1_i \times \cdots \times S^1_p))$ which is the identity near the boundary. The latter further extends to a diffeomorphism of $\mathbb{R}^{p+2}$ by the identity outside the image of k_{1i}. ☐

These automorphisms are not enough for generating $E_{\imath_p}$ when $p \geq 3$. Extra extendable ones come from the following geometric construction.

Lemma 3.12. *$R_{1p} R_{ip} \in E_{\imath_p}$, for $1 < i < p$.*

Proof. According to Lemma 3.2, we will first extend $\eta = R_{1p} R_{ip}$ on T^p over $B^4 \times (S^1_2 \times \cdots \times \hat{S}^1_i \times \cdots \times S^1_p)$, identity near the boundary, then extend it further as a self-diffeomorphism of $\mathbb{R}^{p+2}$ via k_{1i} by the identity outside the image. Since $\eta = \tau \times \mathrm{id}$ as from $(S^1_1 \times S^1_i \times S^1_p) \times (S^1_2 \times \cdots \times \hat{S}^1_i \times \cdots \times S^1_{p-1})$ to itself, essentially one must extend:

$$\tau : S^1_1 \times S^1_i \times S^1_p \to S^1_1 \times S^1_i \times S^1_p,$$

as $B^4 \times S^1_p \to B^4 \times S^1_p$. It is easy to see the matrix of τ is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, the $(1, i, p)$ minor of $R_{1p} R_{ip}$. It follows that each column sum of $R_{1p} R_{ip}$ is odd. Rewrite u_p as w, then we have $\tau((u_1, u_i), w) = (\mu_w(u_1, u_i), w)$, where $\mu_w(u_1, u_i) = (wu_1, wu_i)$ is an action of S^1_p on T^2 by flowing along the diagonal-slope direction.

We extend τ as follows. First, via the inclusion $T^2 = S_1^1 \times S_i^1 \subset S^3$ (in fact, the sphere centered at the origin with radius $\sqrt{2}$), the diagonal-slope fibration on T^2 extends as the Hopf fibration on S^3, and μ_w also extends as $\tilde{\mu}_w : S^3 \to S^3$ by flowing along the fiber loops. Thus τ extends as $\tilde{\tau} : S^3 \times S_p^1 \to S^3 \times S_p^1$, $\tilde{\tau}(x, w) = (\tilde{\mu}_w(x), w)$.

On the other hand, $\tilde{\mu}_w$ can be regarded as $S_p^1 \to \mathrm{Diff}_+(S^3)$ which is given by a Lie group left multiplication, regarding S_p^1 as a subgroup of S^3. Thus $\tilde{\mu}_w$ extends as $S^3 \to \mathrm{Diff}_+(S^3)$ by the Lie group left multiplication. This implies that $\tilde{\mu}_w$ is homotopic to the constant identity in $\pi_1 \mathrm{Diff}_+(S^3)$ (since any circle is null-homotopic in the 3-sphere). Thus $\tilde{\tau}$ is diffeotopic to the identity. By the isotopy extension theorem, $\tilde{\tau}$ can be extended to a diffeomorphism $\bar{\tau}$ of $B^4 \times S_p^1$, which is the identity near the boundary.

Now $\bar{\eta} = \bar{\tau} \times \mathrm{id}$ is a self-diffeomorphism of $(B^4 \times S_p^1) \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_{p-1}^1)$, or diffeomorphically, of $B^4 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1)$, being the identity near the boundary. Finally extend $\bar{\eta}$ to a diffeomorphism $\mathbb{R}^{p+2} \to \mathbb{R}^{p+2}$ via k_{1i} with the identity outside the image. $\square$

We need an elementary matrix lemma below. The technical result (2) is needed in Section 4.

Lemma 3.13. *(1) The subgroup G of $\mathrm{SL}(p, \mathbb{Z})$ generated by R_{1i}^2, Q_{1i} $(1 < i \leq p)$ and $R_{1p}R_{ip}$ $(1 < i < p)$ consists of all the matrices $U \in \mathrm{SL}(p, \mathbb{Z})$ whose entry sum of each column is odd.*

(2) For any $1 \leq i \leq p$, any $U \in \mathrm{SL}(p, \mathbb{Z})$ can be written as KJ such that K is a word in R_{1j}^2, Q_{1j}, $1 < j \leq p$, and J has the minor $J_{ii}^* = 1$.

