Physics Letters A 372 (2008) 6551–6559

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

New embedded pairs of explicit Runge–Kutta methods with FSAL properties adapted to the numerical integration of oscillatory problems

Yonglei Fang [a], Yongzhong Song [a], Xinyuan Wu [b,*]

[a] School of Mathematics and Computer Science, Nanjing Normal University, Nanjing 210097, PR China
[b] State Key Laboratory for Novel Software Technology at Nanjing University, Department of Mathematics, Nanjing University, Nanjing 210093, PR China

ARTICLE INFO

Article history:
Received 5 December 2007
Received in revised form 9 July 2008
Accepted 5 September 2008
Available online 10 September 2008
Communicated by R. Wu

MSC:
65L05
65L06

Keywords:
Embedded pairs
Adapted RK methods
Oscillatory problems
FSAL technique
Absolute stability

ABSTRACT

A higher order Runge–Kutta (pair) method specially adapted to the numerical integration of IVPs with oscillatory solutions is presented. This method is based on the adapted methods proposed by Franco (see Ref. [J.M. Franco, Appl. Numer. Math. 50 (2004) 427]). We give explicit method (up to order 5) as well as pairs of embedded Runge–Kutta methods of order 5 and 4 designed using the FSAL properties. The stability of the new methods is analyzed. The numerical experiments are carried out to show the efficiency and robustness of our methods in comparison with some efficient methods.

1. Introduction

In this Letter we are concerned with the numerical integration of oscillatory problems modeled by initial value problems of the form

$$y' = f(x, y), \qquad y(x_0) = y_0, \quad x \in [x_0, X], \tag{1}$$

where whose solution exhibits a pronounced oscillatory character. This kinds of problems often arise in many fields of applied science such as mechanics, astrophysics, quantum chemistry, electronics. They can be integrated with general purpose methods or using other codes adapted to the special structure of the problem. In [1], Bettis derived explicit Runge–Kutta-type algorithm with 3 and 4 stages for the integration of ODEs with oscillatory solutions. Basing on the rooted trees and B-series theory, furthermore, Franco gave the necessary and sufficient order conditions for this class of Runge–Kutta methods, and constructed explicit methods (up to order 4) as well as the pairs of embedded Runge–Kutta methods of orders 4 and 3 (see Ref. [2]). This motivates some new and robust methods with higher order.

In this Letter, therefore, we present a new Runge–Kutta method of order 5 as well as pairs of embedded Runge–Kutta methods of orders 5 and 4 adapted to the numerical integration of oscillatory problems designed using the FSAL technique (the last evaluation at any step is the same as the first evaluation at the next step). The numerical experiments are reported.

2. Preliminaries and some basic results

In [2], Franco constructed a formula of the Adapted RK methods for solving oscillatory ODEs. This method can be introduced simply as follows.

Applying an s-stage explicit RK method (A, b, c) to the test differential equation

$$y' = i\omega y, \quad \omega \in \mathbb{R}, \tag{2}$$

* Corresponding author.
E-mail addresses: ylfangmath@gmail.com (Y. Fang), yzsong@njnu.edu.cn (Y. Song), xywu@nju.edu.cn (X. Wu).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.09.014

gives

$$y_{n+1} = R(i\omega h) y_n, \tag{3}$$

where

$$R(z) = 1 + zb^T (I - zA)^{-1} e = 1 + zb^T \big(I + zA + z^2 A^2 + \cdots\big) e = 1 + zb^T e + z^2 b^T Ae + z^3 b^T A^2 e + \cdots + z^s b^T A^{s-1} e. \tag{4}$$

Since $y(x) = e^{i\omega x} y_0$ is the exact solution of (2), we have

$$y_{n+1} = e^{i\omega h} y_n. \tag{5}$$

If the explicit RK method is of order p, then $R(z) = e^z + \mathcal{O}(z^{p+1})$, thus we have

$$b^T A^j e = \frac{1}{(j+1)!}, \quad j = 0, 1, \ldots, p-1. \tag{6}$$

Theorem 1. *(See Ref. [2].) The necessary and sufficient conditions for an adapted RK method to be of order p are given by*

$$b^T(v)\Phi(\tau) - \frac{1}{\gamma(\tau)} = \mathcal{O}\big(v^{p+1-\rho(\tau)}\big) = \mathcal{O}\big(h^{p+1-\rho(\tau)}\big), \quad \rho(\tau) = 1, 2, \ldots, p, \tag{7}$$

where $v = h\omega$, τ expresses a rooted tree and the functions $\rho(\tau)$, $\alpha(\tau)$, $\gamma(\tau)$ and $\Phi(\tau)$ are defined in Refs. [3,4].

Theorem 2. *(See Ref. [2].) If we replace the order conditions that correspond to the high trees*

$$b(v)^T A^j e - \frac{1}{(j+1)!} = \mathcal{O}\big(v^{p-j}\big), \quad j = 0, 1, \ldots, p-1, \tag{8}$$

with the following conditions

$$b^T A^j e = \frac{1}{(j+1)!}, \quad j = 0, 1, \ldots, p-3, \qquad b^T A^{p-2} e = \phi_{p-1,s}(v), \qquad b^T A^{p-1} e = \phi_{p,s}(v), \tag{9}$$

where $\phi_{p-1,s}(v)$ and $\phi_{p,s}(v)$ are defined by

$$\phi_{j,s}(v) = \begin{cases} \phi_j(v), & \text{if } s = j \text{ or } j+1, \\ \phi_j(v) + v^2(b^T A^{j+1} e) + \cdots + (-1)^\beta v^{2\beta+2}(b^T A^{j+2\beta+1}), & \text{if } s \geqslant j+2, \end{cases} \tag{10}$$

with $\beta = [\frac{s-j-2}{2}]$, and the ϕ-functions are defined in Ref. [1], then the resulting Adapted RK method is of order p.

