Discrete Applied Mathematics 161 (2013) 2396–2404

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Super-simple balanced incomplete block designs with block size 5 and index 3 ☆

Kejun Chen [a], Guangzhou Chen [b], Wen Li [a], Ruizhong Wei [c,*]

[a] Department of Mathematics, Yancheng Teachers University, Yancheng 224051, PR China

[b] Mathematics and Information Science College, Hebei Normal University, Shijiazhuang, Hebei 050016, PR China

[c] Department of Computer Science, Lakehead University Thunder Bay, ON, P7B 5E1, Canada

ARTICLE INFO

Article history:
Received 18 July 2011
Received in revised form 3 May 2013
Accepted 10 May 2013
Available online 6 June 2013

Keywords:
Super-simple designs
BIBD
GDD

ABSTRACT

Super-simple designs are useful in constructing codes and designs such as superimposed codes and perfect hash families. In this article, we investigate the existence of a super-simple $(v, 5, 3)$ balanced incomplete block design and show that such a design exists if and only if $v \equiv 1, 5 \pmod{20}$ and $v \geq 21$ except possibly when $v = 45, 65$.

1. Introduction

A *group divisible design* (or GDD) is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of a set $\mathcal{X}$ (of *points*) into subsets called *groups*;
2. $\mathcal{B}$ is a set of subsets of $\mathcal{X}$ (called *blocks*) such that a group and a block contain at most one common point;
3. Every pair of points from distinct groups occurs in exactly λ blocks.

The *group type* (or *type*) of a GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We shall use an "exponential" notation to describe types: so type $g_1^{u_1} \cdots g_k^{u_k}$ denotes u_i occurrences of g_i, $1 \leq i \leq k$, in the multiset. A GDD with block sizes from a set of positive integers K is called a (K, λ)-GDD. When $K = \{k\}$, we simply write k for K. When $\lambda = 1$, we simply write K-GDD. A (k, λ)-GDD with group type 1^v is called a *balanced incomplete block design*, denoted by (v, k, λ)-BIBD.

A design is said to be *simple* if it contains no repeated blocks. A design is said to be *super-simple* if the intersection of any two blocks has at most two elements. When $k = 3$, a super-simple design is just a simple design. When $\lambda = 1$, the designs are always super-simple. In this paper, when we talk about super-simple BIBDs, we usually mean the case that $k \geq 4$ and $\lambda > 1$.

Super-simple designs were introduced by Gronau and Mullin [22]. The existence of super-simple designs is an interesting extremal problem by itself, but there are also useful applications. For examples, such designs are used in constructing perfect hash families [32] and coverings [9], in the construction of new designs [8] and in the construction of superimposed codes [29]. There are other useful applications related to super-simple designs [25,19,33,7]. In statistical planning of experiments, super-simple designs are the ones providing samples with maximum intersection as small as possible.

☆ Research supported by NSFC (NO. 11071207), Jiangsu Provincial NSF (No. BK2008198), and NSERC Grant 239135-2011.

* Corresponding author.
E-mail addresses: yctuckj@163.com (K. Chen), rwei@lakeheadu.ca (R. Wei).

0166-218X/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.dam.2013.05.007

It is well known that the following are the necessary conditions for the existence of a super-simple (v, k, λ)-BIBD:

1. $v \geq (k-2)\lambda + 2$;
2. $\lambda(v-1) \equiv 0 \pmod{k-1}$;
3. $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

For the existence of super-simple $(v, 4, \lambda)$-BIBDs, the necessary conditions are known to be sufficient for $\lambda \in \{2, 3, 4, 5, 6, 8, 9\}$ (see [6,13–16,34,12,28,22]). For arbitrary k and λ, the usually necessary conditions are asymptotically sufficient (see [25,23,26]).

For the existence of super-simple $(v, 5, \lambda)$-BIBDs, the necessary conditions for $\lambda = 2, 4, 5$ are proved to be sufficient with two exceptions and one possible exception. Gronau, Kreher, and Ling [21] solved the case of $\lambda = 2$ with 11 unsettled values. Later on, these values were removed by Abel and Bennett [1], and Chen and Wei [18]. The $\lambda = 4, 5$ cases were solved by Chen and Wei [18,17]. We summarize these known results in the following theorem.

Theorem 1.1 (*[18,17,21,1,20]*). *A super-simple $(v, 5, \lambda)$-BIBD exists for $\lambda = 2, 4, 5$ if and only if the following conditions are satisfied:*

1. $\lambda = 2$, $v \equiv 0, 5 \pmod{10}$ *and* $v \neq 5, 15$*;*
2. $\lambda = 4$, $v \equiv 0, 1 \pmod 5$ *and* $v \geq 15$*;*
3. $\lambda = 5$, $v \equiv 1 \pmod 4$ *and* $v \geq 17$, *except possibly when* $v = 21$.

In this article, we investigate the existence of super-simple $(v, 5, 3)$-BIBDs. When $k = 5$ and $\lambda = 3$ the necessary condition becomes $v \equiv 1, 5 \pmod{20}$ and $v \geq 21$. We shall show that this necessary condition is also sufficient with two possible exceptions. Specifically, we shall prove the following.

Theorem 1.2. *A super-simple $(v, 5, 3)$-BIBD exists if and only if $v \equiv 1, 5 \pmod{20}$ and $v \geq 21$ except possibly when $v = 45, 65$.*

Some recursive constructions used in this paper are listed in Section 2. Section 3 gives direct constructions which are based on a computer search. The proof of Theorem 1.2 will be given in Section 4.

2. Recursive constructions

We shall use the following standard recursive constructions. The proofs of these constructions can be found in [14,12].

Construction 2.1 (*Weighting*). *Let $(X, \mathcal{G}, \mathcal{B})$ be a super-simple GDD with index λ_1, and let $w : X \to Z^+ \cup \{0\}$ be a weight function on X, where Z^+ is the set of positive integers. Suppose that for each block $B \in \mathcal{B}$, there exists a super-simple (k, λ_2)-GDD of type $\{w(x) : x \in B\}$. Then there exists a super-simple $(k, \lambda_1\lambda_2)$-GDD of type $\{\sum_{x \in G_i} w(x) : G_i \in \mathcal{G}\}$.*

Construction 2.2 (*Breaking up Groups*). *If there exists a super-simple (k, λ)-GDD of type $h_1^{u_1} \cdots h_t^{u_t}$ and a super-simple $(h_i + \eta, k, \lambda)$-BIBD for each i $(1 \leq i \leq t)$, then there exists a super-simple $(\sum_{i=1}^{t} h_i u_i + \eta, k, \lambda)$-BIBD, where $\eta = 0$ or 1.*

To present the next construction, we need the notation of a (v, w, k, λ)-IBIBD. An *incomplete balanced incomplete block design* (v, w, k, λ)-IBIBD is a triple $(\mathcal{V}, \mathcal{H}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{V}$ is a v-set of points, $\mathcal{H}$ is a w-subset of $\mathcal{V}$ (called a hole) and $\mathcal{B}$ is a collection of k-subsets of $\mathcal{V}$ (called blocks);
2. $|\mathcal{H} \cap B| \leq 1$ for all $B \in \mathcal{B}$;
3. any two points of $\mathcal{V}$ appear either in $\mathcal{H}$ or in λ blocks of $\mathcal{B}$ exactly.