Proof. (1) Because $R_{ij}^2 = Q_{1i} R_{1j}^2 Q_{1i}^{-1}$, $Q_{ij} = Q_{1i} Q_{1j} Q_{1i}^{-1}$, for $1 < i \neq j \leq p$, we have $R_{ij}^2, Q_{ij} \in G$ $(i \neq j)$. Also $R_{1k}R_{jk} = Q_{kp}^{-1} R_{1p} R_{jp} Q_{kp}$ $(k \neq 1, j, p)$, and $R_{ik}R_{jk} = Q_{1i} R_{1k} R_{jk} Q_{1i}^{-1}$ $(i \neq 1, j, k)$, we have $R_{ik}R_{jk} \in G$ $(i, j, k$ mutually different). Multiplying by R_{ij}^2 from the left of a matrix adds twice of the j-th row to the i-th, and by $R_{ik}R_{jk}$ adds the k-th row to both the i-th and the j-th, and by Q_{ij} switches the i-th row and j-th row up to a sign. We claim that any $U \in \mathrm{SL}(p, \mathbb{Z})$ with odd column sums becomes diagonal in ± 1's under finitely many such operations, by a Euclid type algorithm described below.

Suppose $U = (u_{ij})_{1 \leq i, j \leq p}$. In the first column there are an odd number of odd entries, say $u_{i_1,1}, \cdots, u_{i_{2m+1},1}$. Adding the i_1-th row simultaneously to the i_2-th, $\cdots$, i_{2m+1}-th rows (i.e. multiplying U from the left by $R_{i_2 i_1} R_{i_3 i_1} \cdots R_{i_{2m} i_1} R_{i_{2m+1} i_1}$) if necessary, we may assume only one entry is odd, and switching that row with the first row if necessary, we may further assume u_{11} is the only odd entry in the first column. Now there is some nonzero entry with a minimum absolute value, say u_{r1}. As $U \in \mathrm{SL}(p, \mathbb{Z})$, u_{r1} cannot divide all other entries in this column unless $u_{r1} = \pm 1$, so if $u_{r1} \neq \pm 1$, there must be some u_{s1} with $|u_{s1}| > |u_{r1}|$ and u_{r1} cannot divide u_{s1}. By adding (or subtracting) an even times of the r-th row to the s-th, the u_{s1} becomes u'_{s1} such that $-|u_{r1}| < u'_{s1} < |u_{r1}|$. Because this operation does not change parity, we may repeat this process until we get a matrix with $u_{11} = \pm 1$, and other entries in the first column being even. Then add (or subtract) several even times of the first row to other rows, we obtain a matrix U'' with the entries in the first column being zero except $u_{11} = \pm 1$. Apply the process recursively on the $(p-1) \times (p-1)$-submatrix $U''_{11} = (u''_{ij})_{2 \le i,j \le p}$, and use $u_{22} = \pm 1$ to kill other even entries in the second column, and so on. In the end U becomes a diagonal matrix with ± 1's on the diagonal.

Finally, the claim means that any U with odd column sums can be written as $U = KD$ where K is a word in R^2_{ij}, Q_{ij}, $R_{ik}R_{jk}$, and D is diagonal in ± 1's. Moreover, D must have even -1's on the diagonal since the determinant is 1, for example, at the $i_1, \cdots, i_{2m}$ place, $0 \le 2m \le p$, then $D = Q^2_{i_1 i_2} \cdots Q^2_{i_{2m-1} i_{2m}}$. Therefore $U = KD \in G$.

To see any element $U \in G$ has odd column sums, note that this condition is the same as saying that $X\bar{U} = X$ where X is the row vector $(1, \cdots, 1) \in \mathbb{Z}_2^p$, where $\bar{U}$ is the modulo 2 reduction of U. As all the R^2_{1i}, Q_{1i} $(1 < i \le p)$ and $R_{1p}R_{ip}$ $(1 < i < p)$ fix X, so does G. Therefore G consists of elements in $\mathrm{SL}(p, \mathbb{Z})$ with odd column sums.

(2) Instead of doing row operations, do the first step of the algorithm by column operations on the i-th row of U^{-1} to make the (i, r)-th entry ± 1. Switch the i-th and r-th column, and multiply Q^2_{ir} if necessary to make the (i, i)-th entry $+1$. In other words, the (i, i)-th entry of $U^{-1}K$ is 1 for some word K in R^2_{1i}, Q_{1i}. Then let $J = (U^{-1}K)^{-1}$. □

Proof of Theorem 1.5. By Lemma 3.11 and Lemma 3.12, $G \le E_{\imath_p}$, where G is the subgroup of $\mathrm{SL}(p, \mathbb{Z})$ as in Lemma 3.13. By Lemma 3.13 (1), G is the stabilizer of the row vector $(1, \cdots, 1) \in \mathbb{Z}_2^p$ under the right action of $\mathrm{SL}(p, \mathbb{Z})$ on the row vector space $\mathbb{Z}_2^p$. Note that with the chosen product structure of T^p, this action is naturally identified with the action of $\mathrm{Aut}(T^p)$ on $H^1(T^p; \mathbb{Z}_2)$, thus $E_{\imath_p}$ contains the stabilizer of a nontrivial element in $H^1(T^p; \mathbb{Z}_2)$. It follows that the $[\mathrm{Aut}(T^p) : E_{\imath_p}] \le 2^p - 1$ since $\mathrm{Aut}(T^p)$ acts invariantly and transitively on the subset of nontrivial elements of $H^1(T^p; \mathbb{Z}_2)$. □