The ϕ-functions defined in [1] have the following recurrence relation:

$$\phi_0(v) = \cos(v), \qquad \phi_1(v) = \sin(v)/v, \qquad \phi_{j+2}(v) = \left(\frac{1}{j!} - \phi_j(v)\right)\Big/ v^2, \quad j \geqslant 0.$$

In order to avoid the possible round-off error (when $|v|$ is small), the ϕ-functions can also be replaced by the Taylor expansions

$$\phi_j(v) = \sum_{k=0}^{\infty} (-1)^k \frac{v^{2k}}{(2k+j)!}, \quad j \geqslant 0.$$

3. Construction of a higher order explicit adapted RK methods using the FSAL technique

In [2], Franco constructed a formula of the Adapted RK methods for solving ODEs with oscillating solutions. In a slightly different way, we will construct a higher order adapted RK method (up to order 5) as well as pairs of embedded Runge–Kutta methods of orders 5 and 4 based on the Dormand–Prince 5(4) in [5] designed using the FSAL technique.

3.1. A fifth-order adapted RK method

We consider the six-stages explicit RK method displayed by the following Butcher-tableau:

$$\begin{array}{c|cccccc}
0 & 0 & & & & & \\
1/5 & 1/5 & 0 & & & & \\
3/10 & 3/40 & 9/40 & 0 & & & \\
4/5 & 44/45 & -56/15 & 32/9 & 0 & & \\
8/9 & 19372/6561 & -25360/2187 & 64448/6561 & -212/729 & 0 & \\
1 & 9017/3168 & -355/33 & 46732/5247 & 49/176 & -5103/18656 & 0 \\
\hline
 & b_1(v) & b_2(v) & b_3(v) & b_4(v) & b_5(v) & b_6(v)
\end{array}$$

The entries of A and C are chosen from the well known five-order RK method derived by Dormand–Prince in Ref. [5]. Because the entries of A and C are available, we need only to find the weights of the adapted RK method from the order conditions that correspond to homogeneous linear problems (see [2]). With stage $s = 6$ and order $p = 5$ it follows that

$$b^T e = 1, \qquad b^T Ae = 1/2, \qquad b^T A^2 e = 1/6, \qquad b^T A^3 e = \phi_4(v) + v^2\big(b^T A^5 e\big), \qquad b^T A^4 e = \phi_5(v),$$

in which $v = \omega h$, and the ϕ-functions are defined in [11].

Solving these equations with the choice of $b^T c^2 = 1/3$ yields

$$\begin{cases} b_1(v) = (v^2(14 + 675\phi_5(v)) + 10(-23 + 390\phi_4(v) + 1440\phi_5(v)))/(144(4 + v^2)), \\ b_2(v) = 0, \\ b_3(v) = -(28v^2(-53 + 1350\phi_5(v)) + 100(-205 + 1986\phi_4(v) + 7470\phi_5(v)))/(3339(4 + v^2)), \\ b_4(v) = (300(2\phi_4(v) + 15\phi_5(v)) + v^2(11 + 675\phi_5(v)))/(24(4 + v^2)), \\ b_5(v) = -(243(22 - 300\phi_4(v) + 75(-8 + v^2)\phi_5(v)))/(848(4 + v^2)), \\ b_6(v) = -11(-11 + 150\phi_4(v) + 450\phi_5(v))/(21(4 + v^2)). \end{cases} \tag{11}$$

As stated above we have derived an adapted RK method of order fifth, and we denote it as ARK5.

3.2. A new 5(4) pair adapted RK method

Now we are interested in constructing an embedded pair of explicit adapted RK methods based on the fifth-order adapted RK method presented above. Consequently, we consider the following Butcher-tableau:

$$\begin{array}{c|ccccccc} 0 & 0 & & & & & & \\ 1/5 & 1/5 & 0 & & & & & \\ 3/10 & 3/40 & 9/40 & 0 & & & & \\ 4/5 & 44/45 & -56/15 & 32/9 & 0 & & & \\ 8/9 & 19\,372/6561 & -25\,360/2187 & 64\,448/6561 & -212/729 & 0 & & \\ 1 & 9017/3168 & -355/33 & 46\,732/5247 & 49/176 & -5103/18\,656 & 0 & \\ 1 & b_1(v) & b_2(v) & b_3(v) & b_4(v) & b_5(v) & b_6(v) & 0 \\ \hline & b_1(v) & b_2(v) & b_3(v) & b_4(v) & b_5(v) & b_6(v) & 0 \\ \hline & b_1^*(v) & b_2^*(v) & b_3^*(v) & b_4^*(v) & b_5^*(v) & b_6^*(v) & b_7^*(v) \end{array}$$

where the entries $b_i(v)$ $(i = 1, 2, \ldots, 6)$ of b are given by (11). We observe that in the tableau the entries of A from the last row are the same as the weights in the fifth-order method, therefore during the variable step-size program the last evaluation of f in a current step can be re-used for the first evaluation in the following step, that is to say, the FSAL technique can be used. Therefore, with $s = 7$ and $p = 4$ we have

$$b^{*T} e = 1, \qquad b^{*T} Ae = 1/2, \qquad b^{*T} A^2 e = \phi_3(v) + v^2 b^{*T} A^4 e - v^4 b^{*T} A^6 e, \qquad b^{*T} A^3 e = \phi_4(v) + v^2\big(b^{*T} A^5 e\big),$$

in which $v = \omega h$, and the ϕ-functions are defined in [11].

