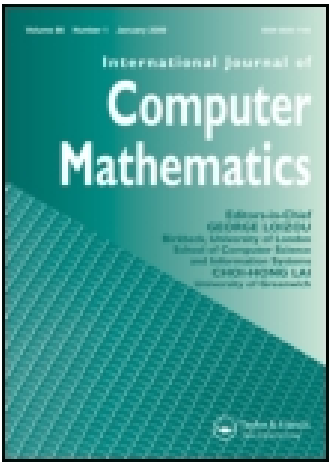

International Journal of Computer Mathematics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcom20

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Published online: 27 Sep 2008.

To cite this article: Zongmin Wu & Zhengchao Xiong (2010) Multivariate quasi-interpolation in $L^p(\mathbb{R}^d)$ with radial basis functions for scattered data, International Journal of Computer Mathematics, 87:3, 583-590, DOI: 10.1080/00207160802158702

To link to this article: http://dx.doi.org/10.1080/00207160802158702

International Journal of Computer Mathematics
Vol. 87, No. 3, March 2010, 583–590

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Multivariate quasi-interpolation in $L^p(\mathbb{R}^d)$ with radial basis functions for scattered data

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(*Received 26 April 2007; revised version received 26 September 2007; second revision received 30 March 2008; accepted 12 April 2008*)

In this paper, quasi-interpolation for scattered data was studied. On the basis of generalized quasi-interpolation for scattered data proposed in [Z.M. Wu and J.P. Liu, *Generalized strang-fix condition for scattered data quasi-interpolation*, Adv. Comput. Math. 23 (2005), pp. 201–214.], we have developed a new method to construct the kernel in the scheme by the linear combination of the scales, instead of the gridded shifts of the radial basis function. Compared with the kernel proposed in [Z.M. Wu and J.P. Liu, *Generalized strang-fix condition for scattered data quasi-interpolation*, Adv. Comput. Math. 23 (2005), pp. 201–214.], the new kernel, which is still a radial function, possesses the feature of polynomial reproducing property. This opens a possibility for us to propose a different technique by obtaining a higher approximation order of the convergence.

Keywords: quasi-interpolation; scattered data approximation; polynomial reproducing; radial basis function

2000 AMS Subject Classification: 41A63; 41A25; 65D10

1. Introduction

Quasi-interpolation method is an important approach in approximation theories and applications. For a uniform grid with spacing h and a set of given basis function $\Phi_{j,h}(x)$, $j \in Z^d$, the quasi-interpolation of a d-variate function f takes the standard form via the linear combination

$$\sum f(jh)\Phi_{j,h}(x) \sim f(x).$$

The most useful and simplest form is Schoenberg's model [14]

$$\sum f(jh)\Phi\left(\frac{x}{h} - j\right) \sim f(x), \quad x \in \mathbb{R}^d, \tag{1}$$

which has been used, for instance, with the Shannon sampling theorem and the B-spline series. Taking $\Phi_{j,h}(x)$ to be scaled shifts of a single kernel function Φ on $\mathbb{R}^d$ benefits the computing and

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ISSN 0020-7160 print/ISSN 1029-0265 online

DOI: 10.1080/00207160802158702
http://www.informaworld.com

storage of the kernel function in computer. In [12,17], the Strang-Fix condition is shown to be a necessary and sufficient condition for the convergence of the quasi-interpolation. In details, the quasi-interpolation possesses an approximation order of n if and only if the Fourier transform of the kernel Φ possesses zero points of order n at $2\pi j,\ j \neq 0$, and $\hat{\Phi} - 1$ possesses a zero point of order n at the origin. If the kernel does not satisfy the Strang-Fix condition that $\hat{\Phi} - 1$ possesses only a lower order of zero at origin, in [16], a new kernel Ψ is introduced, which is a linear combination of shifts of Φ to satisfy the condition. In [20], a full Strang-Fix condition is introduced, by which we are able to take the kernel to satisfy the Strang-Fix condition asymptotically. On the basis of the full Strang-Fix condition, the quasi-interpolation scheme can be constructed for multivariate scattered data as follows:

$$f^*(x) = \sum_j f(x_j)\Psi\left(\frac{x - x_j}{h^q}\right)\frac{\Delta_j}{h^{qd}}, \tag{2}$$

where the kernel Ψ is also constructed as in [16]. With this Scheme (2) and the kernel Ψ in [16], an optimal convergence order of the quasi-interpolation has been presented in [20].