4 Realizing DE attractors of T^p in codimension 2

We prove Theorem 1.4 in this section. To fix the notation, we write $(u_1, \cdots, u_p)$ for the coordinate of a point u in $T^p = S^1_1 \times \cdots \times S^1_p$, and write (u, z) for the coordinate of a point in $T^p \times D^2$. We use T^{p-1}_i to denote $S^1_1 \times \cdots \times \hat{S}^1_i \times \cdots \times S^1_p$.

With the fixed product structure, expanding maps of T^p may be identified with expanding endomorphisms, i.e. which are represented by $p \times p$ integral matrices whose eigenvalues all have absolute values strictly greater than 1. We identify expanding maps and automorphisms of T^p with their matrices, and often write them as ϕ_A, τ_U, etc.

Given an expanding map $\phi : T^p \to T^p$, we wish to use an unknotted embedding to realize an attractor derived from ϕ. It is not hard to lift it as a hyperbolic bundle embedding first. Note that the normal bundles of orientable codimension 2 submanifold of $\mathbb{R}^{p+2}$ must have trivial Euler classes, and hence are trivial bundles (cf. [MS, Corollary 11.4]), we should consider self-embeddings e of trivial disk bundles which lifts an expanding map ϕ. For simplicity, we also assume from now on that ϕ has positive degree, possibly after passing to ϕ^2.

Proposition 4.1. *Let $\phi : T^p \to T^p$ be any orientation-preserving expanding map. There is a hyperbolic bundle embedding $e : T^p \times D^2 \hookrightarrow T^p \times D^2$ lifted from ϕ.*

Proof. Let A be the matrix of ϕ. Recall that a Smith normal form ([Ne] Theorem II.9) of a positive-determinant integral matrix A is a decomposition $A = U\Delta V$, where $U, V \in \mathrm{SL}(p, \mathbb{Z})$, and Δ is a diagonal matrix $\mathrm{diag}(\delta_1, \cdots, \delta_p)$, with positive integral diagonal entries δ_i, such that δ_i divides δ_{i+1} for each $1 \leq i \leq p-1$. Let $\Delta_i = \mathrm{diag}(1, \cdots, \delta_i, \cdots, 1)$, then:

$$A = U\Delta_1 \cdots \Delta_p V.$$

We first lift each factor to a bundle embedding $e_U, e_V, e_{\Delta_i} : T^p \times D^2 \hookrightarrow T^p \times D^2$.

To lift τ_U, define, for example, $e_U(u, z) = (\tau_U(u), u_1^{m_1} \cdots u_p^{m_p} z)$ for chosen integers $m_1, \cdots, m_p$. Similarly lift τ_V to e_V. To lift ϕ_{Δ_i}, first pick a hyperbolic bundle embedding $b_{\delta_i} : S^1_i \times D^2 \hookrightarrow S^1_i \times D^2$ such that $b_{\delta_i}(u_i, z) = (u_i^{\delta_i}, \bar{b}_i(u_i, z))$ sends the solid torus into itself as a connected thickened closed braid with winding number δ_i, shrinking evenly on the disk direction. Then define:

$$e_{\Delta_i}(u, z) = (\phi_{\Delta_i}(u), \bar{b}_i(u_i, z)) = (b_{\delta_i} \times \mathrm{id}_{T_i^{p-1}})(u, z).$$

Finally, take the composition $e = e_U \circ e_{\Delta_1} \circ \cdots \circ e_{\Delta_p} \circ e_V$, and we obtain a hyperbolic bundle embedding lifted from ϕ. □

Although the lifting is far from unique, for our purpose of use we must pick a topologically simple one, composing which does not make the embedding 'knottier' or 'more twisted' in $\mathbb{R}^{p+2}$.

Example 4.2 (The favorite lifting)**.** Define $e_U(u, z) = (\tau_U(u), z)$, $e_V(u, z) = (\tau_V(u), z)$, and $e_{\Delta_i}(u, z) = b_{\delta_i} \times \mathrm{id}_{T_i^{p-1}}$ with:

$$b_{\delta_i}(u_i, z) = (u_i^{\delta_i}, \frac{1}{2}u_i + \frac{1}{\delta_i^2} u_i^{1-\delta_i} z).$$

The chosen braid b_{δ_i} is a $(\delta_i, 1)$-cable with respect to the given trivialization of $S^1_{\delta_i} \times D^2$. It is presented in Figure 2 as $\delta_i = 3$, where the framing change is indicated by $b_{\delta_i}(S^1_i \times \{1\}$ and its image.