Solving these equations with the choice of $b_5^*(v) = -92\,097/339\,200$, $b_6^*(v) = 187/2100$, $b_7^*(v) = 1/40$, we obtain the following weights

$$b_1^*(v) = \big(-279\,463 + 1\,920\,000\phi_3(v) - 600\,000\phi_4(v) + 8v^2\big(1859 + 6000\phi_4(v) - 1875\phi_5(v)\big)\big)/172\,800,$$
$$b_2^*(v) = \big(-625\big(-11 + 96\phi_3(v) - 120\phi_4(v)\big) + v^2\big(-401 - 1500\phi_4(v) + 1875\phi_5(v)\big)\big)/1800,$$
$$b_3^*(v) = \big(-140\,074 + 12\,985v^2 + 2\,226\,000\phi_3(v) - 4\,452\,000\phi_4(v) + v^2\big(55\,650\phi_4(v) - 111\,300\phi_5(v)\big)\big)/100\,170,$$
$$b_4^*(v) = \big(3395 + 60\,000\phi_4(v) + 4v^2\big(17 + 375\phi_5(v)\big)\big)/9600,$$
$$b_5^*(v) = -92\,097/339\,200, \qquad b_6^*(v) = 187/2100, \qquad b_7^*(v) = 1/40.$$

As stated above, we propose an improvement on the classical Runge–Kutta (pair) algorithm basing on the classical Dormand–Prince 5(4) pair in Ref. [3] such that the oscillatory linear problem is solved without truncation errors. And this modification affects only the weights which will be frequency dependent. We denote this fourth order ARK method as ARK(4).

4. Stability analysis

Let us consider the stability analysis of RK methods with v-dependent coefficients. The original definition is due to Coleman and Ixaru [6] about the periodicity of exponentially-fitted symmetric methods for $y'' = f(x, y)$ which is re-considered by Van de Vyver [7] for RK methods.

The stability regions of the new methods can be analyzed by applying them to the test equation

$$y' = i\lambda y, \quad \lambda > 0. \tag{12}$$

When an adapted RK method is applied to test Eq. (12) it gives the difference equation

$$y_{n+1} = R(i\theta; v) y_n, \quad \theta = \lambda h, \tag{13}$$

with $R(i\theta, v) = \frac{\det(I - i\theta A + i\theta e b(v)^T)}{\det(I - i\theta A)}$, $e = (1, \ldots, 1)^T$.

Definition 1. (See [7,8].) For an adapted RK method with stability function $R(i\theta, v)$, the quantities

$$P(\theta) = \theta - \arg\big(R(i\theta; v)\big), \qquad D(\theta) = 1 - \big|R(i\theta; v)\big| \tag{14}$$

are called the phase-lag (dispersion) and dissipation, respectively. The method is said to be of phase-lag order q and dissipation order p if

$$P(\theta) = \mathcal{O}\big(\theta^{q+1}\big), \qquad d(\theta) = \mathcal{O}\big(\theta^{p+1}\big).$$

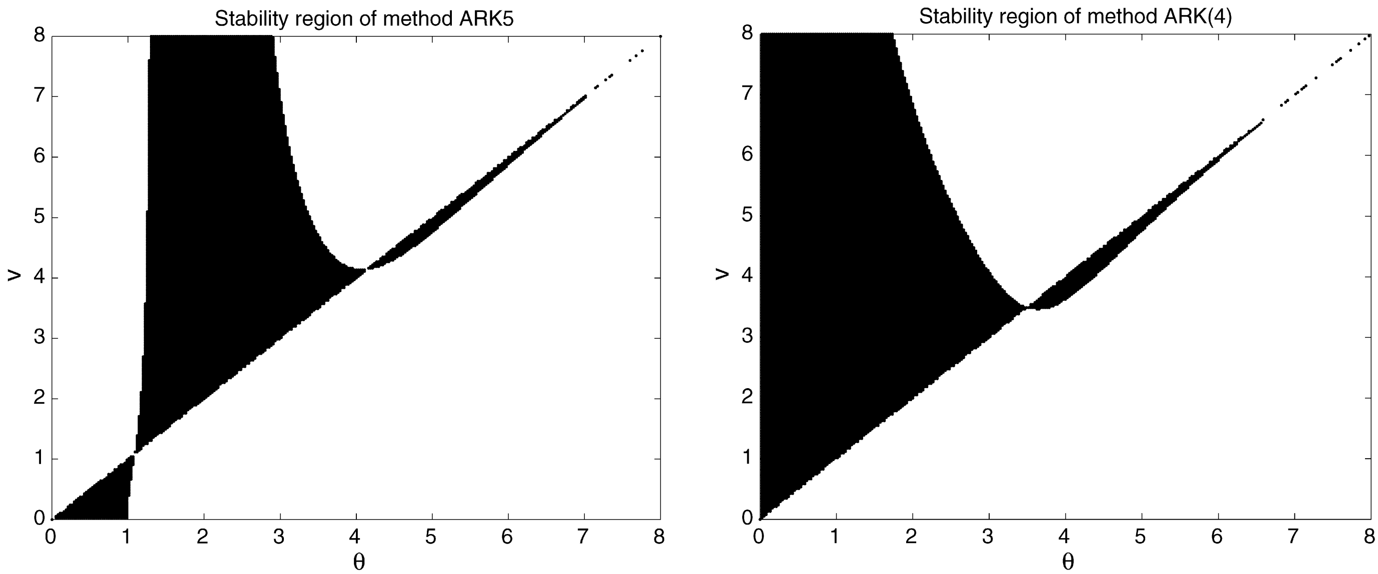


Fig. 1. Regions of imaginary stability (dark parts) of ARK5, ARK(4).