It is obvious that a (v, w, k, λ)-IBIBD is a (v, k, λ)-BIBD indeed when $w \in \{0, 1\}$. So, the following construction can be considered as a generalization of Construction 2.2.

Construction 2.3 (*[12] Filling in Holes*). *Suppose that there exists a super-simple (k, λ)-GDD of type $h_1 h_2 \cdots h_t$, a super-simple $(h_i + s, s, k, \lambda)$-IBIBD for each $i (1 \leq i \leq t-1)$, and a super-simple $(h_t + s, k, \lambda)$-BIBD, then there exists a super-simple $(\sum_{i=1}^{t} h_i + s, k, \lambda)$-BIBD.*

A *transversal design* $\mathrm{TD}_\lambda(k, n)$ is a (k, λ)-GDD of group type n^k. When $\lambda = 1$, we simply write $\mathrm{TD}(k, n)$. A parallel class in a design is a collection of blocks that partition the points of the design. If all the blocks of a design can be partitioned into parallel classes we say that the design is *resolvable*. A resolvable TD is denoted by RTD.

It is well known that a $\mathrm{RTD}(k, n)$ is equivalent to a $\mathrm{TD}(k+1, n)$, and a $\mathrm{TD}(k, n)$ is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order n. For a list of lower bounds on the number of MOLS for all orders up to 10 000 we refer to [3]. We have the following.

Lemma 2.4 (*[3]*).

1. *A* $\mathrm{TD}(q+1, q)$ *exists, consequently, a* $\mathrm{TD}(k, q)$ *exists for any positive integer k $(k \leq q+1)$, where q is a prime power.*
2. *A* $\mathrm{TD}(5, n)$ *exists for all n and* $n \notin \{2, 3, 6, 10\}$.
3. *A* $\mathrm{TD}(6, n)$ *exists for all* $n \geq 5$ *and* $n \notin \{6, 10, 14, 18, 22\}$.
4. *A* $\mathrm{TD}(7, n)$ *exists for all* $n \geq 7$ *and* $n \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$.

In this paper, we shall also make use of the following known results on GDDs and PBDs.

Lemma 2.5 ([2]). *There exists a* 5-*GDD of type* 4^u *for any* $u \geq 5$ *and* $u \equiv 0, 1 \pmod 5$.

Lemma 2.6 ([4,31]). *There exists a* $(v, \{5, 6, 7, 8, 9\}, 1)$-*PBD for any* $v \geq 21$ *and* $v \notin \{22 - 24, 27 - 29, 32 - 34\}$.

An orthogonal array $OA_\lambda(t, k, n)$ is an $k \times \lambda n^t$ array over a n-set G, having the property that every t-tuple with entries from G appears exactly λ times as a column in every $t \times \lambda n^t$ submatrix. The parameters λ and t are the *index* and the *strength* of the orthogonal array, respectively. In this notation, if λ is omitted it is understood to be one, and if t is omitted it is understood to be two. It is well known that an $OA_\lambda(k, n)$ is equivalent to a $TD_\lambda(k, n)$. An $OA_\lambda(t, k, n)$ is called r-*simple* if any two different columns agree in less than r entries. Clearly, a 3-simple $OA_\lambda(k, n)$ leads to a super-simple $TD_\lambda(k, n)$. An $OA_\lambda(t, k, n)$ is said to be *completely reducible* if it is the union of λ $OA(t, k, n)$s. We have the following.

Lemma 2.7 ([11]). *If* q *is a prime power and* $t < q$, *then an* $OA(t, q + 1, q)$ *exists. Moreover, if* $q \geq 4$ *is a power of* 2, *then an* $OA(3, q + 2, q)$ *exists.*

Lemma 2.8 ([24]). *If an* $OA(t, k, n)$ *exists, then there is also a completely reducible* t-*simple* $OA_{n^s}(t - s, k - s, n)$ *for every non-negative integer* $s < t$.

Corollary 2.9. *There exists a super-simple* $TD_3(5, n)$ *for any* $n \in M = \{4, 5, 16, 21, 25, 41, 61, 81, 101, 121\}$.

Proof. For $n = 21$, there exists an $OA(3, 6, 21)$, see [27]. For each $n \in M \setminus \{21\}$, there exists an $OA(3, 6, n)$ by Lemma 2.7. Applying Lemma 2.8 with $s = 1$, we obtain completely reducible 3-simple $OA_n(5, n)$ for each $n \in M$, this leads immediately to a 3-simple $OA_3(5, n)$, consequently, we get a super-simple $TD_3(5, n)$. □

3. Direct constructions

In this section, we shall use direct constructions to obtain super-simple $(v, 5, 3)$-BIBDs for some small values of v and some super-simple (5, 3)-GDDs, which will be used as master designs or input designs in our recursive constructions. All of these designs have been found after computer-assisted searches. In fact, most of them have cyclic groups of automorphism of order v. So, they are cyclic designs. For a cyclic design, we just need to find base blocks and other blocks can be obtained by developing with the automorphism.

The checking for super-simplicity can be done by a computer after developing the designs. But there are more economical ways to check the super-simplicity of cyclic designs. For example, suppose that a design is obtained by developing m base blocks modulo v. In order to check whether the design is super-simple, we form the ten 3-subsets of each base block and develop them modulo v. Thus we get a list of $10mv$ triples. If these $10mv$ triples are pairwise distinct, then the design is super-simple. This criteria can be further reduced as follows. Let $S = \{b_1, b_2, b_3\}$, $b_1 < b_2 < b_3$, be a 3-set contained in a base block. Instead of developing S modulo v we form the following three representatives of the orbit corresponding to S:

$$\{b_1 - b_i, b_2 - b_i, b_3 - b_i\}, \quad i = 1, 2, 3.$$

We get a list of $30m$ triples in this way. It is easy to see that if these $30m$ triples are pairwise distinct, then the design is super-simple. Considerations of this nature have been implemented in all of the computer searches. It should be mentioned that the above approach of checking super-simplicity is essentially the same as in [10].

In most cases, we managed to find a multiplier or partial multiplier with an appropriate order so that the required base blocks can be found in a shorter time. A method we used in computer program is applying multipliers of blocks. Since our constructions are over Z_v, we can use both the addition and the multiplication of Z_v. We say that $w \in Z_v^*$ is a *multiplier* of the design, if for each base block $B = \{x_1, x_2, x_3, x_4, x_5\}$, there exists some $g \in Z_v$ such that $C = w \cdot B + g = \{w \cdot x_1 + g, w \cdot x_2 + g, w \cdot x_3 + g, w \cdot x_4 + g, w \cdot x_5 + g\}$ is also a base block. We say that $w \in Z_v^*$ is a *partial multiplier* of the design, if for each base block $B \in \mathcal{M}$, where $\mathcal{M}$ is a subset of all the base blocks, there exists some $g \in Z_v$ such that $C = w \cdot B + g$ is also a base block.