This specific choice of b_{δ_i} gives us a favorite lifting $e = e_U \circ e_{\Delta_1} \circ \cdots \circ e_{\Delta_p} \circ e_V$ of ϕ.

An embedding $\jmath : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ is, in general, understood by knowing its core restriction $\imath = \jmath|_{T^p \times \{0\}}$, and the framing. We need the following definition.

Definition 4.3. $\jmath : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ is called *unknotted*, if its core restriction $\imath$ is unknotted. The *type* of $\jmath$ is the type of $\imath$. The embedding $\jmath$ is said to have *untwisted framing*, if $\jmath(T^p \times \{1\})$ is null-homologous in the complement of the core $\imath(T^p)$ in $\mathbb{R}^{p+2}$.

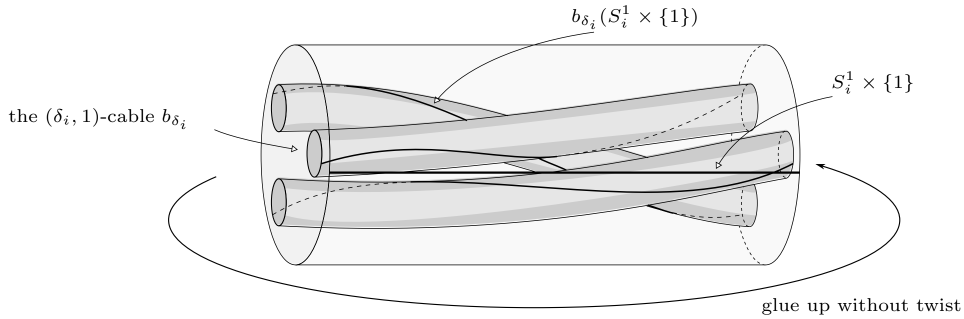

Figure 2: The favorite lifting, illustrated as $\delta_i = 3$.

Proposition 4.4. *Unknotted embeddings with untwisted framing are unique of their types, up to compactly-supported self-diffeomorphisms of* $\mathbb{R}^{p+2}$ *. Namely, for unknotted embeddings* $\jmath_0, \jmath_1$ *with untwisted framing of the same type, there is a compactly-supported self-diffeomorphism* $h : \mathbb{R}^{p+2} \to \mathbb{R}^{p+2}$ *of* $\mathbb{R}^{p+2}$ *such that* $h \circ \jmath_0 = \jmath_1$*.*

Proof. By definition we may first find a compactly supported self-diffeomorphism h of $\mathbb{R}^{p+2}$ so that $\imath_1 = h \circ \imath_0$, and in the smooth category one may assume $h| : \jmath_0(T^p \times D^2) \to \jmath_1(T^p \times D^2)$ is conjugate to the normal-bundle map, namely,

$$\jmath_1(u, z) = h(\jmath_0(u, g_u(z))),$$

for $g_u \in \mathrm{SO}(2)$. Thus there is a continuous map $g : T^p \to \mathrm{SO}(2)$. Note the set $[T^p, \mathrm{SO}(2)]$ of homotopy classes of maps from T^p to $\mathrm{SO}(2)$ is in bijection to $H^1(T^p; \mathbb{Z}) \cong \mathrm{Hom}(H_1(T^p), \mathbb{Z})$, as $\mathrm{SO}(2) \cong S^1$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 1)$. We may assume $g_u(z) = u_1^{m_1} \cdots u_p^{m_p} z$, for some integers $m_1, \cdots, m_p$. In fact, $m_1, \cdots, m_p$ are determined so that $g^*\xi = m_1 [S^1_1]^* + \cdots + m_p [S^1_p]^*$ in $H^1(T^p; \mathbb{Z})$. Here ξ is the generator of $H^1(\mathrm{SO}(2)) \cong \mathbb{Z}$ giving the natural orientation of the normal bundle, and $([S^1_1]^*, \cdots, [S^1_p]^*)$ is the basis of $H^1(T^p)$ dual to the basis $([S^1_1], \cdots, [S^1_p])$ of $H_1(T^p)$. Because $h(\jmath_0(u, 1))$ is null-homologous in $\mathbb{R}^{p+2} - h(\imath_0(T^p))$ by the untwisted-framing assumption of $\jmath_0$, it is not hard to check that the p-torus defined by $h(\jmath_0(u, g_u(1)))$, $u \in T^p$, represents $g^*\xi$ under the Alexander duality $H_p(\mathbb{R}^{p+2} - \imath_1(T^p)) \cong H^1(T^p)$. Because $\jmath_1(u, 1) = h(\jmath_0(u, g_u(1)))$, this p-torus is also null-homologous in $\mathbb{R}^{p+2} - \imath_1(T^p)$ by the untwisted-framing assumption on $\jmath_1$. We see $g^*\xi = 0$, so $m_i = 0$ for $1 \leq i \leq p$. Thus $g_u(z) = z$ for any $z \in D^2$, and $\jmath_1 = h \circ \jmath_0$. □

