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A MODEL FOR TWO SPECIES WITH STAGE STRUCTURE AND FEEDBACK CONTROL

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This paper studies a periodic coefficients predator-prey delay system with mixed functional response, in which the prey has a history that takes them through two stages, immature and mature. Also, the total toxic action on the predator population expressed by an integral term is considered in our system. Furthermore, the feedback control is considered in our system. Sufficient conditions which guarantee the permanence and extinction of the system are obtained. Finally, we give a brief discussion of our results. From a biological point of view, our results can be used to help protect beneficial animals.

Keywords: Predator-prey system; infinite delay; stage structure; functional response; permanence; extinction.

1. Introduction

In the natural world, there are many species whose individual members have a life history that takes them through two stages, immature and mature. In particular, we have mammalian populations and some amphibious animals in mind, which exhibit these two stages. From the view point of mathematics, the description of the age structure of the population in the life history is also an interesting problem in population dynamics. The permanence and extinction of species are significant concepts for those stage-structured population dynamical systems. Recently, stage structure models constructed by ODEs have received much attraction (see [1, 2, 5, 7, 8, 22, 28, 30–32] and the references therein). This is not only because they are much more simple than the models governed by partial differential equations but also they can exhibit phenomena similar to those of partial differential models [4], and many important physiological parameters can be incorporated. To our knowledge, much research has been devoted the models concerning single-species population growth with various stages life history [1, 3, 15]. Two species models with stage structure were investigated by Wang and Chen [34]; Magnusson [25]; Xiao and Chen [35]; Cui and Song [10]. Zhang, Chen and Neumann [36] proposed the following autonomous stage structure predator-prey system

$$\begin{cases} \dot{x}_1 = \alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \beta_1 x_1 x_3, \\ \dot{x}_2 = \beta x_1 - r_2 x_2, \\ \dot{x}_3 = x_3(-r + k\beta_1 x_1 - \eta_1 x_3), \end{cases} \tag{1.1}$$

where $\alpha, \beta, \beta_1, \eta, \eta_1, r, r_1, r_2$ and k are all positive constants, k is a digesting constant. Sufficient conditions which ensure the permanence of two species and extinction of one or two species are obtained.

On the other hand, since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically varying environment are considered as important selective forces on systems in a fluctuating environment. More realistic and interesting models should take into account both the seasonality of the changing environment [14, 22]. This motivated Cui and Song [10] to consider the following periodic nonautonomous predator-prey model with stage structure for prey

$$\begin{cases} \dot{x}_1 = a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p(t)x_1 y, \\ \dot{x}_2 = c(t)x_1 - f(t)x_2^2, \\ \dot{y} = y[-g(t) + h(t)x_1 - q(t)y], \end{cases} \tag{1.2}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $g(t)$, $h(t)$, $p(t)$ and $q(t)$ are all continuous positive ω-periodic functions. x_1 and x_2 denote the density of immature and mature population (prey) respectively, and y is the density of the predator that only prey on x_1(immature prey). They obtained a set of sufficient and necessary conditions which guarantee the permanence of the above system.

It is well-known that past history as well as current conditions can influence population dynamics and such interactions has motivated the introduction of delays in population growth. There are several books [17,18,21,23] devoted to investigation of the dynamic behavior of functional differential equations. Maybe stimulated by the works of Teng and Chen [31], Cui and Sun [11] further incorporated infinite delay to system (1.2) and investigated the following model

$$\begin{cases} \dot{x}_1 = a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p(t)x_1 \displaystyle\int_{-\infty}^{0} k_{12}(s)y(t+s)\,ds, \\ \dot{x}_2 = c(t)x_1 - f(t)x_2^2, \\ \dot{y} = y[-g(t) + h(t) \displaystyle\int_{-\infty}^{0} k_{21}(s)x_1(t+s)\,ds - q(t) \displaystyle\int_{-\infty}^{0} k_{22}(s)y(t+s)\,ds]. \end{cases} \tag{1.3}$$

Under the assumption that the coefficients in (1.3) are all ω-periodic and continuous for $t \geq 0$, $a(t), b(t), c(t)$, $d(t)$ and $f(t)$ are all positive, $p(t), h(t)$ and $q(t)$ are nonnegative, and $\int_0^\omega q(t)\, dt > 0$, $\int_0^\omega g(t)dt \geq 0$. The functions $k_{ij}(s)(i, j = 1, 2)$ defined on $\mathbb{R}_- = (-\infty, 0]$ are nonnegative and integrable, $\int_{-\infty}^0 k_{ij}(s) = 1$. By using analysis technique, they obtained a set of sufficient and necessary conditions which guarantee the permanence of the system.

To our knowledge, seldom did scholars consider the stage structure predator-prey system with functional response and infinite delay. In this paper, we generalize these systems [5, 10, 11, 13, 16, 36, 37] and establish sufficient conditions which guarantee the permanence and extinction of a three species prey-predator system with stage structure. We propose that the life history of prey species is divided into two stages: immature and mature. As for predator species, the predator population feed on the immature prey population and the mature prey population. Furthermore, the total toxic action and feedback control are considered in our model.

The main purpose of this paper is to find a set of easily verifiable sufficient conditions for the permanence and extinction of the system (2.4). The present paper is organized as follows. In Sec. 2, we formulate our model, introduce some notations and definitions and give some preliminary results needed in later sections. In Sec. 3, we state the main results of this paper. We then prove in Sec. 4, the main results of our model by using analysis technique. Finally, in Sec. 5, we give a biological example and a brief discussion of our results.

2. Model Formulation and Preliminaries

In most biological population, the accumulation of metabolic products may seriously inconvenience a population and one of the consequences can be a fall in the birth and an increase in the mortality rate. If we suppose that the total toxic action on birth and death rates is expressed by an integral term in the logistic equation [33], one can then, consider the following integrodifferential equation

$$\frac{dN(t)}{dt} = rN(t) - bN^2(t) - cN(t)\left(\int_0^\infty K(s)N(t-s)ds\right)^n, \tag{2.1}$$

where K denotes the residual intensity of pollution and $n \in (0, \infty)$. A model relate to (2.1) in theme has been numerically studied by Borsellino and Torre [5]. To derive the qualitative results of Borsellino and Torre by means of an analytically manageable model, Cushing [13] has proposed a model of the form

$$\frac{dN(t)}{dt} = N(t)\left\{\alpha - \beta N(t) - \gamma\left(\int_0^\infty K(s)N(t-s)ds\right)^2\right\}, \tag{2.2}$$

where α, β, γ are positive constants. We suppose that it is desired to reduce the equilibrium level of (2.2) and maintain the population size at a reduced level by means of a feedback regulator (or feedback control). We can model such a regulated system by

$$\begin{cases} \dfrac{dN(t)}{dt} = N(t)\left\{\alpha - \beta N(t) - \gamma\left(\displaystyle\int_0^\infty K(s)N(t-s)ds\right)^2 - cF(t)\right\}, \\ \dfrac{dF(t)}{dt} = -aF(t) + bN(t), \end{cases} \tag{2.3}$$

where a, b, $c > 0$ and F denotes an "indirect" feedback control [17].