In the computer program, we first choose a (partial) multiplier w. Our experiences tell us that choosing a w which has long orbits in the multiplication group of Z_v usually gives better results. Then we start to find base blocks in the following way. When a base block B is found, the algorithm requires that $wB, w^2B, \ldots, w^sB$ are also different base blocks, where s is a positive number. If we can find all the base blocks in this way, then w^i, $1 \leq i \leq s$ are multipliers of the design. Otherwise, these are partial multipliers, and the algorithm tries to find the remaining base blocks. To decide the value of s is also important for the success of the algorithm. In practice, we usually let s be as large as possible at the beginning. Then the value of s is reduced if the search time is too long.

Lemma 3.1. *There exists a super-simple* $(v, 5, 3)$-*BIBD for each* $v \in \{21, 41, 61, 81\}$.

Proof. For $v = 21$, let the point set be Z_{21}. The required base blocks are $\{0, 1, 2, 4, 16\}$, $\{0, 1, 5, 11, 13\}$, $\{0, 3, 7, 10, 16\}$. Developing these base block modulo 21, we get a super-simple (21, 5, 3)-BIBD.

For each $v \in \{41, 61, 81\}$, let the point set be Z_v. With a computer program we found the required base blocks, which are divided into two parts, P and R, where P consists of some base blocks with a partial multiplier m of order s, (i.e., each base block of P has to be multiplied by m^i for $0 \le i \le s-1$), and R is the set of the remaining base blocks. We list P, m, s and R below. All base blocks are developed by mod v to form the set of blocks.

$v = 41$
$P: \{0, 1, 2, 7, 16\},\ m = 2,\ s = 3;$
$R: \{0, 1, 13, 24, 34\},\ \{0, 3, 6, 17, 25\},\ \{0, 3, 10, 15, 35\}.$
$v = 61$
$P: \{0, 1, 2, 4, 21\},\ \{0, 1, 8, 10, 16\},\ \{0, 4, 11, 34, 50\},\ m = 13,\ s = 3;$
$R: \emptyset.$
$v = 81$
$P: \{0, 1, 2, 23, 26\},\ m = 2,\ s = 7;$
$R: \{0, 9, 18, 45, 70\},\ \{0, 13, 41, 51, 80\},\ \{0, 18, 31, 36, 41\},\ \{0, 15, 34, 43, 69\}, \{0, 6, 19, 26, 60\}.$ □

Lemma 3.2. *There exists a super-simple* $(25, 5, 3)$*-BIBD.*

Proof. Let the point set be Z_{25} and the required base blocks are listed below. All base block are developed by the automorphism group $\alpha = (0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9)(10\ 11\ 12\ 13\ 14)(15\ 16\ 17\ 18\ 19)(20\ 21\ 22\ 23\ 24)$.

$\{0, 1, 2, 6, 23\}, \{3, 5, 6, 7, 11\}, \{8, 10, 11, 12, 16\}, \{13, 15, 16, 17, 21\}, \{1, 18, 20, 21, 22\},$
$\{0, 1, 3, 11, 19\}, \{5, 6, 8, 16, 24\}, \{4, 10, 11, 13, 21\}, \{1, 9, 15, 16, 18\}, \{6, 14, 20, 21, 23\},$
$\{0, 9, 11, 16, 24\}, \{4, 5, 14, 16, 21\}, \{1, 9, 10, 19, 21\}, \{1, 6, 14, 15, 24\}, \{4, 6, 11, 19, 20\},$
$\{0, 5, 10, 15, 20\}, \{0, 6, 12, 18, 24\}, \{0, 7, 14, 16, 23\}.$ □

Lemma 3.3. *There exists a super-simple* $(v, 5, 3)$*-BIBD for each* $v \in M = \{281, 381, 461\}$.

Proof. For each $v \in M$, we take Z_v as the point set. Below are the required base blocks, each of which has to be multiplied by m^i, $0 \le i \le s-1$. All base blocks are developed by mod v.

$v = 281$:
$\{0, 6, 16, 26, 74\}, \{0, 1, 5, 11, 85\}, \{0, 3, 8, 15, 45\}, \{0, 1, 2, 4, 149\}, \{0, 11, 23, 77, 252\},$
$\{0, 2, 115, 173, 269\}, m = 32, s = 7.$
$v = 381$:
$\{0, 158, 252, 286, 377\}, \{0, 17, 49, 293, 322\}, \{0, 1, 75, 77, 379\}, \{0, 65, 121, 249, 308\},$
$\{0, 81, 197, 323, 371\}, \{0, 54, 166, 300, 361\}, \{0, 2, 161, 175, 348\}, \{0, 8, 26, 157, 207\},$
$\{0, 39, 162, 214, 298\}, \{0, 42, 49, 150, 318\}, \{0, 28, 106, 127, 141\}, \{0, 85, 110, 137, 302\},$
$\{0, 185, 306, 321, 376\}, \{0, 8, 67, 156, 238\}, \{0, 6, 141, 144, 150\}, \{0, 6, 217, 283, 370\},$
$\{0, 128, 193, 259, 353\}, \{0, 185, 220, 318, 362\}, \{0, 5, 52, 215, 293\}, m = 19, s = 3.$
$v = 461$:
$\{0, 1, 2, 4, 8\}, \{0, 1, 11, 19, 36\}, \{0, 5, 29, 64, 77\}, m = 14, s = 23.$ □

The following super-simple GDDs will be used as master designs or input designs in our recursive constructions.

Lemma 3.4. *There exists a super-simple* $(5, 3)$*-GDD of group type* 4^t *for any* $t \in T = \{6, 11, 16, 21, 31\}$.

Proof. For each $t \in T$, we construct a super-simple $(5, 3)$-GDD of group type 4^t. Let the point set be Z_{4t} and let the group set be $\{\{i, i+t, i+2t, i+3t\} : 0 \le i \le t-1\}$. The required base blocks are listed below. All base blocks are developed by mod $4t$.

$t = 6, \{0, 1, 2, 5, 9\}, \{0, 1, 11, 15, 22\}, \{0, 2, 11, 16, 19\}.$
$t = 11, \{0, 13, 26, 29, 34\}, \{0, 9, 18, 24, 41\}, \{0, 5, 8, 9, 25\}, \{0, 17, 18, 30, 37\}, \{0, 2, 25, 39, 40\},$
$\{0, 2, 6, 34, 36\}.$

For $t = 16, 21, 31$, as before, the base blocks are divided into two parts, P and R, such that each of the base blocks of P has to be multiplied by m^i for $0 \le i \le s-1$.

$t = 16$
P: $\{0, 1, 2, 4, 35\}, m = 3, s = 4;$
R: $\{0, 7, 21, 28, 47\}, \{0, 11, 22, 24, 39\}, \{0, 13, 14, 25, 33\}, \{0, 20, 24, 46, 54\}, \{0, 4, 12, 19, 25\}.$
$t = 21$
P: $\{0, 1, 2, 10, 46\}, \{0, 1, 5, 12, 32\}, \{0, 5, 11, 35, 69\}, \{0, 14, 27, 33, 55\}, m = 25, s = 3;$
$R: \emptyset.$
$t = 31$
P: $\{0, 1, 2, 4, 34\}, \{0, 1, 12, 47, 61\}, \{0, 2, 6, 13, 58\}, \{0, 3, 27, 44, 61\}, \{0, 6, 14, 22, 43\},$
$\{0, 8, 17, 36, 59\}, m = 5, s = 3;$
$R: \emptyset.$ □

Lemma 3.5. *There exists a super-simple* $(5, 3)$*-GDD of group type* 5^t *for any* $t \in T = \{9, 13, 17, 29\}$.