Proposition 4.4 says a topologically 'simple' embedding $\jmath$ is determined by its type. We must show our favorite lifting e is topologically simple, namely composing e preserves the untwisted-framing property, the unknotted-ness. Moreover, we must show that any modular-type embedding remains modular after composing e, in order to use the finiteness result of Corollary 3.9. When $p = 1$, there is no type issue, and the rest are the following well-known facts in classical knot theory:

Lemma 4.5. *(1) $b_{\delta_i}(S^1_i \times \{1\})$ is homological to δ_i times $S^1_i \times \{1\}$ in $S^1_i \times D^2 - b_{\delta_i}(S^1_i \times \mathrm{Int}(D^2))$. Hence if $\jmath : S^1_i \times D^2 \hookrightarrow \mathbb{R}^3$ has untwisted framing, then $\jmath \circ b_{\delta_i}$ has untwisted framing.*

(2) Furthermore if $\jmath$ in (1) is also unknotted, then $b_{\delta_i}(S^1_i \times \{0\})$ is the $(\delta_i, 1)$-torus knot, which is unknotted.

We prove the general case in Lemmas 4.6, 4.8.

Lemma 4.6. *If $\jmath : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ has untwisted framing, then $\jmath \circ e$ also has untwisted framing.*

Proof. Obviously the composition with e_U, e_V preserves untwisted framing as they are identity on the fiber D^2. We claim $e_{\Delta_1}, \cdots, e_{\Delta_p}$ preserves untwisted framing, then provided that $\jmath$ has untwisted framing, so does $\jmath \circ e_U$, and hence so does $(\jmath \circ e_U) \circ e_{\Delta_1}$, and so on. Finally $\jmath \circ e$ also has untwisted framing.

Since $e_{\Delta_i} = b_{\delta_i} \times \mathrm{id}_{T^{p-1}_i} : S^1_i \times D^2 \times T^{p-1}_i \hookrightarrow S^1_i \times D^2 \times T^{p-1}_i$, by Lemma 4.5 (1), we have $e_{\Delta_i}(T^p \times \{1\})$ is homological to δ_i times $T^p \times \{1\}$ in $T^p \times D^2 - e_{\Delta_i}(T^p \times \mathrm{Int}(D^2))$. Thus if $\jmath : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ has untwisted framing, then $\jmath \circ e_{\Delta_i}$ has untwisted framing. $\square$

Showing that e also preserves the unknotted-ness needs more effort. We first investigate an essential case when e is just e_{Δ_i}, and $\jmath$ is a special candidate of its type.

We will call $\jmath_p : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ *standard*, if $\jmath_p$ has untwisted framing, and $\jmath_p|_{T^p \times \{0\}} = \imath_p$.

To help visualize how $\jmath \circ e(T^p \times D^2)$ unknots itself, remember that a standard type $\jmath_p$ can be written as $k_i \circ g_i$, for $i = 1, ..., p$, where:

$$g_i : S^1_1 \times \cdots \times S^1_p \times D^2 \subset (S^1_1 \times \cdots \times \hat{S^1_i} \times \cdots \times S^1_p) \times D^3$$

is the identity on factors S^1_j, $j \neq i$, and $S^1_i \times D^2 \subset D^3$ is the thicken-up of the circle lying on the equatorial disc of D^3, centered at the origin with radius one half, (cf. Corollary 3.3).

Lemma 4.7. *Suppose $1 \leq i \leq p$, and $\jmath_J = \jmath_p \circ e_J$, where $J \in \mathrm{SL}(p, \mathbb{Z})$ satisfies the minor $J^*_{ii} = 1$. Then $\jmath_J \circ e_{\Delta_i}$ is also unknotted. Moreover, the type of $\jmath_J \circ e_{\Delta_i}$ is modular.*

Proof. Without loss of generality, we may assume $i = 1$. Denote $g_J = g_1 \circ e_J$. The commutative diagram below has included all the maps involved so far:

$$\begin{tikzcd}
 & & T^{p-1}_1 \times D^3 \arrow[dr, "k_1"] & \\
T^p \times D^2 \arrow[r, "e_{\Delta_1}"] \arrow[d, "\pi"] & T^p \times D^2 \arrow[r, "e_J"] \arrow[d, "\pi"] \arrow[ur, "g_J"] & T^p \times D^2 \arrow[r, "\jmath_p"] \arrow[d, "\pi"] \arrow[u, "g_1"] & \mathbb{R}^{p+2} \\
T^p \arrow[r, "\phi_{\Delta_1}"] & T^p \arrow[r, "\tau_J"] & T^p \arrow[ur, "\imath_p"] &
\end{tikzcd}$$