In this paper, the quasi-interpolation of the form (2) for scattered data is also studied. We have developed a new technique to construct the kernel Ψ. With the new kernel, the convergence order is improved. Our technique is carried out through a linear combination of the scales, instead of the shifts of Φ in [20]. Compared with the method in [20], our technique has several advantages. First, the new kernel possesses a polynomial reproducing property under convolution operator. This makes it possible for us to propose a different technique, which do not base on the Strang-Fix condition, to prove the convergence of the quasi-interpolation of the form (2) and to obtain a higher approximation order. Second, the radial property of the kernel remains. If Φ is radial, Ψ is a radial function too. This is important while dealing with the problems in multidimensional space. Third, the construction technique is independent of the dimension, and we only need to solve a system of n linear equations in order to obtain the kernel, whereas in [20] n^d linear equations are required.

We refer to [1,5–8,11–13,15,16,18,21] and the references therein for more details of quasi-interpolation and related topics. The scheme is also used for numerical solution of partial differential equations (PDEs). For example, see [2,3,9,10,19].

The remainder of the paper is organized as follows. Section 2 is devoted to introduce some notation necessary for the study. In Section 3, we develop a new technique to construct a new kind of kernel. In the end, we complete the error analysis of the new quasi-interpolation and conclude the advantages of the new kind of kernel.

2. Preliminaries

In this section, we list some notation used throughout this paper. For $x^T = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$$

stands for its Euclidean norm. For $\alpha \in \mathbb{Z}_+^d := \{\beta \in \mathbb{Z}^d : \beta \geq 0\}$, we set $\alpha! := \alpha_1! \cdots \alpha_d!$ and $|\alpha| := \sum_{k=1}^d \alpha_k$.

π_k^d stands for the space of all d variables polynomials whose degree does not exceed k. Denote the monomial by

$$V_\alpha(x) := \frac{x^\alpha}{\alpha!}.$$

For a discrete set $X \subset \mathbb{R}^d$, we define the density of X by

$$h := h(X) := \sup_{x\in\mathbb{R}^d} \inf_{x_j\in X} |x - x_j|.$$

For a given function f, denote the scale of f with h of power q by

$$f_{h,q}(x) := h^{-qd} f(h^{-q}x)$$

and the partial derivatives of f by

$$(D^\alpha f)(x) := \frac{\partial^{\alpha_1+\cdots+\alpha_d} f}{\partial x_1^{\alpha_1}\cdots \partial x_d^{\alpha_d}}.$$

Our approximands in this paper are chosen from the Sobolev space

$$W_p^s(\mathbb{R}^d), \quad 1 \le p \le \infty, \quad s \in Z_+,$$

of all functions that the function and its derivatives, whose order does not exceed s, are all in $L^p(\mathbb{R}^d)$. We denote the homogeneous sth order L^p-Sobolev semi-norm by $|\cdot|_{s,p}$, *i.e.*

$$|f|_{s,p} := \sum_{|\alpha|=s} \|D^\alpha f\|_p.$$

Given a sequence $\{\Delta_j\}$ as weights of a quadrature scheme based on discrete set X, $\{\Delta_j\}$ possesses an approximation order of v, if for any function $f \in W_p^v(\mathbb{R}^d)$

$$\left\| \sum_{x_j\in X} f(x_j)\Delta_j - \int_{\mathbb{R}^d} f(x)\mathrm{d}x \right\|_p \le c\cdot |f|_{v,p}\cdot h^v, \quad (1 \le p \le \infty) \tag{3}$$

as h tends to 0, where c is a constant independent of f and h. Details of the method for constructing $\{\Delta_j\}$ based on X can be found in Appendix B of [20]. In the rest of this paper, we assume that $\{\Delta_j\}$ possesses an approximation order of v.