Proof. For each $t \in T$, we construct a super-simple $(5, 3)$-GDD of group type 5^t. Let the point set be Z_{5t} and let the group set be $\{\{i, t+i, 2t+i, 3t+i, 4t+i\} : 0 \le i \le t-1\}$.

When $t = 9$, the required design is generated by developing the following base blocks modulo 45.

$\{0, 15, 16, 21, 44\}, \{0, 7, 19, 30, 32\}, \{0, 8, 20, 37, 39\}, \{0, 3, 5, 10, 24\},$
$\{0, 1, 8, 12, 42\}, \{0, 3, 13, 17, 23\}.$

When $t \in \{13, 17, 29\}$, below are the required base blocks, each of which has to be multiplied by m^i, $0 \le i \le s-1$. All the base blocks are then developed by mod $5t$.

$t = 13, \{0, 1, 2, 4, 60\}, \{0, 2, 9, 14, 44\}, \{0, 3, 22, 37, 43\}, m = 16, s = 3.$
$t = 17, \{0, 1, 56, 64, 83\}, \{0, 2, 33, 40, 72\}, \{0, 5, 38, 61, 79\}, \{0, 12, 18, 54, 65\}, \{0, 14, 15, 50, 76\},$
$\{0, 25, 37, 77, 82\}, m = 13, s = 2;$
$t = 29, \{0, 1, 2, 4, 26\}, \{0, 2, 12, 22, 33\}, \{0, 3, 13, 62, 82\}, m = 4, s = 7.$ □

Lemma 3.6. *There exists a super-simple* $(5, 3)$*-GDD of group type* 8^t *for any* $t \in T = \{6, 11\}$.

Proof. For each $t \in T$, we construct a super-simple $(5, 3)$-GDD of group type 8^t. Let the point set be Z_{8t} and let the group set be $\{\{i, t+i, 2t+i, \ldots, 7t+i\} : 0 \le i \le t-1\}$. The required base blocks are listed below. All the base blocks are developed by mod $8t$.

$t = 6, \{0, 11, 16, 31, 39\}, \{0, 2, 13, 21, 23\}, \{0, 9, 10, 13, 26\}, \{0, 21, 22, 29, 44\}, \{0, 2, 9, 43, 47\},$
$\{0, 3, 17, 32, 37\}.$
$t = 11, 3^i\{0, 1, 2, 4, 19\}, 0 \le i \le 6, \{0, 32, 64, 72, 84\}, \{0, 19, 29, 60, 68\}, \{0, 16, 40, 45, 53\},$
$\{0, 13, 29, 30, 78\}, \{0, 2, 26, 32, 62\}.$ □

Lemma 3.7. *There exists a super-simple* $(5, 3)$*-GDD of group type* $(20)^t$ *for each* $t \in \{7, 8\}$.

Proof. For each $t \in \{7, 8\}$, let the point set be Z_{20t} and let the group set be $\{\{i, t+i, \ldots, 19t+i\} : 0 \le i \le t-1\}$. Below are the required base blocks, which are divided into two parts, P and R, such that each of the base blocks of P has to be multiplied by m^i for $0 \le i \le s-1$. All the base blocks are developed by mod $20t$.

$t = 7$
P: $\{0, 1, 2, 52, 124\}, \{0, 1, 5, 53, 100\}, \{0, 2, 5, 62, 88\}, \{0, 4, 13, 43, 110\}, \{0, 6, 31, 67, 78\},$
$\{0, 55, 80, 86, 124\}, m = 19, s = 3;$
R: $\emptyset$.
$t = 8$
P: $\{0, 2, 6, 11, 15\}, \{0, 5, 11, 30, 63\}, \{0, 10, 27, 44, 109\}, m = 7, s = 4;$
R: $\{0, 3, 22, 103, 109\}, \{0, 31, 81, 83, 157\}, \{0, 20, 58, 102, 159\}, \{0, 4, 35, 53, 122\}, \{0, 87, 98, 99, 100\},$
$\{0, 20, 43, 66, 134\}, \{0, 31, 60, 67, 146\}, \{0, 28, 85, 90, 139\}, \{0, 7, 14, 27, 37\}.$ □

4. The proof of Theorem 1.2

In this section, we shall complete the proof of Theorem 1.2. We shall divide it into two cases, $v \equiv 1 \pmod{20}$ and $v \equiv 5 \pmod{20}$.

Case 1: $v \equiv 1 \pmod{20}$

In this case, we will prove that there exists a super-simple $(v, 5, 3)$-BIBD for any $v \equiv 1 \pmod{20}$ and $v \ge 21$. Let a, b be integers and let $[a, b]^1_{20}$ be the set of positive integers v such that $v \equiv 1 \pmod{20}$ and $a \le v \le b$.

Lemma 4.1. *There exists a super-simple* $(5, 3)$*-GDD of group type* $(20)^t$ *for any* $t \in T = \{5, 6, 9, 10, 11, 13\}$.

Proof. For each $t \in \{5, 9, 13\}$, starting from a super-simple $(5, 3)$-GDD of group type 5^t coming from Corollary 2.9 and Lemma 3.5, applying Construction 2.1 with a TD(5, 4) coming from Lemma 2.4, we obtain a super-simple $(5, 3)$-GDD of group type $(20)^t$.

For each $t \in \{6, 11\}$, starting from a super-simple $(5, 3)$-GDD of group type 4^t (from Lemma 3.4) and applying Construction 2.1 with a TD(5, 5) coming from Lemma 2.4, a super-simple $(5, 3)$-GDD of group type $(20)^t$ is obtained.

For $t = 10$, starting from a 5-GDD of group type 4^{10} coming from Lemma 2.5 and applying Construction 2.1 with a super-simple $(5, 3)$-GDD of type 5^5 coming from Corollary 2.9, a super-simple $(5, 3)$-GDD of group type $(20)^t$ is obtained. □

Lemma 4.2. *There exists a super-simple* $(v, 5, 3)$*-BIBD for any* $v \in M = [101, 261]^1_{20}$.

Proof. For $v = 241$, starting from a super-simple (5, 3)-GDD of group type 8^6 coming from Lemma 3.6 and applying Construction 2.1 with a TD(5, 4) (from Lemma 2.4), we obtain a super-simple (5, 3)-GDD of group type $(40)^6$. Since there exists a super-simple $(40 + 1, 5, 3)$-BIBD from Lemma 3.1, applying Construction 2.2 with $\eta = 1$, we obtain a super-simple (241, 5, 3)-BIBD.

For each other $v \in M$, we can write $v = 20t + 1$, where $t \in \{5, 6, 7, 8, 9, 10, 11, 13\}$. There exists a super-simple (5, 3)-GDD of group type $(20)^t$ from Lemmas 3.7 and 4.1. Since there exists a super-simple $(20 + 1, 5, 3)$-BIBD (from Lemma 3.1), applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(20t + 1, 5, 3)$-BIBD. □

Lemma 4.3. *There exists a super-simple* $(v, 5, 3)$*-BIBD for any* $v \in M = [301, 481]^1_{20} \setminus \{381, 461\}$.