If we set $v = r\theta$ in the stability function $R(i\theta; v)$, the phase-lag and dissipation of the higher order method ARK5 (derived in Section 3.1) are given by

$$P(\theta) = \frac{r^2 - 1}{2100}\theta^7 + \mathcal{O}(\theta^9), \qquad D(\theta) = \frac{1 - r^2}{3600}\theta^6 + \mathcal{O}(\theta^8).$$

Hence, the method is of phase-lag order six and dissipation order five. In the same way, the phase-lag and dissipation of the lower order method ARK(4) (derived in Section 3.2) are determined by

$$P(\theta) = \frac{97(r^2 - 1)}{120\,000}\theta^5 + \mathcal{O}(\theta^7), \qquad D(\theta) = \frac{77(r^2 - 1)}{90\,000}\theta^6 + \mathcal{O}(\theta^8).$$

Therefore, the method is of phase-lag order four and dissipation order five.

Definition 2. (See [7,8].) A region Ω in the θ–v plane is said to be the region of imaginary stability for an adapted RK method with stability function $R(i\theta; v)$, if $|R(i\theta; v)| \leqslant 1$, for any $(\theta, v) \in \Omega$. And any closed curve defined by $|R(i\theta; v)| = 1$ is a stability boundary.

In Fig. 1 we plot the stability regions of the two methods derived in Sections 3.1 and 3.2, respectively.

5. Numerical experiments

In this section, we will compare the numerical performance of the new methods with the Franco's methods and other efficient methods. The criterion used in the numerical comparisons is the decimal logarithm of the maximum global error versus the computational effort measured by the number of function evaluations (in logarithmic scale) required by each method.

5.1. Comparisons with fixed step-size

The methods used in the comparisons are denoted by:

- ARK5: The fifth-order ARK method obtained in Section 3.1.
- ARK4: The fourth-order ARK method obtained in [2].
- ARK3/8: The fourth-order ARK method obtained in [2].
- EFRK4: The exponentially fitted Runge–Kutta method with order fourth derived in [9].
- SRK4: An exponentially fitted Runge–Kutta method with order fourth given in [10].

Problem 1. We consider the linear test problem used in [2]

$$y'' + \omega^2 y = (\omega^2 - 1)\sin(t), \quad t \in [0, t_{\mathrm{end}}], \qquad y(0) = 1, \qquad y'(0) = \omega + 1,$$

whose analytic solution is given by

$$y(t) = \cos(\omega t) + \sin(\omega t) + \sin(t).$$

In our test we choose the parameters values $\omega = 5$, $t_{\mathrm{end}} = 100$. The numerical results are showed in Fig. 2 (on the left).

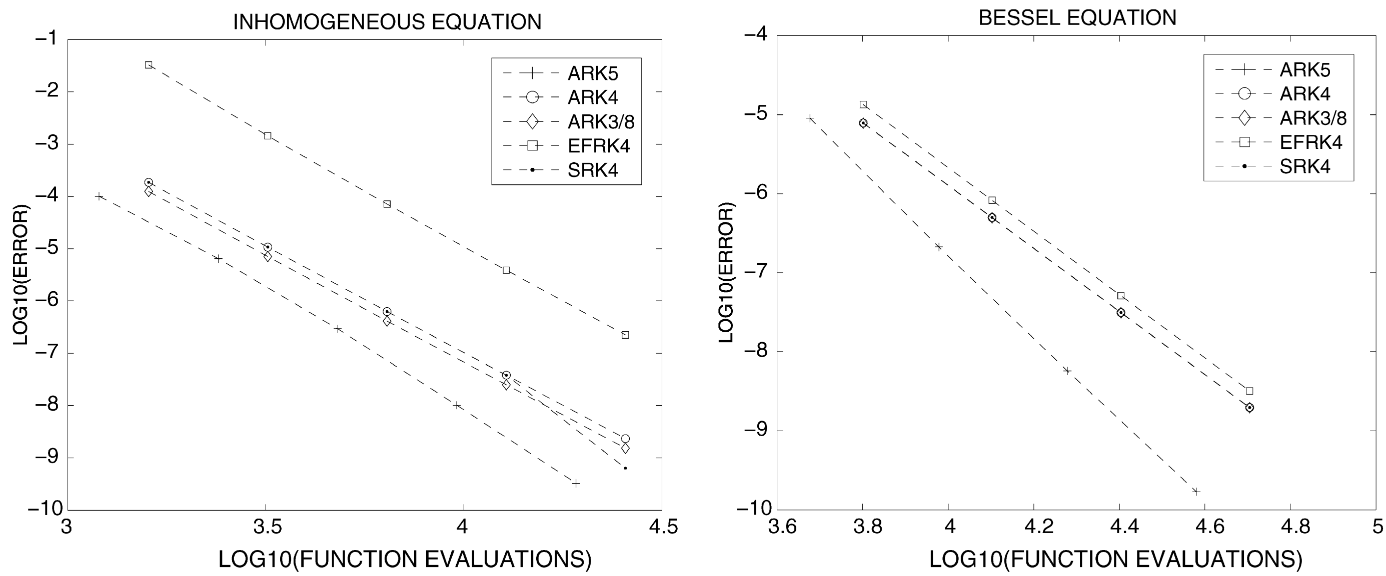


Fig. 2. Efficiency curves in Problems 1–2.