There is a basic picture for $p = 2$ to keep in mind. In this case, we are trying to unknot the core of $\jmath_J \circ e_{\Delta_1}(T^2 \times D^2)$ in $\mathbb{R}^4$. Let $K'|_1$ be a $(\delta_1, 1)$-torus knot in D^3, whose carrier torus is placed parallel to the xy-plane centered at the origin. $K'|_1$ is of course an unknot. Let r be an integer. Imagine an S^1-family of unknots $K'|_w$ in D^3, such that for any $w \in S^1$, $K'|_w$ is obtained by rotating $K'|_1$ about the z-axis by an angle $r \arg(w)$. We may simultaneously cap off all these knots by picking a disk bounded by $K'|_1$ and rotate it about the z-axis so that at the time w it is bounded by $K'|_w$. See Figure 3. This implies that

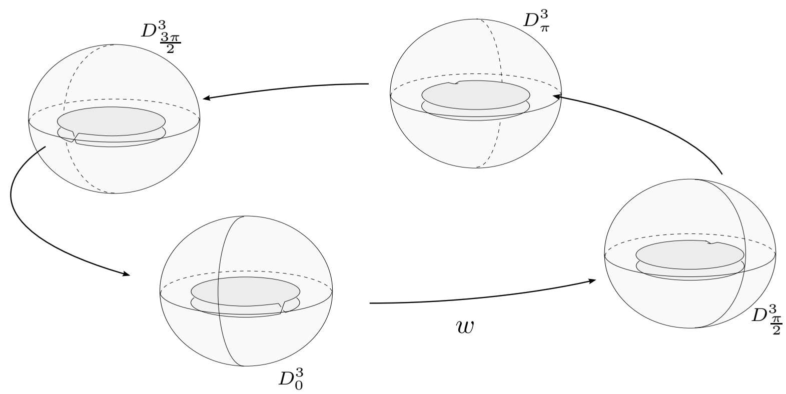

Figure 3: An S^1-family of rotating $(\delta_1, 1)$-torus knots, illustrated as $\delta_1 = 2$.

$K' = \bigcup_{w \in S^1} K'|_w$ is an unknotted torus in $S^1 \times D^3$ (in the sense that it is diffeotopic to $S^1 \times S^1 \subset S^1 \times D^3$). Therefore, if $S^1 \times D^3$ is further embedded in $\mathbb{R}^4$, the image of K' is also an unknotted torus in $\mathbb{R}^4$. As we will see below, K' is exactly $g_J \circ e_{\Delta_1}(T^2 \times \{0\})$. When $p > 2$, there is a similar picture, where we will have a T_1^{p-1}-family instead of just an S^1-family. Another difference is that when $p = 2$, the smooth mapping class group $\pi_0 \mathrm{Diff}_+(T^2)$ is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$ so the new embedding is automatically modular, but in the general case, this is no longer true so we need to analyze more carefully.

Specifically, we wish to diffeotope the core $K' = g_J \circ e_{\Delta_1}(T^p \times \{0\})$ back to $K = g_1(T^p \times \{0\})$ within $T_1^{p-1} \times D^3$, before including the latter into $\mathbb{R}^{p+2}$ via k_1. The $(\delta_1, 1)$-torus knot $K' \subset T_1^{p-1} \times D^3$ may be viewed as a T_1^{p-1}-family of (hopefully) unknotted loops in D^3. One may regard $S^1_2, \cdots, S^1_p$ as independent clocks, and at every 'moment' $(u_2, \cdots, u_p)$ we see a loop in D^3. It turns out that this is a $(\delta_1, 1)$-torus knot rotating around a fixed axis. Then we may simultaneously diffeotope the loops back to the standard place in D^3. The key to reading out this picture is understanding the intersection loci of K' on fibers D^3, namely $K'|_{(u_2, \cdots, u_p)} = K' \cap (\{(u_2, \cdots, u_p)\} \times D^3)$.

Denote $v = \phi(u) = \tau_J \circ \phi_{\Delta_1}(u)$, and $\vec{\omega}(u_1) = u_{1x} \cdot \vec{\varepsilon}_1 + u_{1y} \cdot \vec{\varepsilon}_2$ a rotating vector in $\mathbb{R}^2_{1,2}$. Then:

$$g_1(u, z) = ((u_2, \cdots, u_p),\ \frac{1}{2} \cdot \vec{\omega}(u_1) + \frac{1}{3}[z_x \cdot \vec{\omega}(u_1) + z_y \cdot \vec{\varepsilon}_3]).$$

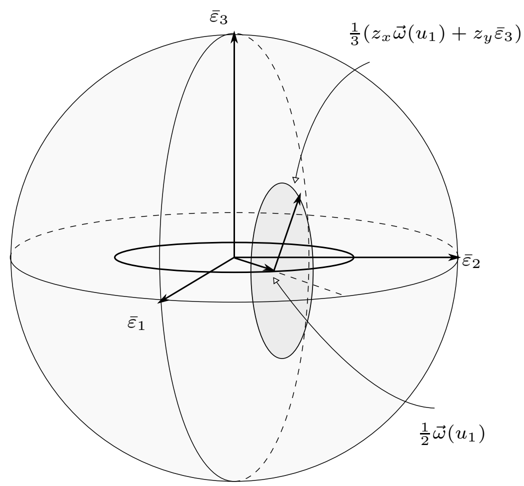

Figure 4: The image of (u_1, z) under $g_1 : S^1_1 \times D^2 \hookrightarrow D^3$.