3. Main result

The aim of this paper is to improve the approximation order of the quasi-interpolation scheme of the form (2). The key point is to replace the kernel introduced in [20] for scheme (2) with a new kernel. Based on this, we propose a new method, which is different from the Strang-Fix conditions, to prove the convergence of the quasi-interpolation. The following part of this section is divided into two steps. First, we approximate f by the convolution operator

$$L_{h,q}(f)(x) := (\Psi_{h,q} * f)(x) := \int_{\mathbb{R}^d} f(y)\Psi\left(\frac{x-y}{h^q}\right)\frac{1}{h^{qd}}\,\mathrm{d}y,$$

where q is a parameter and Ψ the kernel satisfying certain conditions that will be specified later. Second, we apply the quadrature scheme to discretize the convolution based on scattered data X and obtain the quasi-interpolation

$$Q_{h,q}(f)(x) := \sum_{x_j\in X} f(x_j)\Psi_{h,q}(x-x_j)\Delta_j := \sum_{x_j\in X} f(x_j)\Psi\left(\frac{x-x_j}{h^q}\right)\frac{\Delta_j}{h^{qd}}.$$

To estimate the error, we divide $f - Q_{h,q}(f)$ into two parts: $f - L_{h,q}(f)$ and $L_{h,q}(f) - Q_{h,q}(f)$, which will be separately estimated. We need the following lemma to estimate $f - L_{h,q}(f)$.

LEMMA 3.1 *Let $\Psi(x) \in L^1(\mathbb{R}^d)$ and for some positive integer s, let $\Psi * V_\alpha = V_\alpha$ when $|\alpha| \le s$. Then $\Psi_{h,q} * V_\alpha = V_\alpha$ when $|\alpha| \le s$ for all $h > 0$ and $q \in \mathbb{R}$.*

Proof By hypothesis that $\Psi * V_\alpha = V_\alpha$, we have

$$\begin{aligned}
\Psi_{h,q} * V_\alpha &= \int \Psi_{h,q}(y) V_\alpha(x-y)\,\mathrm{d}y \\
&= \int \Psi(y) V_\alpha(x - h^q y)\,\mathrm{d}y \\
&= h^{q|\alpha|} \int \Psi(y) V_\alpha(h^{-q}x - y)\,\mathrm{d}y \\
&= h^{q|\alpha|} V_\alpha(h^{-q}x) = V_\alpha(x).
\end{aligned}$$ ■

THEOREM 3.2 *If $\Psi \in L^1(\mathbb{R}^d)$ and, for some positive integer s, it satisfies*

(a) $V_\alpha \cdot \Psi \in L^1(\mathbb{R}^d)$ *when* $|\alpha| \le s$,
(b) $\Psi * V_\alpha = V_\alpha$ *when* $|\alpha| < s$,

then for $f \in W_p^s(\mathbb{R}^d)$, we have

$$\|L_{h,q}(f) - f\|_p \le \text{const} \cdot h^{sq}.$$

Proof For $f \in W_p^s(\mathbb{R}^d)$, we fix $x \in \mathbb{R}^d$ and let P be the Taylor expansion of degree $s-1$ of the function f at x:

$$P(y) = \sum_{|\alpha|<s} (D^\alpha f)(x) V_\alpha(y-x).$$

By Taylor's theorem, the remainder $R = f - P$ of the expansion is

$$R(y) = \sum_{|\alpha|=s} (D^\alpha f)(\xi_y) V_\alpha(y-x),$$

where $\xi_y = x + t(y-x)$, $(0 \le t \le 1)$. Then

$$\begin{aligned}
\Psi_{h,q} * f - f &= \Psi_{h,q} * (f - P) + (\Psi_{h,q} * P - P) + (P - f) \\
&= \Psi_{h,q} * R + (\Psi_{h,q} * P - P) - R.
\end{aligned}$$