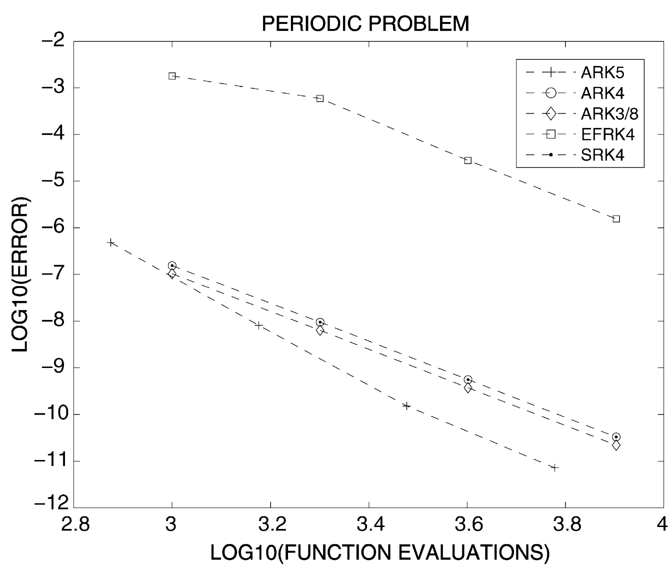


Fig. 3. Efficiency curves in Problem 3.

Problem 2. We consider the Bessel's equation

$$y'' + \left(100 + \frac{1}{4t^2}\right) y = 0, \quad t \in [1, 10],$$

whose exact solution is given by $y(t) = \sqrt{x} J_0(10t)$, where J_0 is the first kind Bessel function of order zero. The initial values of this problem are $y(1) = J_0(10)$ and $y'(1) = \frac{1}{2} J_0(10) - 10 J_1(10)$, where J_1 is the Bessel function of order 1. The numerical results are shown in Fig. 2 (on the right).

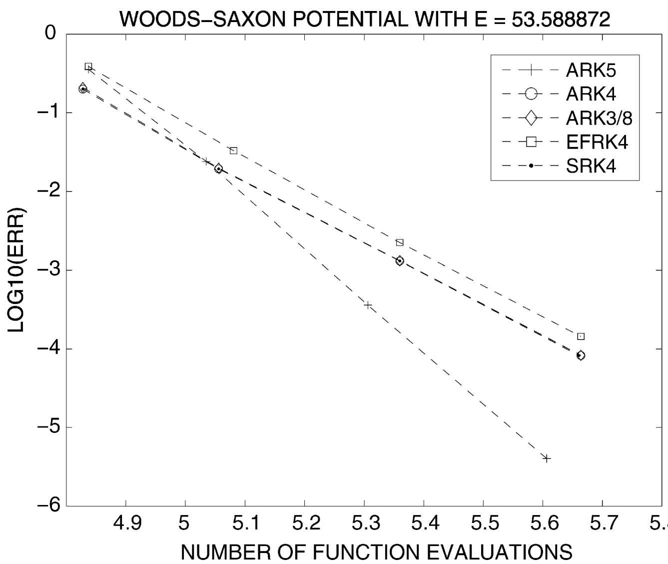
Problem 3. An almost periodic orbital problem studied by Franco and Palacios [11]

$$z'' + z = \epsilon e^{i\psi x}, \quad z(0) = 1, \qquad z'(0) = i,$$

where $\epsilon = 0.001$ and $\psi = 0.01$. The analytical solution $z(x) = u(x) + iv(x)$ is given by

$$u(x) = \frac{1 - \epsilon - \psi^2}{1 - \psi^2} \cos(x) + \frac{\epsilon}{1 - \psi^2} \sin(\psi x), \qquad v(x) = \frac{1 - \epsilon\psi - \psi^2}{1 - \psi^2} \sin(x) + \frac{\epsilon}{1 - \psi^2} \cos(\psi x). \tag{15}$$

The equation has been solved in the interval $[0, 1000]$ with $\omega = 1$. The numerical results are showed in Fig. 3.



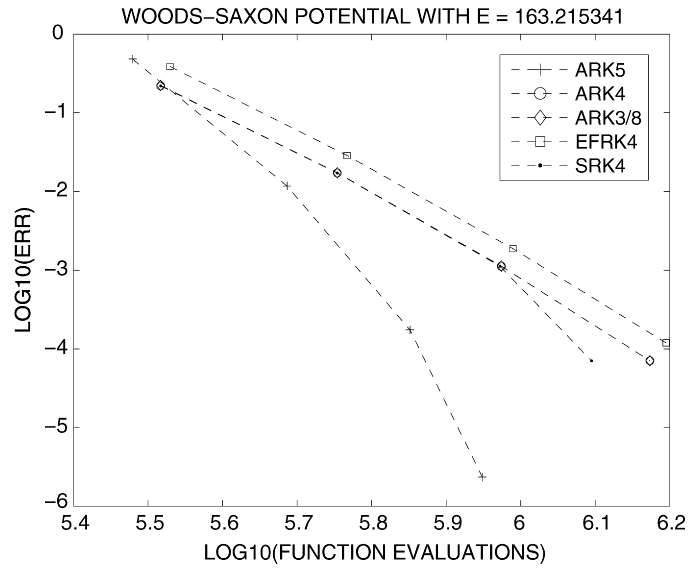


Fig. 4. Efficiency curves in Problem 4.