In this section, we consider a periodic coefficients predator-prey delay system with mixed functional response and feedback control, in which the prey has a history that takes them through two stages: immature and mature.

$$\begin{cases} \dot{x}_1 = a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p_1(t)\dfrac{x_1^3}{P_1 + x_1^3}\displaystyle\int_{-\infty}^0 k_{11}(s)y(t+s)ds, \\ \dot{x}_2 = c(t)x_1 - f(t)x_2^2 - p_2(t)\dfrac{x_2^3}{P_2 + x_2^3}\displaystyle\int_{-\infty}^0 k_{12}(s)y(t+s)ds, \\ \dot{y} = y\left[g(t) + \displaystyle\sum_{i=1}^2 h_i(t)\int_{-\infty}^0 k_{2i}(s)\dfrac{x_i^3(t+s)(1-e^{-\gamma_i x_i(t+s)})}{P_i + x_i^3(t+s)}ds\right. \\ \qquad \left. - \beta(t)y(t-\tau_1) - q(t)\left(\displaystyle\int_{-\infty}^0 k_{23}(s)y(t+s)ds\right)^n - b_1(t)F(t)\right], \\ \dot{F} = -c_1(t)F(t) + d_1(t)y(t-\tau_2), \end{cases} \tag{2.4}$$

where x_1 and x_2 denote, respectively, the density of immature and mature population (prey). y represents the density of the predator that prey on x_1 and x_2. Also, $n \in (0, \infty)$. F describes a feedback regulator. The following assumptions are made for deriving the mathematical model:

(A_1) The birth rate of the immature prey population is proportional to the living mature prey population with a proportionality function $a(t)$. For the immature prey population, the death rate is proportional to the existing immature prey population with a proportionality function $b(t)$. The variable parameter $d(t)$ represents that the immature prey population is density restriction.

(A_2) The predator population y can prey on the immature prey population x_1 and the mature prey population x_2. When its favorite food is severely scarce, population y can eat other resources. The predator species y is density restriction. We maintain the predator population size at a reduced level by applying a feedback control.

(A_3) $\frac{x_i^3}{P_i+x_i^3}$(*Holling type*) and $1 - e^{-\gamma_i x_i}$(*Ivlev type*), $i = 1, 2$, describe the functional response of predator to prey. k_{1i}, $i = 1, 2$, denotes a distribution of the tensity of all the past life of the predator species y on its present ability of preying. k_{2j}, $j = 1, 2$, denotes a distribution of the tensity of all the past life of the predator species y on its conversion efficiency of prey into predator. k_{23} describes the residual intensity of pollution.

(A_4) The coefficients in (2.4) are all continuous positive ω-periodic for $t \geq 0$. P_i, γ_i and τ_i are positive constants for $i = 1, 2$. The functions $k_{ij}(s)(i = 1, 2, j = 1, 2, 3)$ defined on $\mathbb{R}_- = (-\infty, 0]$ are nonnegative and integrable, $\int_{-\infty}^{0} k_{ij}(s) = 1$.

The biological background for system (2.4) can be found in Gopalsamy [17]; Zhang *et al.* [36]; Teng and Chen [31]; Cui and Song [10].

Let $C_+ = \{\phi = (\phi_1, \phi_2, \phi_3, \phi_4) : \phi_i(t)$ is continuous and nonnegative on $\mathbb{R}_-$ and $\phi_i(0) > 0,\ i = 1, 2, 3, 4\}$. In this paper, we always assume that solutions of (2.4) satisfy the following initial conditions

$$\begin{aligned} x_i(s) = \varphi_i(s)\ \ (i = 1, 2), \quad y(s) = \psi_1(s), \\ F(s) = \psi_2(s), (\varphi_1, \varphi_2, \psi_1, \psi_2) \in C_+, \quad s \in (-\infty, 0]. \end{aligned} \tag{2.5}$$

Let $f(t)$ be a continuous ω-periodic function defined on $[0, +\infty)$, we set

$$\mathbb{A}_\omega(f) = \omega^{-1} \int_0^\omega f(t)dt, \quad f^U = \max_{t \in [0,\omega]} f(t),$$
$$f^L = \min_{t \in [0,\omega]} f(t), \quad h(t) = h_1(t) + h_2(t).$$

Definition 2.1. The system $\dot{x} = F(t, x)$, $x \in \mathbb{R}^n$ is said to be permanent if there are constants $M \geq m > 0$ such that every positive solution $x(t) = (x_1(t), \dots, x_n(t)) \in R^n_+ = \{(x_1, \dots, x_n) : x_i > 0,\ i = 1, \dots, n\}$ of this system, satisfies

$$m \leq \liminf_{t\to\infty} x_i(t) \leq \limsup_{t\to\infty} x_i(t) \leq M \quad \text{for } \forall i \in [1, n].$$

Lemma 2.2 [12]. *The system*

$$\begin{cases} \dot{x}_1 = a(t)x_2 - b(t)x_1 - d(t)x_1^2, \\ \dot{x}_2 = c(t)x_1 - f(t)x_2^2, \end{cases} \tag{2.6}$$

has a positive ω-periodic solution $(x_1^*(t), x_2^*(t))$ which is globally asymptotically stable with respect to $R^2_{+0} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$.

Lemma 2.3. *For the following nonautonomous differential equation*

$$\dot{u} = u[a(t) - b(t)u - c(t)u^n], \tag{2.7}$$

where $a(t)$, $b(t)$ and $c(t)$ are ω-periodic continuous functions, c^L, $b^L \geq 0$ and $\mathbb{A}_\omega(b) > 0$, $n \in (0, \infty)$, there is a constant $M > 0$ such that every positive solution $u(t)$ of (2.7) *satisfies* $\limsup_{t\to\infty} u(t) \leq M$.