Note that $R(x) = 0$, and according to Lemma 3.1, we get

$$\begin{aligned}
|(\Psi_{h,q} * f - f)(x)| = |(\Psi_{h,q} * R)(x)| &= \left|\int_{\mathbb{R}^d} \Psi_{h,q}(x-y) R(y)\,\mathrm{d}y\right| \\
&\le \int_{\mathbb{R}^d} |\Psi_{h,q}(x-y)| \sum_{|\alpha|=s} |(D^\alpha f)(\xi_y)\|V_\alpha(y-x)|\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} |\Psi(y)| \sum_{|\alpha|=s} |D^\alpha f(\xi'_{x,y})\|V_\alpha(-h^q y)|\,\mathrm{d}y \\
&= h^{sq} \cdot \sum_{|\alpha|=s} \int_{\mathbb{R}^d} |D^\alpha f(\xi'_{x,y})||\Psi(y) V_\alpha(y)|\,\mathrm{d}y,
\end{aligned}$$

where $\xi'_{x,y} = x - ty$. Then, a direct calculation using the generalized Minkowski inequality (for the case $1 \le p < \infty$) yields

$$\|\Psi_{h,q} * f - f\|_p \le \text{const} \cdot h^{sq} \cdot \sum_{|\alpha|=s} \|D^\alpha f\|_p \int_{\mathbb{R}^d} |\Psi(y)V_\alpha(y)| \,\mathrm{d}y.$$

By condition (a) of the theorem, the integrals in the last expression are finite and their sum should be bounded by a constant. ■

Remark 1 Condition (a) indicates that the kernel Ψ must possess a polynomial decay at least of order $s + d$ in ∞, and condition (b) requires the polynomial reproducing properties under convolution operator. The idea of convolution operator with the polynomial reproducing property comes from [4]. We generalize it to the scale of the kernel with parameters h^q.

Theorem 3.2 provides a convolution operator with high approximation power. Based on it, we improve the approximation order of the quasi-interpolation scheme of the form (2).

THEOREM 3.3 *Let $\Psi \in C^u(\mathbb{R}^d)$ and, for some positive integer s, let it satisfy*

(a) $\Psi \cdot V_\alpha \in L^1(\mathbb{R}^d)$ *when* $|\alpha| \le s$,
(b) $\Psi * V_\alpha = V_\alpha$ *when* $|\alpha| < s$.

If $\{\Delta_j\}$ are weights of a quadrature scheme with approximation order of v $(u \ge v)$, then for $f \in W_p^s(\mathbb{R}^d)$ $(s \ge v)$, we can put $q = v/(s+v)$ to construct a quasi-interpolation with error bound

$$\left\| Q_{h,q}(f) - f \right\|_p \le O(h^{sv/(s+v)}). \tag{4}$$

Proof As introduced above,

$$f - Q_{h,q}(f) = (f - L_{h,q}(f)) + (L_{h,q}(f) - Q_{h,q}(f)).$$

By Theorem 3.2, we have

$$\|f - L_{h,q}(f)\|_p = \|f(x) - (\Psi_{h,q} * f)(x)\|_p \le \text{const} \cdot h^{sq}.$$