Problem 4. We consider the numerical integration of the Schrödinger equation

$$y''(x) = \big(v(x) - E\big)y(x),$$

with the well-known Woods–Saxon potential

$$v(x) = c_0 z\big(1 - a(1 - z)\big),$$

in which $z = (\exp(a(x - b) + 1))^{-1}$, $c_0 = -50$, $a = 5/3$, $b = 7$. The domain of numerical integration is [0,15]. It is appropriate to choose ω as follows (see [12])

$$\omega = \begin{cases} \sqrt{50 + E}, & x \in [0, 6.5], \\ \sqrt{E}, & x \in [6.5, 15]. \end{cases}$$

In the numerical experiments we consider the resonance problem ($E > 0$), the numerical results were compared with the analytical solution of the Woods–Saxon potential, rounded to six decimal places. In Fig. 4, we plot the error of $|E_{\text{analytical}} - E_{\text{calculated}}|$ versus the number of function evaluations (in logarithmic scale) for $E_{\text{analytical}} = 53.588872, 163.215341$.

5.2. Comparisons with variable step-size

The codes used in the comparison are denoted by

- ADormand-Prince5(4): The pair of ARK methods obtained in Section 3.2.
- AZonneveld4(3): The pair of the ARK methods obtained in [2].
- AMerson4(3): The pair of ARK methods obtained in [2].
- MAMmerson4(3): The modified pair of the ARK methods obtained in [2].
- EFRK4(3): The pair of exponentially fitted RK methods obtained in [13].

Problem 5. We consider the nonlinear system

$$y_1'' = -4t^2 y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, \quad y_1(0) = 1, \qquad y_1'(0) = 0,$$
$$y_2'' = -4t^2 y_2 + \frac{2y_1}{\sqrt{y_1^2 + y_2^2}}, \quad y_2(0) = 0, \qquad y_2'(0) = 0,$$

whose analytic solution is given by

$$y_1(t) = \cos(t^2), \qquad y_2(t) = \sin(t^2).$$

The equation has been solved in the interval [0, 10] with $\omega = 2t_n$ $(n \geqslant 1)$ at each step. The numerical results are showed in Fig. 5 (on the left).

Problem 6. We consider the well-known two-body problem

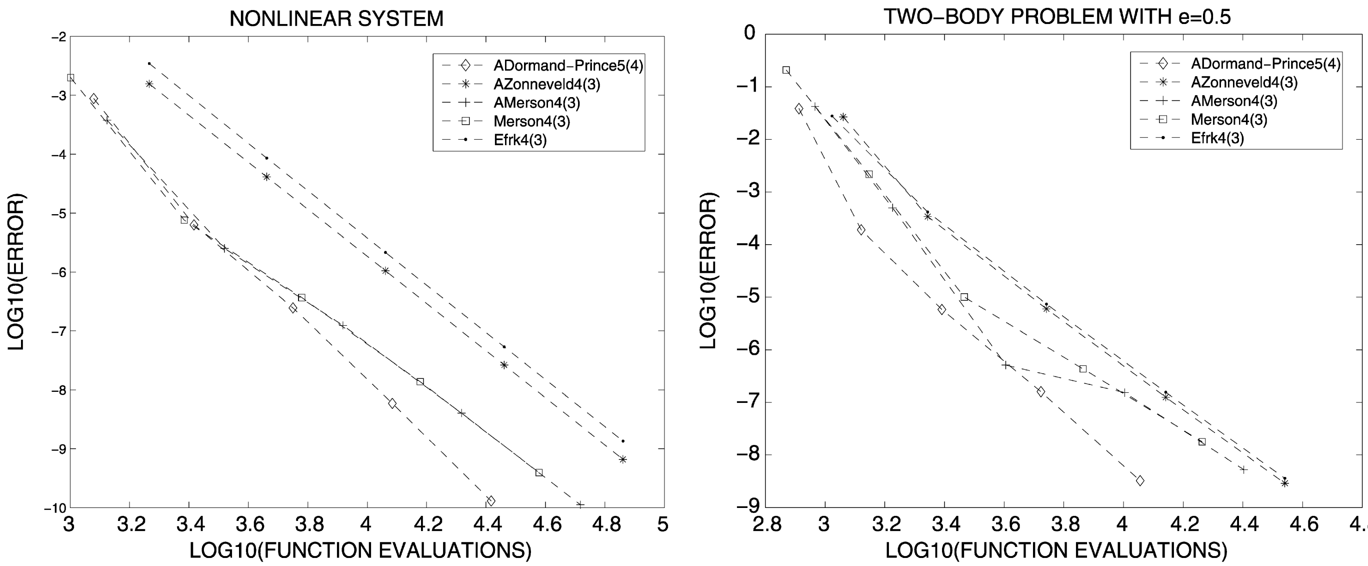


Fig. 5. Efficiency curves in Problems 5–6.

$$y_1'' = -y_1/r^3, \quad y_1(0) = 1 - e, \qquad y_1'(0) = 0,$$
$$y_2'' = -y_2/r^3, \quad y_2(0) = 0, \qquad y_2'(0) = \sqrt{\frac{1+e}{1-e}},$$

in which $r = \sqrt{y_1^2 + y_2^2}$, $e(\geqslant 0)$ is the eccentricity. The analytic solution of the problem

$$y_1(t) = \cos(u) - e, \qquad y_2(t) = \sqrt{1-e^2}\sin(u),$$

where u is the solution of Kepler's equation $u = t + e\sin(u)$. The equation has been solved in the interval $[0, 10\pi]$ with $\omega = 1$, $e = \frac{1}{2}$. The numerical results are showed in Fig. 5 (on the right).