Proof. The proof is obvious, in fact, $\dot{u} = u[a(t) - b(t)u - c(t)u^n] \leq u[a(t) - b(t)u]$. From Teng [32], we have that there exists a constant M such that the solution $x(t)$

of the Logistic equation $\dot{x} = x[a(t) - b(t)x]$ satisfies $\limsup_{t\to\infty} x(t) \le M$. Using the comparison theorem of the scalar ODE, this completes the proof. $\square$

Lemma 2.4. *Consider the following differential inequality*

$$\dot{v}(t) \ge (\le)\widehat{b} - \widehat{a}v(t),$$

where $\widehat{a},\ \widehat{b} > 0$, $v(t_0) > 0$. *One has*

$$v(t) \ge (\le)\frac{\widehat{b}}{\widehat{a}}\left\{1 + \left(\frac{\widehat{a}v(t_0)}{\widehat{b}} - 1\right)\exp(-\widehat{a}(t - t_0))\right\} \quad \text{for } t \ge t_0.$$

3. Main Results

Theorem 3.1. *Let* $\theta(>0)$ *be a positive constant which depends on* d_1^U, c_1^L *and the maximum size of the predator population. Assume*

$$\mathbb{A}_\omega\left(g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\infty}^{0} k_{2i}(s)\frac{x_i^{*^3}(t+s)(1 - e^{-\gamma_i x_i^*(t+s)})}{P_i + x_i^{*^3}(t+s)}ds - \theta b_1(t)\right) > 0, \tag{3.1}$$

where $(x_1^*(t), x_2^*(t))$ *is the positive* ω*-periodic solution of* (2.6). *Then system* (2.4) *is permanent.*

Theorem 3.2. *Let* $\widehat{\theta}(>0)$ *be a positive constant which depended on the initial condition* $F(0)$, *and* $b_2(t) = e^{-c_1^U t}b_1(t)$. *Assume*

$$\mathbb{A}_\omega\left(g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\infty}^{0} k_{2i}(s)\frac{x_i^{*^3}(t+s)(1 - e^{-\gamma_i x_i^*(t+s)})}{P_i + x_i^{*^3}(t+s)}ds - \widehat{\theta}b_2(t)\right) < 0 \tag{3.2}$$

and

$$l = \int_{-\infty}^{0} k_{23}(s)\exp\{\lambda^U s\}ds < \infty, \tag{3.3}$$

where $(x_1^*(t), x_2^*(t))$ *is the positive* ω*-periodic solution of* (2.6). *And*

$$\lambda(t) = g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\infty}^{0} k_{2i}(s)$$

$$\frac{(x_i^*(t+s)+\epsilon)^2(1 - e^{-\gamma_i(x_i^*(t+s)+\epsilon)})}{P_i + (x_i^*(t+s)+\epsilon)^3}ds + h(t)\epsilon - \widehat{\theta}b_2(t),$$

in which $\epsilon(\ll 1)$ *is some positive constant. Then for any solution* (x_1, x_2, y, F) *of* (2.4), $y(t) \to 0$ *as* $t \to \infty$.

4. Proof of Main Results

Lemma 4.1. *There exist positive constants M_x, M_y and M_F such that*

$$\limsup_{t\to\infty} x_i(t) \le M_x \quad (i = 1, 2),$$
$$\limsup_{t\to\infty} y(t) \le M_y,$$
$$\limsup_{t\to\infty} F(t) \le M_F.$$

The proof of this lemma is trivial, so we omit it.

Lemma 4.2. *There is a positive constant ρ_x ($\rho_x < M_x$) such that*

$$\liminf_{t\to\infty} x_i(t) \ge \rho_x \quad (i = 1, 2).$$

Proof. By Lemma 4.1, there exists a positive constant $T_1 > T_0 + 2\tau$ such that

$$\begin{aligned} &0 < x_i(t) \le M_x \quad (i = 1, 2),\\ &0 < y(t) \le M_y, \quad t \ge T_1. \end{aligned} \tag{4.1}$$

Obviously, there exists a constant $\sigma > 0$ such that

$$H_0 \int_{-\infty}^{-\sigma} k_{1i}(s)\,ds < M_y \quad (i = 1, 2),$$

where $H_0 = \sup\{y(t+s) \mid t \ge 0, s \le 0\}$. Hence, by (4.2), for all $t \ge T_1 + \sigma$, we have

$$\begin{aligned}
\dot{x}_1 &= a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p_1(t)\frac{x_1^3}{P_1 + x_1^3}\int_{-\infty}^{-\sigma} k_{11}(s)y(t+s)ds\\
&\quad - p_1(t)\frac{x_1^3}{P_1 + x_1^3}\int_{-\sigma}^{0} k_{11}(s)y(t+s)ds\\
&\ge a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p_1(t)H_0\frac{x_1^3}{P + x_1^3}\int_{-\infty}^{-\sigma} k_{11}(s)ds\\
&\quad - p_1(t)M_y\frac{x_1^3}{P_1 + x_1^3}\int_{-\sigma}^{0} k_{11}(s)ds\\
&\ge a(t)x_2 - b(t)x_1 - d(t)x_1^2 - \frac{2M_y p_1(t)M_x}{P_1}x_1^2,\\
\dot{x}_2 &= c(t)x_1 - f(t)x_2^2 - p_2(t)\frac{x_2^3}{P_2 + x_2^3}\int_{-\infty}^{0} k_{12}(s)y(t+s)ds\\
&\ge c(t)x_1 - f(t)x_2^2 - \frac{2M_y p_2(t)M_x}{P_2}x_2^2.
\end{aligned}$$

Consider the following auxiliary system

$$\begin{cases} \dot{u}_1 = a(t)u_2 - b(t)u_1 - \left[d(t) + \dfrac{2M_xM_yp_1(t)}{P_1}\right]u_1^2, \\ \dot{u}_2 = c(t)u_1 - \left[f(t) + \dfrac{2M_xM_yp_2(t)}{P_2}\right]u_2^2. \end{cases} \tag{4.2}$$

Let $(u_1(t), u_2(t))$ is the solution of system (4.2) with the initial condition $(u_1(T_1+\sigma), u_2(T_1+\sigma)) = (x_1(T_1+\sigma), x_2(T_1+\sigma))$, then for all $t \geq T_1+\sigma$,

$$x_i(t) \geq u_i(t).$$

By Lemma 4.1, (4.2) has a positive ω-periodic solution $(\widehat{u}_1^*(t), \widehat{u}_2^*(t))$, which is globally asymptotically stable. By the global asymptotic stability of $\widehat{u}_i^*(t)$ $(i = 1, 2)$, for any a sufficiently small $\varepsilon^*(> 0)$, there exists $T_2 > T_1 + \sigma$ such that

$$u_i(t) \geq \widehat{u_i}^*(t) - \varepsilon^*$$

for all $t \geq T_2$. Hence, for all $t \geq T_2$, $x_i(t) \geq \rho_x \doteq \min_{0\leq t\leq\omega}\{\widehat{u_i}^*(t) - \varepsilon^*\}$. So we have

$$\liminf_{t\to\infty} x_i(t) \geq \rho_x.$$

This completes the proof. □

Lemma 4.3. *Suppose that* (3.1) *holds. Then there exists a positive constant* ϱ_y $(\varrho_y < M_y)$ *such that*

$$\limsup_{t\to\infty} y(t) \geq \varrho_y. \tag{4.3}$$

Proof. By (3.1), we can choose a positive constant $\varepsilon_0 < \frac{1}{2}\min_{t\in[0,\omega]}\{x_i^*(t), i = 1, 2\}$, where $(x_1^*(t), x_2^*(t))$ is the unique positive solution of system (2.6) such that

$$A_\omega(\psi_{\varepsilon_0}(t)) > 0, \tag{4.4}$$

where

$$\psi_{\varepsilon_0}(t) = g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\infty}^{0} k_{2i}(s)$$

$$\frac{(x_i^*(t+s)-\varepsilon_0)^3(1-e^{-\gamma_i(x_i^*(t+s)-\varepsilon_0)})}{P_i + (x_i^*(t+s)-\varepsilon_0)^3}ds - (2\varepsilon_0)^n q(t) - \varepsilon_0\beta(t) - \theta b_1(t).$$