Proof. For $v \in \{301, 321, 401, 421\}$, we can write $v = 20t + 1$, where $t \in \{15, 16, 20, 21\}$. Starting from a 5-GDD of group type 4^t (from Lemma 2.5) and applying Construction 2.1 with a super-simple (5, 3)-GDD of type 5^5 (from Corollary 2.9), we obtain a super-simple (5, 3)-GDD of group type $(20)^t$. Since there exists a super-simple $(20 + 1, 5, 3)$-BIBD from 3.1, applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(v, 5, 3)$-BIBD.

For $v = 341, 361$, starting from a super-simple (5, 3)-GDD of group type 5^{17} or 5^9 (from Lemma 3.5) and applying Construction 2.1 with weight 4 or 8, respectively, we obtain a super-simple (5, 3)-GDD of group type $(20)^{17}$ or $(40)^9$. Here, the input design TD(5, 4) and TD(5, 8) both come from Lemma 2.4. Since there exist a super-simple $(20 + 1, 5, 3)$-BIBD and a super-simple $(40 + 1, 5, 3)$-BIBD from Lemma 3.1, applying Construction 2.2 with $\eta = 1$, we obtain a super-simple (341, 5, 3)-BIBD and a super-simple (361, 5, 3)-BIBD.

For $v = 441, 481$, starting from a super-simple (5, 3)-GDD of group type 8^{11} or 4^6 (from Lemmas 3.6 or 3.4), respectively, and applying Construction 2.1 with a weight 5 or 20, respectively, we obtain a super-simple (5, 3)-GDD of group type $(40)^{11}$ or $(80)^6$. Here, the input designs TD(5, 5) and TD(5, 20) both come from Lemma 2.4. Since there exist a super-simple $(40 + 1, 5, 3)$-BIBD and a super-simple $(80 + 1, 5, 3)$-BIBD (from Lemma 3.1), applying Construction 2.2 with $\eta = 1$, we obtain a super-simple (441, 5, 3)-BIBD and a super-simple (481, 5, 3)-BIBD. □

Lemma 4.4. *There exists a super-simple* $(v, 5, 3)$*-BIBD for any* $v \in M = [501, 601]^1_{20}$.

Proof. By removing $5 - a$ points from the last group of a TD(6, 5), we get a {5, 6}-GDD of group type 5^5a^1, where $a \in [0, 5]$. Starting from this GDD and applying Construction 2.1 with weight 20, we obtain a super-simple (5, 3)-GDD of group type $(100)^5(20a)^1$. Here, the input super-simple (5, 3)-GDD of group types $(20)^5$ and $(20)^6$ both come from Lemma 4.1. Since there exist a super-simple $(100 + 1, 5, 3)$-BIBD from Lemma 4.2 and a super-simple $(20a + 1, 5, 3)$-BIBD from Lemma 3.1, applying Construction 2.2 we obtain a super-simple $(500 + 20a + 1, 5, 3)$-BIBD. It is easy to calculate that for each $v \in M$, we can write $v = 500 + 20a + 1$, where $a \in [0, 5]$. □

Lemma 4.5. *There exists a super-simple* $(v, 5, 3)$*-BIBD for any* $v \in \{621, 641, 661, 681\}$.

Proof. For $v = 621$ or 661, starting from a super-simple (5, 3)-GDD of group type 4^{31} or 4^{11} (from Lemma 3.4) and applying Construction 2.1 with weight 5 or 15 respectively, we get a super-simple (5, 3)-GDD of group type $(20)^{31}$ or $(60)^{11}$ respectively. Here, the used TD(5, 5) and TD(5, 15) both come from Lemma 2.4. Since there exist a super-simple $(20+1, 5, 5)$-BIBD and a super-simple $(60+1, 5, 3)$-BIBD (from Lemma 3.1), a super-simple (621, 5, 5)-BIBD and a super-simple (661, 5, 3)-BIBD are obtained by Construction 2.2.

For $v = 641$, starting from a super-simple (5, 3)-GDD of group type $(20)^8$ from Lemma 3.7 and applying Construction 2.1 with a TD(5, 4), we get a super-simple (5, 3)-GDD of group type $(80)^8$. Since there exists a super-simple $(80+1, 5, 5)$-BIBD from Lemma 3.1, a super-simple (641, 5, 5)-BIBD is obtained by Construction 2.2.

For $v = 681$, starting from a super-simple (5, 3)-GDD of group type 5^{17} (from Lemma 3.5) and applying Construction 2.1 with a TD(5, 8), we get a super-simple (5, 3)-GDD of group type $(40)^{17}$. Since there exists a super-simple $(40+1, 5, 5)$-BIBD from Lemma 3.1, applying Construction 2.2, we obtain a super-simple (681, 5, 5)-BIBD. □

Theorem 4.6. *There exists a super-simple* $(v, 5, 3)$*-BIBD for any* $v \equiv 1 \pmod{20}$ *and* $v \geq 21$.

Proof. For $v \in [21, 681]^1_{20}$, the desired designs are given in Lemmas 3.1, 3.3 and 4.2–4.5.

For $v \geq 700$ and $v \equiv 1 \pmod{20}$, we can write $v = 20u + 1$, where $u \geq 35$. Starting from a $(u, \{5, 6, 7, 8, 9\}, 1)$-PBD (from Lemma 2.6), applying Construction 2.1 with weight 20, we obtain a super-simple (5, 3)-GDD of type $(20)^u$. Here, the input super-simple (5, 3)-GDDs of types $(20)^t$ with $t \in \{5, 6, 7, 8, 9\}$ come from Lemmas 3.7 and 4.1. Since there exists a super-simple $(20 + 1, 5, 3)$-BIBD, a super-simple $(v, 5, 3)$-BIBD is obtained by Construction 2.2. □

Case 2: $v \equiv 5 \pmod{20}$

In this case, we shall show that there exists a super-simple $(v, 5, 3)$-BIBD for any $v \equiv 5 \pmod{20}$ and $v \geq 25$ except possibly when $v = 45, 65$.

Lemma 4.7. *There exists a super-simple* $(21, 5, 5, 3)$*-IBIBD.*

Proof. We first construct a special super-simple $(16, 4, 3)$-BIBD over Z_{16}. The blocks are listed below, which are partitioned into 5 classes C_1, C_2, C_3, C_4, C_5 such that each point occurs in exactly 3 times in each class.

Let $\mathcal{H} = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$, $\mathcal{V} = Z_{16} \cup \mathcal{H}$ and $\mathcal{B} = \{B \cup \{\infty_i\} : B \in C_i, i = 1, 2, \ldots, 5\}$. It is readily checked that $(\mathcal{V}, \mathcal{H}, \mathcal{B})$ is a super-simple $(21, 5, 5, 3)$-IBIBD.