Problem 7. We consider the numerical integration of the Schrödinger equation

$$y''(x) = \left(\frac{l(l+1)}{x^2} + v(x) - k^2\right)y(x),$$

with the well-known Lennard-Jones potential (see [12])

$$v(x) = 500\left(\frac{1}{x^{12}} - \frac{1}{x^6}\right),$$

and we compute some phase-shifts for this potential. In the numerical results, we compute the phase-shifts correct to four decimal places for the energies $k^2 = 25$ and $k^2 = 100$. We choose the fitted frequency $\omega = k$. For the calculation of the phase-shifts, we show the number of function evaluations as a function of $l = 0, \ldots, 10$ in Fig. 6.

Problem 8. We consider the Fermi–Pasta–Ulam problem in Ref. [14]. The Hamiltonian function can be given

$$H = \frac{1}{2}\sum_{i=1}^{n}\left(u_i'^2 + v_i'^2\right) + \frac{\omega^2}{2}\sum_{i=1}^{n} v_i^2 + \frac{1}{4}\sum_{i=1}^{n}(u_{i+1} - v_{i+1} - u_i - v_i)^4,$$

where $u_0 = v_0 = u_{n+1} = v_{n+1} = 0$. This leads to a nonlinear oscillatory problem of the form

$$\frac{d^2u_i}{dt^2} = -\frac{\partial H_p}{\partial u_i}, \quad i = 1, \ldots, n, \qquad \frac{d^2v_i}{dt^2} + \omega^2 v_i = -\frac{\partial H_p}{\partial v_i}, \quad i = 1, \ldots, n,$$

where

$$H_p = \frac{1}{4}\sum_{i=1}^{n}(u_{i+1} - v_{i+1} - u_i - v_i)^4.$$

Since the analytic solution is unknown, we study the total energy H along the numerical solution on the interval $0 \leqslant t \leqslant t_{\text{end}}$. In our numerical experiments we consider the case of $n = 3$, $t_{\text{end}} = 10$ and as initial values we take

$$u_1(0) = 1, \qquad u_1'(0) = 1, \qquad v_1(0) = 1/\omega, \qquad v_1'(0) = 1,$$

and the remaining initial values are set zeros. Fig. 7 show the number of significant digits in the error propagation of the total energy $(\text{LOG10(ERROR)}) = \log_{10}(\max|H(0) - H_n|)$ versus the number of function evaluations required by each pair with $\omega = 5, 10$.

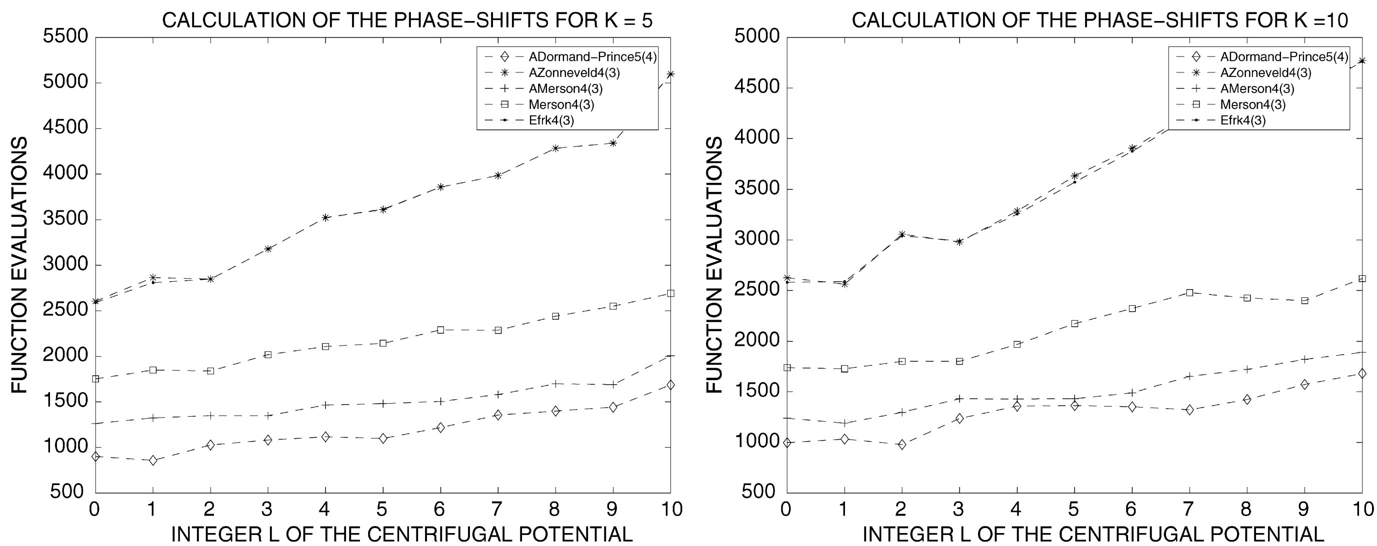


Fig. 6. Efficiency curves in Problem 7 with $k = 5, 10$.