Consider the following equations with a positive parameter μ

$$\begin{cases} \dot{x}_1 = a(t)x_2 - b(t)x_1 - \left[d(t) + \dfrac{2\mu M_xp_1(t)}{P_1}\right]x_1^2, \\ \dot{x}_2 = c(t)x_1 - \left[f(t) + \dfrac{2\mu M_xp_2(t)}{P_2}\right]x_2^2. \end{cases} \tag{4.5}$$

By Lemma 4.1, (4.5) has a positive ω-periodic solution $(x^*_{1\mu}(t), x^*_{2\mu}(t))$, which is globally asymptotically stable. Let $(x_{1\mu}(t), x_{2\mu}(t))$ be the solution of (4.5) with initial condition $x_{i\mu}(0) = x^*_i(0)$, where $x^*(t) = (x^*_1(t), x^*_2(t))$ is the positive periodic solution of (2.6). Hence, for the above ε_0, there exists $T_3 > T_2$ such that

$$|x_{i\mu}(t) - x^*_{i\mu}(t)| < \varepsilon_0/4 \tag{4.6}$$

for $t \geq T_3$, $i = 1, 2$. According to the continuity of the solution in the parameter μ, we have $x_{i\mu}(t) \to x^*_i(t)(i = 1, 2)$ uniformly in $[T_3, T_3 + \omega]$ as $\mu \to 0$. Hence for $\varepsilon_0 > 0$, there exists $\mu_0 = \mu_0(\varepsilon_0)$ $(0 < \mu_0 < \varepsilon_0)$ such that

$$|x_{i\mu}(t) - x^*_i(t)| < \varepsilon_0/4, \quad 0 \leq \mu \leq \mu_0, \tag{4.7}$$

$t \in [T_3, T_3 + \omega]$, $i = 1, 2$. Thus from (4.6) and (4.7), we get

$$|x^*_{i\mu}(t) - x^*_i(t)| < \varepsilon_0/2, \quad 0 \leq \mu \leq \mu_0,$$

$t \in [T_3, T_3 + \omega]$, $i = 1, 2$. Since $x^*_{i\mu}(t)$ and $x^*_i(t)$ are all ω-periodic, we have

$$|x^*_{i\mu}(t) - x^*_i(t)| < \varepsilon_0/2, \quad 0 \leq \mu \leq \mu_0, \tag{4.8}$$

$t \geq 0, i = 1, 2$.

Choose a constant $\mu_1(0 < \mu_1 < \mu_0, \mu_1 < \varepsilon_0)$, from (4.8), we derive

$$x^*_{i\mu_1}(t) \geq x^*_i(t) - \varepsilon_0/2, \quad t \geq 0, \quad i = 1, 2. \tag{4.9}$$

Suppose that (4.3) is not true, then for the above ε_0, there exists $\phi \in C_+$ such that

$$\limsup_{t\to\infty} y(t, \phi) < \mu_1,$$

where $(x_1(t, \phi), x_2(t, \phi), y(t, \phi), F(t, \phi))$ is the solution of (2.4) with the initial condition $(x_1(\widehat{\theta}), x_2(\widehat{\theta}), y(\widehat{\theta}), F(\widehat{\theta})) = (\phi(\widehat{\theta}), \phi(\widehat{\theta}), \phi(\widehat{\theta}), \phi(\widehat{\theta}))$. So there exists a constant $T_4(> T_3)$ such that

$$y(t, \phi) < \mu_1, \quad t \geq T_4. \tag{4.10}$$

On the other hand, Lemma 4.1 shows that there exists an enough large constant $T_5(> T_4)$ such that

$$x_i(t, \phi) < M_x, \quad t \geq T_5, \quad i = 1, 2. \tag{4.11}$$

Also, from $\int_{-\infty}^0 k_{ij}(s)ds = 1$ $(i = 1, 2, j = 1, 2, 3)$, we choose a positive constant τ_0 such that

$$H_0 \int_{-\infty}^{-\tau_0} k(s)ds < \mu_1, \tag{4.12}$$

where $k(t) = k_{11}(t) + k_{12}(t) + k_{21}(t) + k_{22}(t) + k_{23}(t)$ and H_0 is defined in the proof of Lemma 4.2. For any $t \geq T_5 + \tau_0$, we have

$$\begin{aligned}
\dot{x}_1(t,\phi) &= a(t)x_2(t,\phi) - b(t)x_1(t,\phi) - d(t)x_1^2(t,\phi) \\
&\quad - p_1(t)\frac{x_1^3(t,\phi)}{P_1 + x_1^3(t,\phi)} \int_{-\infty}^{-\tau_0} k_{11}(s)y(t+s)ds \\
&\quad - p_1(t)\frac{x_1^3(t,\phi)}{P_1 + x_1^3(t,\phi)} \int_{-\tau_0}^{0} k_{11}(s)y(t+s)ds \\
&\geq a(t)x_2(t,\phi) - b(t)x_1(t,\phi) - d(t)x_1^2(t,\phi) \\
&\quad - p_1(t)H_0\frac{x_1^3(t,\phi)}{P_1 + x_1^3(t,\phi)} \int_{-\infty}^{-\tau_0} k_{11}(s)ds \\
&\quad - p_1(t)\mu_1\frac{x_1^3(t,\phi)}{P_1 + x_1^3(t,\phi)} \int_{-\tau_0}^{0} k_{11}(s)ds \\
&\geq a(t)x_2(t,\phi) - b(t)x_1(t,\phi) - d(t)x_1^2(t,\phi) \\
&\quad - 2\mu_1 p_1(t)\frac{M_x}{P_1}x_1^2(t,\phi), \\
\dot{x}_2(t,\phi) &= c(t)x_1(t,\phi) - f(t)x_2^2(t,\phi) \\
&\quad - p_2(t)\frac{x_2^3(t,\phi)}{P_2 + x_2^3(t,\phi)} \int_{-\infty}^{-\tau_0} k_{12}(s)y(t+s)ds \\
&\quad - p_2(t)\frac{x_2^3(t,\phi)}{P_2 + x_2^3(t,\phi)} \int_{-\tau_0}^{0} k_{12}(s)y(t+s)ds \\
&\geq c(t)x_1(t,\phi) - f(t)x_2^2(t,\phi) - 2\mu_1 p_2(t)\frac{M_x}{P_2}x_2^2(t,\phi).
\end{aligned}$$