By formula (3),

$$\begin{aligned}
\left\| Q_{h,q}(f) - L_{h,q}(f) \right\|_p &= \left\| \sum_j \Psi\left(\frac{x - x_j}{h^q}\right) f(x_j) \frac{\Delta_j}{h^{qd}} - \Psi_{h,q} * f \right\|_p \\
&= \left\| \sum_j \Psi\left(\frac{x - x_j}{h^q}\right) f(x_j) \frac{\Delta_j}{h^{qd}} - \int \frac{1}{h^{qd}} \Psi(\frac{x-y}{h^q}) f(y)\,\mathrm{d}y \right\|_p \\
&\le O(h^{v(1-q)}).
\end{aligned}$$

Let $q = v/(s+v)$, *i.e.* $sq = v(1-q)$, we obtain the optimal error bounds $O(h^{sv/(s+v)})$. ■

Theorem 3.3 shows that we can get a new quasi-interpolation if there exists a kernel with the polynomial reproducing property under convolution operators. The rest of the paper will concentrate on the construction of such kind of kernel. At first, we introduce the notion of 'admissible coefficients'.

Definition 3.4 *The coefficients* $\{a_i, t_i : t_i \neq 0\}$ *are termed admissible for* $\pi_{s-1}(\mathbb{R})$ *if they satisfy the following condition*:

$$p(0) = \sum_{i=1}^{m} a_i p(t_i), \quad (\forall p \in \pi_{s-1}(\mathbb{R})). \tag{5}$$

We note that there exist admissible coefficients when $m \geq s$. For example, if $t_1, t_2, \ldots, t_s$ are s different non-zero points in $\mathbb{R}$, we could find the appropriate coefficients a_i by using the Lagrange interpolation formula with t_i as nodes. Another simple way to obtain admissible coefficients is to use the 'forward difference functional'. See [4] for more details.

Now let us turn to the construction of the kernel. We use the linear combination of the scales instead of the gridded shifts of the function Φ, which is used in [20], to construct the new kernel. This technique is first proposed in [4]. The new kernel has the following property.

Lemma 3.5 *Let* $\Phi(x) \in L^1(\mathbb{R}^d)$ *and* $\int_{\mathbb{R}^d} \Phi(x)\mathrm{d}x = 1$, *and assume that for some positive integer* s, $\Phi \cdot V_\alpha \in L^1(\mathbb{R}^d)$ *when* $|\alpha| \leq s$. *The new kernel is constructed as follows*:

$$\Psi(x) = \sum_{i=1}^{m} a_i \Phi\left(\frac{x}{t_i}\right) t_i^{-d}, \quad (x \in \mathbb{R}^d), \tag{6}$$

where $\{a_i, t_i : t_i \neq 0\}$ *are admissible. Then* $\Psi * V_\alpha = V_\alpha$ *for* $|\alpha| < s$, *and* $\Psi * V_\alpha - V_\alpha$ *is a constant function for* $|\alpha| = s$.

Proof If $|\alpha| < s$,

$$\begin{aligned}(\Psi * V_\alpha)(x) &= \sum_{i=1}^{m} a_i \int \Phi\left(\frac{y}{t_i}\right) V_\alpha(x-y) t_i^{-d}\,\mathrm{d}y = \int \Phi(y) \sum_{i=1}^{m} a_i V_\alpha(x - t_i y)\,\mathrm{d}y \\ &= \int \Phi(y) V_\alpha(x)\,\mathrm{d}y = V_\alpha(x).\end{aligned}$$

If $|\alpha| = s$, then we use the binomial theorem to write

$$\begin{aligned}(\Psi * V_\alpha)(x) &= \int \Phi(y) \sum_{i=1}^{m} \sum_{\beta \leq \alpha} a_i t_i^{|\beta|} V_\beta(-y) V_{\alpha-\beta}(x)\,\mathrm{d}y \\ &= \int \Phi(y) \sum_{\beta \leq \alpha} A_\beta V_\beta(-y) V_{\alpha-\beta}(x)\,\mathrm{d}y,\end{aligned}$$