C_1: {0, 1, 2, 3}, {0, 4, 8, 12}, {0, 5, 10, 15}, {1, 4, 11, 14}, {1, 5, 9, 13}, {2, 6, 10, 14},
{2, 7, 8, 13}, {3, 7, 11, 15}, {3, 6, 9, 12}, {4, 5, 6, 7}, {8, 9, 10, 11}, {12, 13, 14, 15},
C_2: {0, 1, 4, 5}, {0, 2, 8, 10}, {0, 3, 12, 15}, {1, 2, 13, 14}, {1, 3, 9, 11}, {2, 3, 6, 7},
{4, 6, 12, 14}, {4, 7, 8, 11}, {5, 6, 9, 10}, {5, 7, 13, 15}, {8, 9, 12, 13}, {10, 11, 14, 15},
C_3: {0, 1, 6, 7}, {0, 2, 9, 11}, {0, 3, 13, 14}, {1, 2, 12, 15}, {1, 3, 8, 10}, {2, 3, 4, 5},
{4, 6, 13, 15}, {4, 7, 9, 10}, {5, 6, 8, 11}, {5, 7, 12, 14}, {8, 9, 14, 15}, {10, 11, 12, 13},
C_4: {0, 4, 9, 13}, {0, 6, 8, 14}, {0, 7, 11, 12}, {1, 5, 8, 12}, {1, 6, 10, 13}, {1, 7, 9, 15},
{2, 4, 10, 12}, {2, 5, 9, 14}, {2, 6, 11, 15}, {3, 4, 8, 15}, {3, 5, 11, 13}, {3, 7, 10, 14},
C_5: {0, 5, 11, 14}, {0, 6, 9, 15}, {0, 7, 10, 13}, {1, 4, 10, 15}, {1, 6, 11, 12}, {1, 7, 8, 14},
{2, 4, 11, 13}, {2, 5, 8, 15}, {2, 7, 9, 12}, {3, 4, 9, 14}, {3, 5, 10, 12}, {3, 6, 8, 13}. □

Lemma 4.8. *There exists a super-simple $(v, 5, 3)$-BIBD for $v \in M = \{85, 165, 245, 485, 565, 645, 965\}$.*

Proof. For $v = 85$, using a super-simple $(5, 3)$-GDD of type $(16)^5$ coming from Corollary 2.9, a super-simple $(21, 5, 5, 3)$-IBIBD (from Lemma 4.7) and a super-simple $(21, 5, 3)$-BIBD (from Lemma 3.1), and applying Construction 2.3, we obtain a super-simple $(85, 5, 3)$-BIBD.

For other $v \in M$, we can write $v = 16t + 5$, $t \in \{10, 15, 30, 35, 40, 60\}$. There exists a 5-GDD of type 4^t from Lemma 2.5. Applying Construction 2.1 with a super-simple $(5, 3)$-GDD of type 4^5 from Corollary 2.9, we get a super-simple $(5, 3)$-GDD of type $(16)^t$. Since there exist a super-simple $(21, 5, 5, 3)$-IBIBD (from Lemma 4.7) and a super-simple $(21, 5, 3)$-BIBD (from Lemma 3.1), applying Construction 2.3, we obtain a super-simple $(v, 5, 3)$-BIBD. □

Lemma 4.9. *There exists a super-simple $(v, 5, 3)$-BIBD for $v \in M = \{105, 205, 305, 405, 505, 605\}$.*

Proof. For each $v \in M$, we can write $v = 5g$, $g \in \{21, 41, 61, 81, 101, 121\}$. Using a super-simple $(5, 3)$-GDD of type g^5 (from Corollary 2.9) and a super-simple $(g, 5, 3)$-BIBD (from Theorem 4.6), and applying Construction 2.2 with $\eta = 0$, we get a super-simple $(v, 5, 3)$-BIBD. □

Lemma 4.10. *There exists a super-simple $(v, 5, 3)$-BIBD for $v \in \{125, 145, 225, 265, 765, 865\}$.*

Proof. For $v = 125$, using a super-simple $(5, 3)$-GDD of type $(25)^5$ (from Corollary 2.9) and a super-simple $(25, 5, 3)$-BIBD (from Lemma 3.2), and applying Construction 2.2 with $\eta = 0$, we get a super-simple $(125, 5, 3)$-BIBD.

For $v = 145$. A $\{5, 6\}$-GDD of type 6^6 can be obtain by removing a block from a TD(6, 7) (from Lemma 2.4). Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(24)^6$, here the required super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exists a super-simple $(25, 5, 3)$-BIBD, applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(145, 5, 3)$-BIBD.

For $v \in \{225, 765\}$, we write $v = 45m$, $m \in \{5, 17\}$. Starting from a super-simple $(5, 3)$-GDD of type 5^9 coming from Lemma 3.5, applying Construction 2.1 with a TD$(5, m)$ (from Lemma 2.4), we get a super-simple $(5, 3)$-GDD of type $(5m)^9$. Since there exists a super-simple $(5m, 5, 3)$-BIBD (from Lemmas 3.2 or 4.8), applying Construction 2.2 with $\eta = 0$, a super-simple $(v, 5, 3)$-BIBD is obtained.

For $v = 265$, by Lemma 2.4 there exists a RTD(6, 11). Taking a parallel class as groups we can get a $\{6, 11\}$-GDD of type 6^{11}. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(24)^{11}$, here the required super-simple $(5, 3)$-GDDs of types 4^6 and 4^{11} come from Lemma 3.4. Since there exists a super-simple $(25, 5, 3)$-BIBD, applying Construction 2.2 with $\eta = 1$, a super-simple $(265, 5, 3)$-BIBD is obtained.

For $v = 865$. Using a super-simple $(5, 3)$-GDD of type 4^6 from Lemma 3.4, and applying Construction 2.1 with a TD(5, 36) (from Lemma 2.4), we obtain a super-simple $(5, 3)$-GDD of type $(144)^6$. Since there exists a super-simple $(145, 5, 3)$-BIBD from above, applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(865, 5, 3)$-BIBD. □

Lemma 4.11. *There exists a super-simple $(v, 5, 3)$-BIBD for $v \in M = \{185, 285, 345, 365\}$.*

Proof. For $v = 185$. A $\{5, 6\}$-GDD of type $5^8 6^1$ is given in [30]. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(20)^8(24)^1$, here the required super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exists a super-simple $(21, 5, 3)$-BIBD and a $(25, 5, 3)$-BIBD (from Theorem 4.6 and Lemma 3.2), applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(185, 5, 3)$-BIBD.

For $v = 285$. Let $(\mathcal{V}, \mathcal{B})$ be a resolvable $(65, 5, 1)$-BIBD given in [5], which has 16 parallel classes $P_0, P_1, \ldots, P_{15}$. Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$ and let $\mathcal{V}' = \mathcal{V} \cup X$, $\mathcal{G}' = \{G : G \in P_0 \cup X\}$, $\mathcal{B}' = \{B \cup \{\infty_i\} : B \in P_i, i = 1, 2, \ldots, 6\} \cup \{B \in P_i : i = 7, \ldots, 15\}$. Then $(\mathcal{V}', \mathcal{G}', \mathcal{B}')$ is a $\{5, 6\}$-GDD of type $5^{13}6^1$. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(20)^{13}(24)^1$, where the required super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exist a super-simple $(21, 5, 3)$-BIBD and a super-simple $(25, 5, 3)$-BIBD, applying Construction 2.2 with $\eta = 1$, a super-simple $(285, 5, 3)$-BIBD is obtained.