Let $(u_{1\mu_1}, u_{2\mu_1})$ be the solution of (4.5) with $\mu = \mu_1$ and $(u_{1\mu_1}(T_5+\tau_0), u_{2\mu_1}(T_5+\tau_0)) = (x_1(T_5+\tau_0), x_2(T_5+\tau_0))$, then by the vector comparison theorem, we obtain

$$x_i(t,\phi) \geq u_{i\mu_1}(t), \quad i = 1, 2, \tag{4.13}$$

$t \geq T_5 + \tau_0$. By the global asymptotic stability of $(x_{1\mu_1}^*(t), x_{2\mu_1}^*(t))$, for the given $\varepsilon_0 > 0$, there exists $T_6 > T_5 + \tau_0$ such that

$$u_{i\mu_1}(t) > x_{i\mu_1}^*(t) - \varepsilon_0/2, \quad t \geq T_6, \quad i = 1, 2$$

and hence, by (4.9), we derive

$$x_i(t,\phi) > x_i^*(t) - \varepsilon_0, \quad t \geq T_6, \quad i = 1, 2. \tag{4.14}$$

Let $\theta = \frac{3d_1^U M_y}{2c_1^L}$. Therefore, for $t \geq T_6 + \tau_0 + \tau$, we have

$$\begin{aligned}
&\dot{y}(t,\phi)\\
&= y(t,\phi)\Bigg[g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\infty}^{0} k_{2i}(s)\frac{(x_i(t+s,\phi))^3(1-e^{-\gamma_i x_i(t+s,\phi)})}{P_i + (x_i(t+s,\phi))^3}ds\\
&\quad - \beta(t)y(t-\tau_1,\phi) - q(t)\left(\int_{-\infty}^{0} k_{23}(s)y(t+s,\phi)ds\right)^n - b_1(t)F(t,\phi)\Bigg]\\
&\geq y(t,\phi)\Bigg[g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\infty}^{0} k_{2i}(s)\frac{(x_i^*(t+s,\phi)-\varepsilon_0)^3(1-e^{-\gamma_i(x_i^*(t+s,\phi)-\varepsilon_0)})}{P_i + (x_i^*(t+s,\phi)-\varepsilon_0)^3}ds\\
&\quad - \beta(t)\mu_1 - (2\mu_1)^n q(t) - \frac{3d_1^U M_y}{2c_1^L}b_1(t)\Bigg]\\
&\geq y(t,\phi)\Bigg[g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\infty}^{0} k_{2i}(s)\frac{(x_i^*(t+s,\phi)-\varepsilon_0)^3(1-e^{-\gamma_i(x_i^*(t+s,\phi)-\varepsilon_0)})}{P_i + (x_i^*(t+s,\phi)-\varepsilon_0)^3}ds\\
&\quad - \varepsilon_0\beta(t) - (2\varepsilon_0)^n q(t) - \theta b_1(t)\Bigg]\\
&= y(t,\phi)\psi_{\varepsilon_0}(t).
\end{aligned}$$

Integrating the above inequality from $T_6 + \tau_0 + \tau$ to t yields

$$y(t,\phi) \geq y(T_6+\tau_0+\tau)\exp\left(\int_{T_6+\tau_0+\tau}^{t}\psi_{\varepsilon_0}(s)ds\right).$$

It follows from (4.4) that $y(t,\phi) \to \infty$ as $t \to \infty$, which is a contradiction. This completes the proof. □

Lemma 4.4. *Assume that* (3.1) *holds. Then there exists a positive constant* δ_y ($\delta_y < M_y$) *such that any solution* (x_1, x_2, y, F) *of system* (2.4) *with initial condition satisfies*

$$\liminf_{t\to\infty} y(t) \geq \delta_y. \tag{4.15}$$

Proof. Suppose that (4.15) is not true, there must exists a sequence $\{\phi_k\} \subset C_+$ such that

$$\liminf_{t\to\infty} y(t,\phi_k) < \frac{\varrho_y}{(k+1)^2}, \quad k = 1, 2, \cdots,$$

and by Lemma 4.3, we have $\limsup_{t\to\infty} y(t,\varphi_k) \geq \varrho_y$, $k = 1, 2, \cdots$. Hence, for each k, we choose two time sequences $\{s_q^{(k)}\}$ and $\{t_q^{(k)}\}$, satisfying $0 < s_1^{(k)} < t_1^{(k)} < s_2^{(k)} < t_2^{(k)} < \cdots < s_q^{(k)} < t_q^{(k)} < \cdots$ and $s_q^{(k)} \to \infty$ as $q \to \infty$, and

$$y(s_q^{(k)},\phi_k) = \frac{\varrho_y}{k+1}, \quad y(t_q^{(k)},\phi_k) = \frac{\varrho_y}{(k+1)^2}, \tag{4.16}$$

$$\frac{\varrho_y}{(k+1)^2} < y(t,\phi_k) < \frac{\varrho_y}{k+1}, \quad t \in (s_q^{(k)}, t_q^{(k)}). \tag{4.17}$$

By Lemma 4.1, for a given positive integer k, there exists $\widetilde{T}^{(k)} > 0$ such that $x_i(t,\phi_k) \le M_x$ $(i = 1,2)$ and $y(t,\phi_k) \le M_y$ for all $t \ge \widetilde{T}^{(k)}$. Further, there is a constant $\sigma^{(k)} > 0$ such that

$$H_1^{(k)} \int_{-\infty}^{-\sigma^{(k)}} k(s)\,ds < M_y,$$

where $H_1^{(k)} = \sup\{y(t+s,\phi_k) : t \ge 0, s \le 0\}$ and $k(s)$ is given in Lemma 4.3. In view of $s_q^{(k)} \to \infty$ as $q \to \infty$, there exists a positive integer $K_1^{(k)}$ such that $s_q^{(k)} > \widetilde{T}^{(k)} + \sigma^{(k)}$ as $q \ge K_1^{(k)}$. For any $t \ge \widetilde{T}^{(k)} + \sigma^{(k)} + \tau$, we have

$$\dot{y}(t,\phi_k) \ge y(t,\phi_k)\left[-\beta(t)M_y - (2M_y)^n q(t) - \frac{3d_1^U M_y}{2c_1^L} b_1(t)\right].$$