where we have put $A_\beta \triangleq \sum_{i=1}^{m} a_i t_i^{|\beta|}$. By Equation (5), $A_\beta = 1$ if $\beta = 0$ and $A_\beta = 0$ if $0 < |\beta| < s$. Hence, only the terms corresponding to $\beta = 0$ and $\beta = \alpha$ remain in the sum above, and we have

$$\begin{aligned}(\Psi * V_\alpha)(x) &= \int \Phi(y)[V_\alpha(x) + A_\alpha V_\alpha(-y)]\mathrm{d}y \\ &= V_\alpha(x) + A_\alpha \int \Phi(y) V_\alpha(-y)\,\mathrm{d}y. \qquad \blacksquare\end{aligned}$$

We notice that the key point of the construction is to find the admissible coefficients. As discussed above, it can be obtained by solving a linear system of equation only of order n to obtain the kernel, whereas in [20], the construction needs n^d equations . Moreover, we keep the radial property of the kernel; that is, if Φ is a radial function, the new kernel Ψ is a radial function too. Furthermore, the construction in [20] depends on the dimension d; however our construction is dimension independent.

The given kernel $\Phi(x) \in L^1(\mathbb{R}^d)$ can be normalized by multiplying a constant factor c, where $c^{-1} = \int \Phi(x)\mathrm{d}x$. The normalized kernel satisfies $\int \Phi(x)\mathrm{d}x = 1$. We also denote it by Φ. Lemma 3.5 states that if the coefficients $\{a_i, t_i\}$ are admissible, the new kernel Ψ constructed by Equation (6) satisfies condition (b) in Theorem 3.3. After all, we obtain the main results of the paper.

THEOREM 3.6 *Let $\Phi \in C^u(\mathbb{R}^d)$ satisfy the following conditions*:

(a) $\int_{\mathbb{R}^d} \Phi(x)\mathrm{d}x \neq 0$,
(b) $|\Phi| < o(1+|x|)^{-s-d}$ *as* $x \to \infty$.

Assume that $\{\Delta_j\}$ are weights of a quadrature scheme that possesses an approximation order v as above, where $u \geq v$. Then for every $f \in W_p^s(\mathbb{R}^d)$ $(s \geq v)$, we can assign $q = v/(s+v)$ to construct a quasi-interpolation with the error bound

$$\left\| \sum f(x_j)\Psi\left(\frac{x-x_j}{h^q}\right)\frac{\Delta_j}{h^{qd}} - f(x)\right\|_p \leq O(h^{sv/(s+v)}),$$

where Ψ is constructed by Equation (6).

Proof Based on assumption (a), assume that $\int \Phi(x)\,\mathrm{d}x = 1$. Otherwise, it can be normalized by multiplying a constant factor c, where $c^{-1} = \int \Phi(x)\mathrm{d}x$. Assumption (b) implies that for all $|\alpha| \leq s$, $\Phi \cdot V_\alpha \in L^1(\mathbb{R}^d)$. By Lemma 3.5, we can construct a new kind of kernel Ψ as Equation (6) through admissible coefficients $\{a_i, t_i\}$, and the new kernel possesses the polynomial reproducing property under convolution operators. Therefore, Theorem 3.6 is the direct consequence of Theorem 3.3. ■

Remark 2 Recalling Theorem 3 of [20], the conditions for the kernel and approximated functions are related with their Fourier transform. But in Theorem 3.6, the conditions have nothing to do with the Fourier transform and can be easily verified. Moreover, the approximation order is increased from $sv/(2s+v)$ to $sv/(s+v)$. The new kind of kernel Ψ is much simpler and much stronger in approximation.

Acknowledgements

The authors thank the referees for their valuable comments, both for mathematics and English formulation, that helped to complete the final version. This study was supported by NSFC No. 10125102, NBRPC 973-2006CB303102.