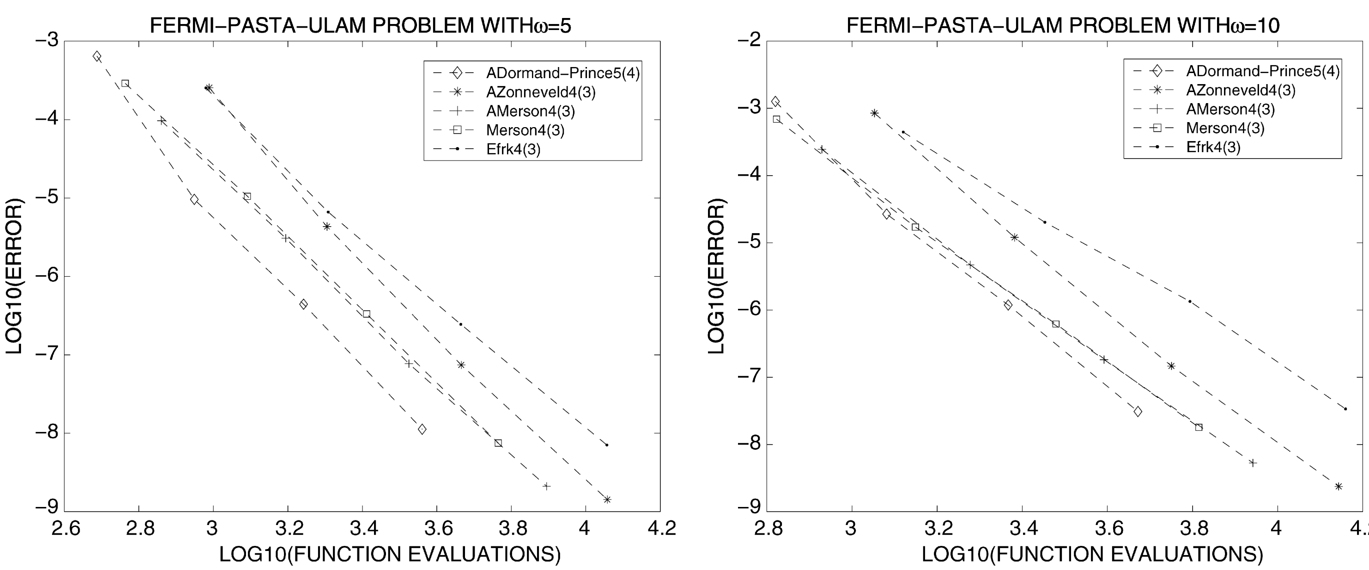


Fig. 7. Efficiency curves in Problem 8 with $\omega = 5, 10$.

6. Conclusion

In this Letter, inspiring by Franco's methods in [2], we present a new Runge–Kutta method of order 5 as well as pairs of embedded Runge–Kutta methods of orders 5 and 4 adapted to the numerical integration of oscillatory problems using the FSAL technique. The analysis of stability is given and the numerical experiments are carried to show the efficiency and robustness of our new methods in comparison with the methods proposed in [2]. In view of the numerical results, it can be concluded that the higher order methods perform more efficiently than the adapted Runge–Kutta methods derived in [2] and the other methods proposed by Franco and Simos.

Acknowledgements

The authors are deeply grateful to the anonymous referees for their valuable comments and suggestions. We would like to thank Hans Van de Vyver for sending us Matlab codes. This research is supported partly by National Science Foundation of China under Grant 10771099, the Foundation for the Authors of the National Excellent Doctoral Thesis Award of China under grant 200720, the Natural Science Foundation of Jiangsu Province of China under grant BK2006725 and Jiangsu Planned Project for Postdoctoral Research Funds 0702046C.

References

[1] D.G. Bettis, J. Appl. Math. Phys. (ZAMP) 30 (1979) 699.
[2] J.M. Franco, Appl. Numer. Math. 50 (2004) 427.
[3] E. Hairer, S.P. Nørsett, S.P. Wanner, Solving Ordinary Differential Equations I, Nonstiff Problems, Springer, Berlin, 1993.

[4] J.C. Butcher, The Numerical Analysis of Ordinary Differential Equations, John Wiley & Sons, Chichester, 2003.
[5] J.R. Dormand, P.J. Prince, J. Comput. Appl. Math. 6 (1980) 19.
[6] J.P. Coleman, L.G. Ixaru, IMA J. Numer. Anal. 16 (1996) 179.
[7] H. Van de Vyver, Comput. Phys. Commun. 173 (2005) 115.
[8] P.J. van der Houwen, B.P. Sommeijer, SIAM J. Numer. Anal. 24 (1987) 595.
[9] G. Vanden Berghe, H. De Meyer, M. Van Daele, T. Van Hecke, Comput. Phys. Commun. 123 (1999) 7.
[10] T.E. Simos, Comput. Phys. Commun. 115 (1998) 1.
[11] J.M. Franco, M. Palacios, J. Comput. Appl. Math. 30 (1990) 1.
[12] H. Van de Vyver, Phys. Lett. A 352 (2006) 278.
[13] J.M. Franco, J. Comput. Appl. Math. 149 (2002) 407.
[14] L. Galgani, A. Giorgilli, A. Martinoli, S. Vanzini, Physica D 59 (1992) 334.