Integrating the above inequality from $s_q^{(k)}$ to $t_q^{(k)}$, for any $q \ge K_1^{(k)}$, then we have

$$y(t_q^{(k)},\phi_k) \ge y(s_q^{(k)},\phi_k)\exp\left(\int_{s_q^{(k)}}^{t_q^{(k)}} \left[-\beta(t)M_y - (2M_y)^n q(t) - \frac{3d_1^U M_y}{2c_1^L} b_1(t)\right] dt\right).$$

Obviously, we derive

$$\int_{s_q^{(k)}}^{t_q^{(k)}} \left[\beta(t)M_y + (2M_y)^n q(t) + \frac{3d_1^U M_y}{2c_1^L} b_1(t)\right] dt \ge \ln(k+1) \quad \text{for } q \ge K_1^{(k)}.$$

Hence, in view of the periodicity of $\beta(t)$, $q(t)$ and $b_1(t)$, we get

$$t_q^{(k)} - s_q^{(k)} \to \infty \tag{4.18}$$

as $k \to \infty,\ q \ge K_1^{(k)}$. By (4.4), (4.16) and (4.18), there are positive constants T and N_0 such that

$$y(s_q^{(k)},\phi_k) = \frac{\varrho_y}{k+1} < \varepsilon_0, \tag{4.19}$$

$$t_q^{(k)} - s_q^{(k)} > 2T \tag{4.20}$$

and

$$\int_0^\kappa \psi_{\varepsilon_0}(t)dt > 0 \tag{4.21}$$

for $k \ge N_0$, $q \ge K_1^{(k)},$ *and* $\kappa > T$. (4.19) implies that

$$y(t,\phi_k) < \varepsilon_0,\ t \in [s_q^{(k)}, t_q^{(k)}], \tag{4.22}$$

for $k \ge N_0$, $q \ge K_1^{(k)}$. Noticing that $s_q^{(k)} \to \infty$ as $q \to \infty$ and $\int_{-\infty}^0 k_{ij}(s)ds = 1$ $(i,j = 1,2)$, for any k there exists $K_2^{(k)} > K_1^{(k)}$ such that for all $q > K_2^{(k)}$, we obtain

$$H_1^{(k)} \int_{-\infty}^{\widetilde{T}^{(k)} - s_q^{(k)} - \sigma^0} k(s)ds < \frac{1}{2}\varepsilon_0 \tag{4.23}$$

and

$$M_y \int_{-\infty}^{-\sigma^0} k(s)ds < \frac{1}{2}\varepsilon_0, \tag{4.24}$$

where $\sigma^0 > 0$ and $k(t) = k_{12} + k_{21}(t) + k_{22}(t)$. By (4.18), there exists a positive integer N_1 such that

$$t_q^{(k)} - s_q^{(k)} > \sigma^0, \text{ for } k > N_1,\ q \geq K_2^{(k)}.$$

For $k > N_1,\ q \geq K_2^{(k)}$ and $s_q^{(k)} + \sigma^0 \leq t \leq t_q^{(k)}$, from (4.22)–(4.24), we have

$$\begin{aligned}
\frac{dx_1(t,\phi_k)}{dt} &= a(t)x_2(t,\phi_k) - b(t)x_1(t,\phi_k) - d(t)x_1^2(t,\phi_k) \\
&\quad - \frac{p_1(t)x_1^3(t,\phi_k)}{P_1 + x_1^3(t,\phi_k)} \int_{-\infty}^{\widetilde{T}^{(k)}} k_{11}(u-t)y(u,\phi_k)\,du \\
&\quad - \frac{p_1(t)x_1^3(t,\phi_k)}{P_1 + x_1^3(t,\phi_k)} \int_{\widetilde{T}^{(k)}}^{s_q^{(k)}} k_{11}(u-t)y(u,\phi_k)\,du \\
&\quad - \frac{p_1(t)x_1^3(t,\phi_k)}{P_1 + x_1^3(t,\phi_k)} \int_{s_q^{(k)}}^{t} k_{11}(u-t)y(u,\phi_k)du \\
&\geq a(t)x_2(t,\phi_k) - b(t)x_1(t,\phi_k) - d(t)x_1^2(t,\phi_k) \\
&\quad - \frac{p_1(t)M_x x_1^2(t,\phi_k)}{P_1} H_1^{(k)} \int_{-\infty}^{\widetilde{T}^{(k)}-t} k_{11}(s)\,ds \\
&\quad - \frac{p_1(t)M_x x_1^2(t,\phi_k)}{P_1} M_y \int_{-\infty}^{s_q^{(k)}-t} k_{11}(s)\,ds \\
&\quad - \frac{p_1(t)_1 M_x x_1^2(t,\phi_k)}{P_1} \varepsilon_0 \int_{-\infty}^{0} k_{11}(s)\,ds \\
&= a(t)x_2(t,\phi_k) - b(t)x_1(t,\phi_k) \\
&\quad - \left[d(t) + \frac{2\varepsilon_0 p_1(t)M_x}{P_1}\right] x_1^2(t,\phi_k), \\
\frac{dx_2(t,\phi_k)}{dt} &\geq c(t)x_1(t,\phi_k) - \left[f(t) + \frac{2\varepsilon_0 p_2(t)M_x}{P_2}\right] x_2^2(t,\phi_k).
\end{aligned}$$

Let $(u_{1\varepsilon_0}, u_{2\varepsilon_0})$ be the solution of (4.5) with $\mu = \varepsilon_0$ and $(u_{1\mu_1}(s_q^{(k)} + \sigma^0), u_{2\mu_1}(s_q^{(k)} + \sigma^0)) = (x_1(s_q^{(k)} + \sigma^0), x_2(s_q^{(k)} + \sigma^0))$, then by the vector comparison theorem, we obtain

$$x_i(t,\phi_k) \geq u_{i\varepsilon_0}(t), \tag{4.25}$$

$i = 1, 2,\ t \in [s_q^{(k)} + \sigma^0, t_q^{(k)}]$. From $\lim_{q\to\infty} s_q^{(k)} = \infty$ and Lemmas 4.1 and 4.2, we obtain that for any k there is a $K_3^{(k)} > K_2^{(k)}$ such that

$$\rho_x \leq x_i(s_q^{(k)} + \sigma^0, \phi_k) \leq M_x$$

for any $q \geq K_3^{(k)},\ i = 1, 2$. For $\mu = \varepsilon_0$, Eq. (4.5) has a globally asymptotically stable positive ω-periodic solution $(x_{1\mu}^*(t), x_{2\mu}^*(t))$. From the periodicity of (4.5), we know that the periodic solution $(x_{1\mu}^*(t), x_{2\mu}^*(t))$ also is globally uniformly asymptotically stable. Hence, there exists a $T_7 > T$, and T_7 is independent of any k and q, such that

$$u_{i\varepsilon_0}(t) > x_{i\mu}^*(t) - \frac{\varepsilon_0}{2}$$

for all $t \geq T_7 + s_q^{(k)} + \sigma^0$ and $q \geq K_3^{(k)}$. Consequently, by (4.9),