For $v = 365$. Since there exists a resolvable $(85, 5, 1)$-BIBD coming from [5], using a similar method of the case of $v = 285$, we can get a super-simple $(365, 5, 3)$-BIBD.

For $v = 345$. By Lemma 2.4, there exists a RTD$(5, 16)(\mathcal{V}, \mathcal{G}, \mathcal{B})$ with 16 parallel classes $P_0, P_1, \ldots, P_{15}$. Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$ and let $\mathcal{V}' = \mathcal{V} \cup X$, $\mathcal{G}' = \{G : G \in P_0 \cup X\}$, $\mathcal{B}' = \{B \cup \{\infty_i\} : B \in P_i, i = 1, 2, \ldots, 6\} \cup \{B \in P_i : i = 7, \ldots, 15\} \cup \{B \in \mathcal{G}\}$. Then $(\mathcal{V}', \mathcal{G}', \mathcal{B}')$ is a $\{5, 6, 16\}$-GDD of type $5^{16}6^1$. Using Construction 2.1 with weight 4 we get a super-simple $(5, 3)$-GDD of type $(20)^{16}(24)^1$, here the input designs of super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exist a super-simple $(21, 5, 3)$-BIBD and a super-simple $(25, 5, 3)$-BIBD, applying Construction 2.2 with $\eta = 1$, a super-simple $(345, 5, 3)$-BIBD is obtained. □

Lemma 4.12. *There exists a super-simple $(v, 5, 3)$-BIBD for $v \in M = \{325, 425, 525, 625, 725, 825, 925\}$.*

Proof. For each $v \in M$, we can write $v = 20n + 25$, where $n \in \{15, 20, 25, 30, 35, 40, 45\}$. By removing $n - 6$ points from the last group of a TD$(6, n)$ coming from 2.4 we obtain a $\{5, 6\}$-GDD of type $n^{5}6^1$. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(4n)^5(24)^1$, here the input super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exist a super-simple $(4n + 1, 5, 3)$-BIBD and a super-simple $(25, 5, 3)$-BIBD (from Theorem 4.6 and Lemma 3.2 respectively), applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(v, 5, 3)$-BIBD. □

Lemma 4.13. *There exists a super-simple $(v, 5, 3)$-BIBD for $v \in \{385, 465, 665\}$.*

Proof. For $v = 385$. By Lemma 2.4, there exists a RTD$(6, 16)$, taking a parallel class as groups we can get a $\{6, 16\}$-GDD of type 6^{16}. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(24)^{16}$, here the input super-simple $(5, 3)$-GDDs of types 4^6 and 4^{16} come from Lemma 3.4. Since there exists a super-simple $(25, 5, 3)$-BIBD from Lemma 3.2, applying Construction 2.2 with $\eta = 1$, a super-simple $(385, 5, 3)$-BIBD is obtained.

For $v = 465$. Deleting 10 points from the last group of a RTD$(6, 21)$ (from Lemma 2.4) and taking a parallel class as groups, we can get a $\{5, 6, 11, 21\}$-GDD of type $6^{11}5^{10}$. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(24)^{11}(20)^{10}$, here the input super-simple $(5, 3)$-GDDs of type 4^s, $s \in \{5, 6, 11, 21\}$, come from Corollary 2.9 and Lemma 3.4. Since there exist a super-simple $(21, 5, 3)$-BIBD and a super-simple $(25, 5, 3)$-BIBD (from Theorem 4.6 and Lemma 3.2 respectively), applying Construction 2.2 with $\eta = 1$, a super-simple $(465, 5, 3)$-BIBD is obtained.

For $v = 665$. Deleting 20 points from the last group of a RTD$(6, 31)$ (from Lemma 2.4) and taking a parallel class as groups, we obtain a $\{5, 6, 11, 31\}$-GDD of type $6^{11}5^{20}$. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(24)^{11}(20)^{20}$, here the input super-simple $(5, 3)$-GDDs of type 4^w, $w \in \{5, 6, 11, 31\}$, come from Corollary 2.9 and Lemma 3.4. Since there exist a super-simple $(21, 5, 3)$-BIBD and a super-simple $(25, 5, 3)$-BIBD, applying Construction 2.2 with $\eta = 1$, a super-simple $(665, 5, 3)$-BIBD is obtained. □

Lemma 4.14. *There exists a super-simple $(v, 5, 3)$-BIBD for $v \in M = \{445, 545, 745, 845\}$.*

Proof. For each $v \in M$, we can write $v = 20m+24+1$, where $m \in \{21, 26, 36, 41\}$. Delete $m-6$ points from the last group of a TD$(6, m)$, we get a $\{5, 6\}$-GDD of type $m^{5}6^1$. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(4m)^5(24)^1$, here the input super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exist a super-simple $(25, 5, 3)$-BIBD and a super-simple $(4m + 1, 5, 3)$-BIBD from Lemmas 3.2, 4.8–4.10, applying Construction 2.2 with $\eta = 1$, a super-simple $(v, 5, 3)$-BIBD is obtained. □

Lemma 4.15. *There exists a super-simple $(v, 5, 3)$-BIBD for any $v \in \{705, 805, 905\}$.*

Proof. For each $v \in \{705, 805, 905\}$, we can write $v = 20n + 105$, where $n \in \{30, 35, 40\}$. By removing $n - 26$ points from the last group of a TD$(6, n)$ coming from 2.4, we obtain a $\{5, 6\}$-GDD of type $n^5(26)^1$. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(4n)^5(104)^1$, here the input super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exist a super-simple $(4n + 1, 5, 3)$-BIBD and a super-simple $(105, 5, 3)$-BIBD coming from Theorem 4.6 and Lemma 4.9, respectively, applying Construction 2.2 with $\eta = 1$ we get a super-simple $(v, 5, 3)$-BIBD. □

Lemma 4.16. *There exists a super-simple $(945, 5, 3)$-BIBD.*

Proof. We first obtain a $\{5, 6\}$-GDD of type $(40)^5(36)^1$ by removing 4 points from the last group of a TD$(6, 40)$. Applying Construction 2.1 with 4, we get a super-simple $(5, 3)$-GDD of type $(160)^5(144)^1$, here the input super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exists a super-simple $(161, 5, 3)$-BIBD and a super-simple $(145, 5, 3)$-BIBD (from Theorem 4.6 and Lemma 4.10), applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(945, 5, 3)$-BIBD. □

Lemma 4.17. *There exists a super-simple $(v, 5, 3)$-BIBD for $v \in M = \{585, 685, 785, 885\}$.*

Proof. For each $v \in M$, we can write $v = 20n + 85$, where $n \in \{25, 30, 35, 40\}$. We obtain a $\{5, 6\}$-GDD of type $n^5(21)^1$ by removing $n - 21$ points from the last group of a TD$(6, n)$. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(4n)^5(84)^1$, here the input super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exist a super-simple $(4n + 1, 5, 3)$-BIBD and a super-simple $(85, 5, 3)$-BIBD (from Theorem 4.6 and Lemma 4.8 respectively), applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(v, 5, 3)$-BIBD. □

Theorem 4.18. *There exists a super-simple* $(v, 5, 3)$*-BIBD for* $v \equiv 5 \pmod{20}$ *and* $v \geq 25$ *except possibly when* $v = 45, 65$.