Composing $e_J(u, z) = (\tau_J(u), z)$ with $e_{\Delta_1}(u, z) = (\phi_{\Delta_1}(u), \frac{1}{2}u_1 + \frac{1}{\delta_1^2}u_1^{1-\delta_1}z)$, we have:

$$g_J \circ e_{\Delta_1}(u, 0) = ((v_2, \cdots, v_p),\ b(u)),$$

where v_j means the S^1_j-component of $v = \phi(u)$, and:

$$b(u) = \frac{1}{2} \cdot \vec{\omega}(v_1) + \frac{1}{6}[u_{1x} \cdot \vec{\omega}(v_1) + u_{1y} \cdot \vec{\varepsilon}_3].$$

Because $J^*_{11} = 1$ is the $(1,1)$-th entry of J^{-1}, $u_1^{\delta_1} = v_1 v_2^{-r_2} \cdots v_p^{-r_p}$ for some integers $r_2, \cdots, r_p$, so $v_1 = u_1^{\delta_1} v_2^{r_2} \cdots v_p^{r_p}$. Therefore at the 'moment' $(v_2, \cdots, v_p) \in T_1^{p-1}$, $K'|_{(v_2,\cdots,v_p)}$ is a $(\delta_1, 1)$-torus knot in D^3 defined by $b(u)$, and as $v_j = e^{i\theta_j}$, $K'|_{(v_2,\cdots,v_p)}$ rotates about the $\vec{\varepsilon}_3$-axis by an angle $r_j\theta_j$.

Note that a $(\delta_1, 1)$-torus knot in D^3 is unknotted, so there is a diffeotopy $h_t : D^3 \to D^3$ supported in the interior, such that $h_0 = \mathrm{id}_{D^3}$ and $h_1(K'|_{(1,\cdots,1)}) = K|_{(1,\cdots,1)}$. To unknot K' simultaneously on fibers, define $\rho : T_1^{p-1} \to \mathrm{SO}(3)$, with $\rho(v_2, \cdots, v_p)$ being the rotation about the $\vec{\varepsilon}_3$-axis by an angle $r_2\theta_2 + \cdots + r_p\theta_p$, where $v_j = e^{i\theta_j}$. The 'unknotting' diffeotopy may be defined as:

$$H_t : T_1^{p-1} \times D^3 \to T_1^{p-1} \times D^3$$

with:

$$H_t((v_2, \cdots, v_p), \vec{x}) = ((v_2, \cdots, v_p), \rho(v_2, \cdots, v_p) \circ h_t \circ (\rho(v_2, \cdots, v_p))^{-1}(\vec{x})).$$

Since H_t is supported in the interior, when we embed $T_1^{p-1} \times D^3$ into $\mathbb{R}^{p+2}$ by k_1, H_t induces a diffeotopy on $k_1(T_1^{p-1} \times D^3)$ supported in the interior, which extends as a diffeotopy of $\mathbb{R}^{p+2}$. This diffeotopy takes $\jmath_J \circ e_{\Delta_1}(T^p \times \{0\})$ back to $\jmath_p(T^p \times \{0\})$, so the former is unknotted too.

To see the 'moreover' part, note that both $g_1, H_1 \circ g_J \circ e_{\Delta_1}$ embed $T^p \times \{0\}$ into $T_1^{p-1} \times D^3$ with the same image. Let B denote the matrix obtained from J by multiplying each entry of the first column by δ_1 and then replacing the first row by $(1, 0, \cdots, 0)$. Since $J_{11}^* = 1$, $B \in \mathrm{SL}(p, \mathbb{Z})$. By comparing:

$$H_1 \circ g_J \circ e_{\Delta_1}(u, 0) = (v_2, \cdots, v_p, \rho(v_2, \cdots, v_p) \circ h_1 \circ (\rho(v_2, \cdots, v_p))^{-1} \circ b(u)),$$

with:

$$g_1 \circ e_B(u, 0) = (v_2, \cdots, v_p, \frac{1}{2}\vec{\omega}(u_1)),$$

we see that the self-diffeomorphism $(H_1 \circ g_J \circ e_{\Delta_1}) \circ (g_1 \circ e_B)^{-1} : T^p \to T^p$ can be written as $F : T_1^{p-1} \times S^1 \to T_1^{p-1} \times S^1$, such that $F((v_2, \cdots, v_p), z) = ((v_2, \cdots, v_p), f(v_2, \cdots, v_p)(z))$, where $f : T_1^{p-1} \to \mathrm{Diff}_+(S^1)$. Because $\mathrm{Diff}_+(S^1) \simeq \mathrm{SO}(2)$, f is homotopic to $f_0 : T_1^{p-1} \to \mathrm{SO}(2)$, where $f_0(v_2, \cdots, v_p) = v_2^{m_2} \cdots v_p^{m_p}$ for some integers $m_2, \cdots, m_p$, and F is diffeotopic to the automorphism $F_0 : T_1^{p-1} \times S^1 \to T_1^{p-1} \times S^1$, $F_0((v_2, \cdots, v_p), z) = ((v_2, \cdots, v_p), v_2^{m_2} \cdots v_p^{m_p} z)$. Let M denote the matrix obtained from the $p \times p$ identity matrix by replacing the first row by $(1, m_2, \cdots, m_p)$. Then the type of $\jmath_J \circ e_{\Delta_1}$ is a modular transformation of the standard type by MB. $\square$

The general case that composing e preserves the unknotted-ness and remains modular is as follows.

Lemma 4.8. *If $\jmath$ is unknotted of modular type with untwisted framing, then $\jmath \circ e : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ is also unknotted of modular type.*

Proof. Clearly $\hat{\jmath} = \jmath \circ e_U$ is unknotted with modular type and untwisted framing. Note $\hat{\jmath} \circ e_{\Delta_1}$ is unknotted if and only if so is $h \circ \hat{\jmath} \circ e_{\Delta_1}$ for any $\mathbb{R}^{p+2}$ self-diffeomorphism h. By Proposition 4.4, the unknotted-ness of $\hat{\jmath} \circ e_{\Delta_1}$ depends only on the type of $\hat{\jmath}$. Since $\hat{\jmath}$ is of modular type, suppose $\jmath_J = \jmath_p \circ e_J$ has the same type as $\hat{\jmath}$. Moreover, J may be picked so that $J_{11}^* = 1$ by Lemma 3.13(2). Then by Lemma 4.7, $\jmath_J \circ e_{\Delta_1}$ remains unknotted with modular type. Therefore $\hat{\jmath} \circ e_{\Delta_1}$ is unknotted with modular type. By Lemma 4.6, $\hat{\jmath} \circ e_{\Delta_1}$ has untwisted framing.

Repeat this argument so $(\hat{\jmath} \circ e_{\Delta_1}) \circ e_{\Delta_2}$ is unknotted with modular type and untwisted framing, and so on we see that $\hat{\jmath} \circ e_{\Delta_1} \cdots e_{\Delta_p}$ is also unknotted with modular type. Finally $(\jmath \circ e_U \circ e_{\Delta_1} \circ \cdots \circ e_{\Delta_p}) \circ e_V$ is also unknotted since it has the same image of the core as $\jmath \circ e_U \circ e_{\Delta_1} \circ \cdots \circ e_{\Delta_p}$, and the type change is modular. We conclude that $\jmath \circ e$ is still unknotted of modular type. $\square$

Proof of Theorem 1.4. Pick an unknotted embedding $\jmath : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ of modular type with untwisted framing. (We can in fact require the core image to be any unknotted T^p in $\mathbb{R}^{p+2}$, by Lemma 3.7.) By Lemma 4.6 and Lemma 4.8, $\jmath \circ e^i$ $(i \geq 0)$ are all unknotted of modular type with untwisted framing. By Corollary 3.9, at least two of $\jmath \circ e^i$ $(0 \leq i \leq 2^p - 1)$ are of the same type. Suppose $\jmath \circ e^k$ and $\jmath \circ e^l$ $(0 \leq k < l \leq 2^p - 1)$ are of the same type. Pick the embedding $\hat{\jmath} = \jmath \circ e^k : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ instead of $\jmath$, and let $d = l - k$. There is a compactly-supported self-diffeomorphism h of $\mathbb{R}^{p+2}$ such that $h \circ \hat{\jmath} = \hat{\jmath} \circ e^d$ by Proposition

4.4. This is to say, e^d can be realized by an embedding $\hat{\jmath} : T^p \times D^2 \hookrightarrow \mathbb{R}^{p+2}$ with extension h.

Therefore,

$$\Lambda = \bigcap_{i=0}^{\infty} e^i(T^p \times D^2) = \bigcap_{i=0}^{\infty} e^{di}(T^p \times D^2),$$

embeds into $\mathbb{R}^{p+2}$ by $\hat{\jmath}$ as an attractor of h, so we have realized an expanding attractor derived from ϕ. $\square$

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