$$u_{i\varepsilon_0}(t) > x_i^*(t) - \varepsilon_0,\ i = 1, 2 \tag{4.26}$$

for all $t \geq T_7 + s_q^{(k)} + \sigma^0$ and $q \geq K_3^{(k)}$. By (4.20), there is a $N_2 \geq N_1$ such that $t_q^{(k)} - s_q^{(k)} \geq 2T$ for all $k \geq N_2$ and $q \geq K_3^{(k)}$, where $T \geq T_7 + \sigma^0$. Hence, from (4.25) and (4.26), we obtain

$$x_i(t, \phi_k) \geq x_i^*(t) - \varepsilon_0 \tag{4.27}$$

for all $t \in [T + s_q^{(k)}, t_q^{(k)}]$, $k \geq N_2$ and $q \geq K_3^{(k)}$. Since, for any $t \in [T + s_q^{(k)} + \sigma_0 + \tau, t_q^{(k)}]$, $k \geq N_2$ and $q \geq K_3^{(k)}$, by (2.4), (4.23) and (4.24), we have

$$\begin{aligned}
\frac{dy(t, \phi_k)}{dt} &= y(t, \phi_k)\left[g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\sigma^0}^{0} k_{2i}(s)\frac{x_i^3(t+s, \phi_k)(1 - e^{-\gamma_i x_i(t+s,\phi_k)})}{P_i + x_i^3(t+s, \phi_k)}\,ds\right.\\
&\quad - \beta(t)y(t - \tau_1, \phi_k) - q(t)\left(\int_{-\infty}^{\widetilde{T}^{(k)}} k_{23}(u - t)y(u, \phi_k)du\right.\\
&\quad + \int_{\widetilde{T}^{(k)}}^{s_q^{(k)}} k_{23}(u - t)y(u, \phi_k)du\\
&\quad \left.\left. + \int_{s_q^{(k)}}^{t} k_{23}(u - t)y(u, \phi_k)du\right)^n - b_1(t)F(t, \phi_k)\right]\\
&\geq y(t, \phi_k)\left[g(t) + \sum_{i=1}^{2} h_i(t)\int_{-\sigma^0}^{0} k_{2i}(s)\frac{(x_i^*(t+s, \phi_k) - \varepsilon_0)^3}{P_i + (x_i^*(t+s, \phi_k) - \varepsilon_0)^3}\right.\\
&\quad \left.\times (1 - e^{-\gamma_i(x_i^*(t+s,\phi_k) - \varepsilon_0)})ds - \varepsilon_0\beta(t) - (2\varepsilon_0)^n q(t) - \theta b_1(t)\right]\\
&= y(t, \phi_k)\psi_{\varepsilon_0}(t).
\end{aligned}$$

Integrating from $T + s_q^{(k)} + \sigma^0 + \tau$ to $t_q^{(k)}$, for any $k \geq N_2$ and $q \geq K_3^{(k)}$, we obtain

$$y(t_q^{(k)}, \phi_k) \geq y(T + s_q^{(k)} + \sigma^0 + \tau, \phi_k)\exp\int_{T+s_q^{(k)}+\sigma^0+\tau}^{t_q^{(k)}} \psi_{\varepsilon_0}(t)dt.$$

Hence, it follows from (4.16) and (4.17) that

$$\frac{\varrho_y}{(k+1)^2} \geq \frac{\varrho_y}{(k+1)^2} \exp \int_{T+s_q^{(k)}+\sigma^0+\tau}^{t_q^{(k)}} \psi_{\varepsilon_0}(t)dt > \frac{\varrho_y}{(k+1)^2},$$

which leads to a contradiction. This completes the proof. □

Lemma 4.5. *Assume that* (3.1) *holds. Then there exists a positive constant* δ_F ($\delta_F < M_F$) *such that any solution* (x_1, x_2, y, F) *of system* (2.4) *with initial condition satisfies*

$$\liminf_{t\to\infty} F(t) \geq \delta_F. \tag{4.28}$$

Proof. According to Lemma 4.4 and the fourth equation in system (2.4), we note that there exists a $T_8 > 0$ such that

$$\dot{F} \geq -c_1^U F + d_1^L \delta_y \text{ for } t \geq T_8.$$

It is easy to follow from Lemma 2.3 that there exists a positive constant δ_F ($\delta_F < M_F$) such that any solution (x_1, x_2, y, F) of system (2.4) with initial condition satisfies $\liminf_{t\to\infty} F(t) \geq \delta_F$. This completes the proof. □

Proof of Theorem 3.1. This theorem now follows from Lemmas 4.1–4.5. □

Proof of Theorem 3.2. Assume

$$\int_0^\omega [g(t) + \sum_{i=1}^2 h_i(t) \int_{-\infty}^0 k_{2i}(s) \frac{x_i^{*^3}(t+s)(1-e^{-\gamma_i x_i^*(t+s)})}{P_i + x_i^{*^3}(t+s)} ds - \widehat{\theta} b_2(t)]dt \leq 0.$$

We will show that $\lim_{t\to\infty} y(t) = 0$. In fact, we know that for any given $0 < \varepsilon < 1$, there exist $\epsilon < \varepsilon$ and $\epsilon_0 > 0$ such that

$$\begin{aligned}\int_0^\omega \Bigg[g(t) + \sum_{i=1}^2 h_i(t) \int_{-\infty}^0 k_{2i}(s) \frac{(x_i^*(t+s)+\epsilon)^3}{P_i + (x_i^*(t+s)+\epsilon)^3} (1 - e^{-\gamma_i (x_i^*(t+s)+\epsilon)}) ds \\ + h(t)\epsilon - \widehat{\theta} b_2(t) - \frac{1}{2} q(t) l \epsilon^n \Bigg] dt < -\epsilon_0,\end{aligned} \tag{4.29}$$

where $l = \int_{-\infty}^0 k_{23}(s) \exp(\lambda^U s)\, ds$. Since

$$\begin{cases} \dot{x}_1 \leq a(t)x_2 - b(t)x_1 - d(t)x_1^2, \\ \dot{x}_2 \leq c(t)x_1 - f(t)x_2^2, \end{cases}$$

for all $t \geq 0$. Let $(\overline{x}_1(t), \overline{x}_2(t))$ be the solution of (2.6) with initial condition $\overline{x}_i(0) = x_i(0)$ $(i = 1, 2)$. By the vector comparison theorem, we obtain $x_i(t) \leq \overline{x}_i(t)$ $(i = 1, 2)$, $t \geq 0$. Obviously, by the global asymptotic stability of $x^*(t)$, there is a $\overline{T}$, for all $t \geq \overline{T}$, we have

$$x_i(t) \leq x_i^*(t) + \epsilon \quad (i = 1, 2). \tag{4.30}$$

Obviously, from (2.4), we derive

$$\dot{F(t)} \geq -c_1(t)F(t),$$

which implies

$$F(t) \geq F(0)e^{-c_1^U t} \quad \text{for } t > T_9. \tag{4.31}$$

Choose a constant $\tau_1^* > 0$ such that

$$\int_{-\infty}^{-\tau_1^*} k(s)\, ds < \epsilon, \tag{4.32}$$

$$\int_{-\tau_1^*}^{0} k_{23}(s) \exp(\lambda^U s)\, ds > \sqrt[n]{\frac{l}{2}}. \tag{4.33}$$