Proof. For $v \in [25, 965]_{20}^{5} \setminus \{45, 65\}$, the associated super-simple designs are given in Lemma 3.2 and Lemmas 4.8–4.17.

For $v \equiv 5 \pmod{20}$ and $v \geq 985$, it can be written as $v = 20n + u + 1$, where $n \geq 45, n \equiv 0 \pmod 5$ and $u + 1 \in \{85, 105, 125, 145, 165\}$. By removing $n - u/4$ points from the last group of a TD$(6, n)$, we can obtain a $\{5, 6\}$-GDD of type $n^5(u/4)^1$. Applying Construction 2.1 with weight 4, we get a super-simple $(5, 3)$-GDD of type $(4n)^5u^1$, here the input super-simple $(5, 3)$-GDDs of types 4^5 and 4^6 come from Corollary 2.9 and Lemma 3.4 respectively. Since there exist a super-simple $(4n+1, 5, 3)$-BIBD and a super-simple $(u+1, 5, 3)$-BIBD (from Theorem 4.6 and Lemmas 4.8–4.10), applying Construction 2.2 with $\eta = 1$, we obtain a super-simple $(v, 5, 3)$-BIBD. □

The proof of Theorem 1.2. The proof follows directly from Theorems 4.6 and 4.18. □

References

[1] R.J.R. Abel, F.E. Bennett, Super-simple steiner pentagon systems, Discrete Math. 156 (2008) 780–793.
[2] R.J.R. Abel, F.E. Bennett, G. Ge, Super-simple holey steiner pentagon systems and related designs, J. Combin. Des. 16 (2008) 301–328.
[3] R.J.R. Abel, C.J. Colbourn, J.H. Dinitz, Latin squares, in: C.J. Colbourn, J.H. Dinitz (Eds.), Handbook of Combinatorial Designs, second ed., Chapman & Hall/CRC, 2007, pp. 135–228.
[4] R.J.R. Abel, C.J. Colbourn, J.H. Dinitz, Pairwise balanced designs, in: C.J. Colbourn, J.H. Dinitz (Eds.), Handbook of Combinatorial Designs, second ed., Chapman & Hall/CRC, 2007, pp. 229–270.
[5] R.J.R. Abel, M. Greig, Some new RBIBDs with block size 5 and PBDs with block sizes $\equiv 1$ mod 5, Australas. J. Combin. 15 (1997) 177–202.
[6] P. Adams, D. Bryant, A. Khodkar, On the existence of super-simple designs with block size 4, Aequationes Math. 52 (1996) 230–246.
[7] T.L. Alderson, Keith E. Mellinger, 2-dimensional optical orthogonal codes from singer groups, Discrete Appl. Math. 157 (2009) 3008–3019.
[8] I. Bluskov, New designs, J. Combin. Math. Combin. Comput. 23 (1997) 212–220.
[9] I. Bluskov, H. Hamalainen, New upper bounds on the minimum size of covering designs, J. Combin. Des. 6 (1998) 21–41.
[10] I. Bluskov, K. Heinrich, Super-simple designs with $v \leq 32$, J. Statist. Plann. Inference 95 (2001) 121–131.
[11] K.A. Bush, Orthogonal arrays of index unity, Ann. Math. Stat. 23 (1952) 426–434.
[12] H. Cao, K. Chen, R. Wei, Super-simple balanced incomplete block designs with block size 4 and index 5, Discrete Math. 309 (2009) 2808–2814.
[13] K. Chen, On the existence of super-simple $(v, 4, 3)$-BIBDs, J. Combin. Math. Combin. Comput. 17 (1995) 149–159.
[14] K. Chen, On the existence of super-simple $(v, 4, 4)$-BIBDs, J. Statist. Plann. Inference 51 (1996) 339–350.
[15] K. Chen, Z. Cao, R. Wei, Super-simple balanced incomplete block designs with block size 4 and index 6, J. Statist. Plann. Inference 133 (2005) 537–554.
[16] K. Chen, Y.G. Sun, Y. Zhang, Super-simple balanced incomplete block designs with block size 4 and index 8, Util. Math. (2013) in press.
[17] K. Chen, R. Wei, Super-simple $(v, 5, 5)$ designs, Des. Codes Cryptogr. 39 (2006) 173–187.
[18] K. Chen, R. Wei, Super-simple $(v, 5, 4)$ designs, Discrete Appl. Math. 155 (2007) 904–913.
[19] F.R.K. Chung, J.A. Salehi, V.K. Wei, Optical orthogonal codes: design, analysis and applications, IEEE Trans. Inform. Theory 35 (1989) 595–604.
[20] H.-D.O.F. Gronau, Super-simple designs, in: C.J. Colbourn, J.H. Dinitz (Eds.), Handbook of Combinatorial Designs, second ed., Chapman & Hall/CRC, 2007, pp. 633–635.
[21] H.-D.O.F. Gronau, D.L. Kreher, A.C.H. Ling, Super-simple $(v, 5, 2)$ designs, Discrete Appl. Math. 138 (2004) 65–77.
[22] H.-D.O.F. Gronau, R.C. Mullin, On super-simple 2-$(v, 4, \lambda)$ designs, J. Combin. Math. Combin. Comput. 11 (1992) 113–121.
[23] S. Hartmann, Asymptotic results on suborthogonal $\overline{G}$-decompositions of complete di-graphs, Discrete Appl. Math. 95 (1999) 311–320.
[24] S. Hartmann, On simple and super-simple transversal designs, J. Combin. Des. 8 (2000) 311–320.
[25] S. Hartmann, Superpure digraph designs, J. Combin. Des. 10 (2000) 239–255.
[26] S. Hartmann, U. Schumacher, Suborthogonal double covers of complete graphs, Congr. Numer. 147 (2000) 33–40.
[27] L. Ji, J. Yin, Constructions of new orthogonal arrays and covering arrays of strength three, J. Combin. Theory Ser. A 117 (2010) 236–247.
[28] A. Khodkar, Various super-simple designs with block size four, Australas. J. Combin. 9 (1994) 201–210.
[29] H.K. Kim, V. Lebedev, On optimal superimposed codes, J. Combin. Des. 12 (2004) 79–91.
[30] A.C.H. Ling, Pairwise balanced designs and related codes, Ph.D. Thesis, University of Waterloo, 1996.
[31] A.C.H. Ling, X.J. Zhu, C.J. Colbourn, R.C. Mullin, Pairwise balanced designs with consecutive block sizes, Des. Codes Cryptogr. 10 (1997) 203–222.
[32] D.R. Stinson, R. Wei, L. Zhu, New constructions for perfect hash families and related structures using related combinatorial designs and codes, J. Combin. Des. 8 (2000) 189–200.
[33] G.C. Yang, W.C. Kwong, Performance comparison of multiwavelength CDMA and WDMA+CDMA for fiber-optic networks, IEEE Trans. Commun. 45 (1997) 1426–1434.
[34] Y. Zhang, K. Chen, Y. Sun, Super-simple balanced incomplete block designs with block size 4 and index 9, J. Statist. Plann. Inference 139 (2009) 3612–3624.