References

[1] M.D. Buhmann, N. Dyn, and D. Levin, *On quasi-interpolation radial basis functions with scattered centers*, Construct. Approx. 11 (1995), pp. 239–254.
[2] R.H. Chen and Z.M. Wu, *Applying multiquadratic quasi-interpolation to solve Burgers' equation*, Appl. Math. Comput. 172 (2006), pp. 472–484.
[3] ———, *Solving hyperbolic conservation laws using multiquadric quasi-interpolation*, Numer. Methods Partial Differ. Equ. 22 (2006), pp. 776–796.
[4] E.W. Cheney, W.A. Light, and Y. Xu, *On kernels and approximation orders, Approximation Theory* (Memphis, TN), Lecture Notes in Pure and Applied Mathematics, Vol. 138, Dekker, New York, 1992, pp. 227–242.
[5] C. de Boor, *Quasi-interpolants and approximation power of multivariate splines*, in *Computation of Curves and Surfaces*, M. Gasca and C.A. Micchelli, eds., Nato Advanced Science Institute Series C:Mathematical and Physical Sciences, Vol. 307, Kluwer Academic, Dordrecht, 1990, pp. 313–345.

[6] ———, *Approximation order without quasi-interpolants*, in *Approximation Theory*, C. Chui, L. Schumaker, and J. Ward, eds., Vol. VII, Academic Press, Boston, MA, 1993, pp. 1–18.
[7] N. Dyn, I.R.H. Jackson, D. Levin, and A. Ron, *On multivariate approximation by integer translates of a basis function*, Israel J. Math. 78 (1992), pp. 95–130.
[8] N. Dyn and A. Ron, *Radial basis functions: from gridded centers to scattered centers*, Proc. London Math. Soc. 71 (1995), pp. 76–108.
[9] G. Fasshauer, *Solving differential equations with radial basis functions: multilevel methods and smoothing*, Adv. Comp. Math. 11 (1999), pp. 139–159.
[10] Y.C. Hon and Z.M. Wu, *A quasi-interpolation method for solving stiff ordinary differential equations*, Int. J. Numer. Methods Eng. 48 (2000), pp. 1187–1197.
[11] R.Q. Jia, *Partition of unity and density: a counterexample*, Constr. Approx. 13 (1997), pp. 251–260.
[12] R.Q. Jia and J.J. Lei, *A new version of the strang-fix conditions*, J. Approx. Theory 74 (1993), pp. 221–225.
[13] X. Li and C. Micchelli, *Approximation by radial basis functions and neural networks*, Numer. Algorithms 25(2000), pp. 241–262.
[14] C. Rabut, *An introduction to Schoenbergs approximation*, Comput. Math. Appl. 24 (1992), pp. 149–175.
[15] R. Schaback, *Remarks on meshless local construction of surfaces*, in *Proceedings of the IMA Mathematics of Surfaces Conference*, Cambridge, 2000.
[16] R. Schaback and Z.M. Wu, *Construction techniques for highly accurate quasi-interpolation operators*, J. Approx. Theory 91 (1997), pp. 320–331.
[17] G. Strang and G. Fix, *A Fourier analysis of the finite-element method*, in *Constructive Aspects of Functional Analysis*, G. Geymonat, ed., CIME, Rome, 1973, pp. 793–840.
[18] Z.M. Wu, *Compactly supported positive definite radial basis functions and the strang-fix condition*, Appl. Math. Comput. 84 (1997), pp. 115–124.
[19] ———, *Dynamical knot shape parameter setting for simulating shock wave by using multi-quadric quasi-interpolation*, Eng. Anal. Bound. Elem. 29(2005), pp. 354–358.
[20] Z.M. Wu and J.P. Liu, *Generalized strang-fix condition for scattered data quasi-interpolation*, Adv. Comput. Math. 23 (2005), pp. 201–214.
[21] J. Yoon, *Approximation in $L_p(R^d)$ from a space spanned by the scattered shifts of radial basis functions*, Construct. Approx. 17 (2001), pp. 227–247.