Let $\widehat{\theta} = F(0)$. For any $t \geq \overline{T} + \tau_1^*$, by (4.31)–(4.33), we have

$$\begin{aligned}
\dot{y} \leq y\Bigg[&g(t) + \sum_{i=1}^{2} h_i(t) \int_{-\infty}^{0} k_{2i}(s) \\
&\times \frac{(x_i(t+s))^3(1 - e^{-\gamma_i x_i(t+s)})}{P_i + (x_i(t+s))^3} ds - b_1(t)F(t)\Bigg] \\
\leq y\Bigg[&g(t) + \sum_{i=1}^{2} h_i(t) \int_{-\tau_1^*}^{0} k_{2i}(s) \\
&\times \frac{(x_i^*(t+s)+\epsilon)^3(1 - e^{-\gamma_i (x_i^*(t+s)+\epsilon)})}{P_i + (x_i^*(t+s)+\epsilon)^3} ds + h(t)\epsilon - \widehat{\theta} b_2(t)\Bigg] \\
\leq y&\lambda(t).
\end{aligned}$$

Hence, by (4.33), for any $t \geq t + s \geq \overline{T} + \tau_1^* + \tau$, we obtain

$$\begin{aligned}
\dot{y} &\leq y\left[\lambda(t) - q(t)\left(\int_{-\tau_1^*}^{0} k_{23}(s) y(t+s)\, ds\right)^n\right] \\
&\leq y\left[\lambda(t) - q(t)\left(\int_{-\tau_1^*}^{0} k_{23}(s) \exp(\lambda^U s)\, ds\right)^n y^n\right] \\
&\leq y\left[\lambda(t) - \frac{1}{2} l q(t) y^n\right].
\end{aligned}$$

If $y(t) \geq \epsilon$ for all $t \geq \overline{T} + 2\tau_1^* + \tau$, then we have

$$\dot{y} \leq y \left[\lambda(t) - \frac{1}{2} l q(t) \epsilon^n \right]. \tag{4.34}$$

Consequently, from (4.29), we get

$$y(t) \leq y(\overline{T} + 2\tau_1^* + \tau) \exp \int_{\overline{T}+2\tau_1^*+\tau}^{t} \left[\lambda(s) - \frac{1}{2} l q(s) \epsilon^2 \right] ds \to 0$$

as $t \to \infty$, which leads to a contradiction. Hence, there is a $t_1 \geq \overline{T} + 2\tau_1^* + \tau$ such that $y(t_1) < \epsilon$.

Let $M(\epsilon) = \max_{t \geq 0}\{|\lambda(t)| + \frac{1}{2} l q(t) \epsilon^n\}$. We know that $M(\epsilon)$ is bounded for $\epsilon \in [0, 1]$. We then show that

$$y(t) \leq \epsilon \exp(M(\epsilon)\omega), \quad t \geq t_1. \tag{4.35}$$

Otherwise, there are $t_3 > t_2 > t_1$ such that $y(t_3) > \epsilon \exp(M(\epsilon)\omega)$, $y(t_2) = \epsilon$ and $y(t) > \epsilon$ for all $t \in (t_2, t_3]$. Let $\theta^* \geq 0$ be an integer such that $t_3 \in (t_2 + \theta^*\omega, t_2 + (\theta^* + 1)\omega]$. From (4.34), we then obtain

$$\begin{aligned}
\epsilon \exp(M(\epsilon)\omega) &< y(t_3) \\
&\leq y(t_2) \exp \int_{t_2}^{t_3} \left[\lambda(t) - \frac{1}{2} l q(t) \epsilon^n \right] dt \\
&= \epsilon \exp \left(\int_{t_2}^{t_2+\theta^*\omega} + \int_{t_2+\theta^*\omega}^{t_3} \right) \left[\lambda(t) - \frac{1}{2} l q(t) \epsilon^n \right] dt \\
&< \epsilon \exp \left(\int_{t_2+\theta^*\omega}^{t_3} \left[\lambda(t) - \frac{1}{2} l q(t) \epsilon^n \right] dt \right) \\
&\leq \epsilon \exp(M(\epsilon)\omega).
\end{aligned}$$

This leads to a contradiction. Hence, inequality (4.35) holds. Further, by the arbitrariness of ϵ we obtain $y(t) \to 0$ as $t \to \infty$. This completes the proof. □

5. Discussion

In this paper, we have discussed a periodic coefficients predator-prey system with functional response and infinite delay, in which the predator is controlled by a feedback regulator, and the prey has a history that takes them through two stages, immature and mature. Some sufficient conditions which guarantee the permanence and extinction of the system have been obtained.

Our results could provide a useful insight into the conservation of beneficial animals, especially rare animals. As an example, we depict the case of the red-eared slider (Trachemys scripta) and Rana omeimontis, a rare species of frog found near Mountain Omei in Sichuan, China. The red-eared slider is a native of the Mississippi Valley area of the United States [22, 27]. Since 1970s, large numbers of red-eared sliders have been produced on turtle farms in the USA for the international pet trade. Red-eared turtles are traded as pet animals, and have been introduced to many countries. They are omnivorous and will eat insects, crayfish, shrimp, worms, snails, amphibians and small fish as well as aquatic plants, and hardly may be controlled by a natural enemy. Frogs are beneficial to humans because they eat so many insect pests. In our model, the variables $x_1(t)$ and $x_2(t)$ represent the density of tadpoles of the frog species and adult frogs, respectively. The variable $y(t)$ describes the density of the red-eared slider at time t. Ironically, although red-eared sliders have been widely introduced throughout the world, concern has been expressed regarding their future well-being over portions of their natural range within the Gulf ecosystem [9]. Chen and Lue [8] reported detrimental effects of red-eared sliders on aquatic vegetation. Now Trachemys scripta has been banned from import in China. So we incorporate the variable $F(t)$, which denotes a regulator for the red-eared slider population size at time t, in our model. The number of red-eared sliders removed at time t is $b_1(t)F(t)y(t)$. Also, the term $d_1(t)y(t-\tau_2)$ represents that the measure of the regulator F is strengthened at time t as the number of red-eared sliders increases at time $t-\tau_2$. We hope our results can be used to help protect beneficial animals found in their habitats.

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