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Distance transformation for the numerical evaluation of nearly singular integrals on triangular elements

Y. Miao, W. Li, J.H. Lv*, X.H. Long

School of Civil Engineering and Mechanics, Huazhong University of Science and Technology, Wuhan 430074, China

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ABSTRACT

The accurate numerical evaluation of nearly singular boundary integrals is a major concerned issue in the implementation of the boundary element method (BEM). In this paper, the previous distance transformation method is extended into triangular elements both in polar and Cartesian coordinate systems. A new simple and efficient method using an approximate nearly singular point is proposed to deal with the case when the nearly singular point is located outside the element. In general, the results obtained using the polar coordinate system are superior to that in the Cartesian coordinate system when the nearly singular point is located inside the element. Besides, the accuracy of the results is influenced by the locations of the nearly singular point due to the special topology of triangular elements. However, when the nearly singular point is located outside the element, both the polar and Cartesian coordinate systems can get acceptable results.

Contents

1. Introduction

Nearly singular integrals are involved in many boundary element method (BEM) analyses of engineering problems, such as sensitivity problems [1], the unknowns around crack tips [2], contact problems [3] and thin structures [4]. Accurate and efficient evaluation of nearly singular integrals with various kernels is crucial for successful implementation of the boundary type numerical methods. Near singularities arise when a source point is very close to but not on the integration elements. Although these integrals are actually regular in nature, they cannot be evaluated accurately by the standard Gaussian quadrature, because the denominator, the distance between the source and the field point, is close to zero but not zero, resulting in a drastically spiked integrand.

Effective computation of nearly singular integrals has received intensive attention in recent years and various numerical techniques have been proposed to remove the near singularities [5–8], among which the most popular approaches are based on the various nonlinear transformations, such as cubic polynomial transformation [9], coordinate optimization transformation [10], rational transformation [11], sigmoidal transformation [12], the

* Corresponding author. Tel.: +86 27 87540172; fax: +86 27 87542231.
E-mail address: kaixinhust@gmail.com (J.H. Lv).

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PART method [13], exponential transformation [14,15], sinh transformation [16–18] and the distance transformation [19–23]. The key aspect of the transformation methodology is to cluster more Gaussian points towards the 'nearly singular point' (the projection point from the source point to the element), which benefit from the strategies for singular integrals [24,25].

Among these techniques, the distance transformation method proposed by Ma seems to be a more promising method for dealing with different orders of nearly singular integrals. This method employs an asymptotic distance function via Taylor expansion and has been successfully applied into the evaluation of nearly singular integrals in 2D and 3D BEM. However, only quadrilateral element is concerned in 3D BEM, and when the source point is located outside of the element, conventional triangle subdivision method is complicated and cannot obtain satisfactory result. In this paper, we extend the distance transformation method both with the polar and Cartesian coordinate system into triangular elements. Besides, a new strategy using an approximate nearly singular point is proposed to deal with the case when the nearly singular point is located outside the element. Numerical examples prove the simplicity and efficiency of the presented method.

This paper is organized as follows. The general forms of nearly singular integrals are described in Section 2. The distance transformation method based on the triangular element is briefly reviewed in Section 3. Section 4 presents how to split into subtriangles and a new simple and efficient method is proposed to deal with the case when the nearly singular point is located outside the element. Numerical examples are given in Section 5 to verify the efficiency and accuracy of the presented method. The paper ends with conclusions in Section 6.

2. General descriptions

Considering 3D potential problems in the domain Ω enclosed by boundary Γ, the two boundary integrals concerned in the present work are written in the usual forms in terms of the potential u and the flux q on the boundary as follows:

$$c(\mathbf{y})u(\mathbf{y}) = \int_\Gamma q(\mathbf{x})u^*(\mathbf{x},\mathbf{y})\,d\Gamma(\mathbf{x}) - \int_\Gamma u(\mathbf{x})q^*(\mathbf{x},\mathbf{y})\,d\Gamma(\mathbf{x}) \tag{1}$$

$$c(\mathbf{y})u_k(\mathbf{y}) = \int_\Gamma q(\mathbf{x})u_k^*(\mathbf{x},\mathbf{y})\,d\Gamma(\mathbf{x}) - \int_\Gamma u(\mathbf{x})q_k^*(\mathbf{x},\mathbf{y})\,d\Gamma(\mathbf{x}) \tag{2}$$

where $\mathbf{y}$ and $\mathbf{x}$ are the source and the field points, respectively. c is a coefficient depending on the smoothness of the boundary at $\mathbf{y}$. $u^*(\mathbf{x},\mathbf{y})$ represents the fundamental solution for 3D potential problems:

$$u^*(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\frac{1}{r} \tag{3}$$

and $u_k^*(\mathbf{x},\mathbf{y})$, $q^*(\mathbf{x},\mathbf{y})$ and $q_k^*(\mathbf{x},\mathbf{y})$ are the derived fundamental solutions:

$$u_k^*(\mathbf{x},\mathbf{y}) = \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial x_k}, \quad q^*(\mathbf{x},\mathbf{y}) = \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}}, \quad q_k^*(\mathbf{x},\mathbf{y}) = \frac{\partial q^*(\mathbf{x},\mathbf{y})}{\partial x_k} \tag{4}$$

where r denotes the Euclidean distance between the source and the field point and $\mathbf{n}$ is the unit outward normal on the boundary.

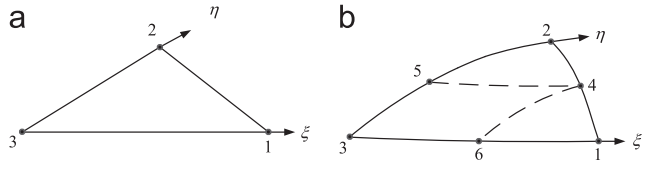

Fig. 1. Node distributions for triangular elements: (a) planar triangular element; (b) curved triangular element.

When the source point $\mathbf{y}$ moves towards the boundary, the boundary integrals in Eqs. (1) and (2) become nearly singular with different orders, namely, the nearly weak singularity with kernel u^*, the nearly strong singularity with the kernels q^* and u_k^* and the nearly hyper-singularity with the kernels q_k^*. After the boundary discretization and the coordinate transformation to the local system, the boundary integrals can be generalized into the following form:

$$\mathrm{I} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{f(\xi_1,\xi_2)}{r^\chi}\,d\xi_1\,d\xi_2 \tag{5}$$

where χ denotes the orders of singularities, $\chi = 1, 2, 3$ and $f(\xi_1,\xi_2)$ is a well-behaved function, consisting of the shape function, Jacobian and coefficients from the derivation of the kernels.

In this paper, both the planar and curved triangular boundary elements are concerned. The distributions of element nodes are shown in Fig. 1. The corresponding expressions of the shape functions and Jacobian can be found in Ref. [26].

3. Distance transformation method

In this section, we will briefly review the distance transformation method based on the triangular element. First, the projection from the source point $\mathbf{y}$ to the triangular element is taken as shown in Fig. 2, where $\mathbf{x}^c$ is the projection point, herein we call it 'the nearly singular point' and r_0 is the minimum distance from the source point to the triangular element. By employing the one-order Taylor expansion in the neighborhood of the nearly singular point $\mathbf{x}^c$, we have

$$\begin{aligned} x_k - y_k &= x_k - x_k^c + x_k^c - y_k \\ &= \frac{\partial x_k}{\partial \xi_1}|_{\xi_1=c_1}(\xi_1 - c_1) + \frac{\partial x_k}{\partial \xi_2}|_{\xi_2=c_2}(\xi_2 - c_2) \\ &\quad + r_0 n_k(c_1,c_2) + O(\rho^2) \\ &= \rho A_k(\theta) + r_0 n_k(c_1,c_2) + O(\rho^2) \end{aligned} \tag{6}$$

where

$$A_k(\theta) = \frac{\partial x_k}{\partial \xi_1}|_{\xi_1=c_1}\cos\theta + \frac{\partial x_k}{\partial \xi_2}|_{\xi_2=c_2}\sin\theta \tag{7}$$

and

$$\xi_1 = c_1 + \rho\cos\theta, \quad \xi_2 = c_2 + \rho\sin\theta \tag{8}$$

where (c_1,c_2) are the local coordinates of the nearly singular point $\mathbf{x}^c$ as shown in Fig. 3.

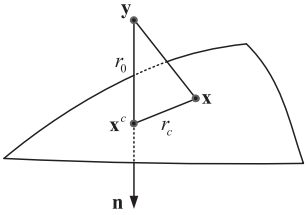

Fig. 2. The projection from the source point $\mathbf{y}$ to the triangular element.

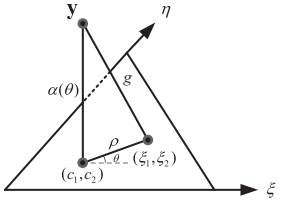

Fig. 3. The distance function g in the local parametric system.

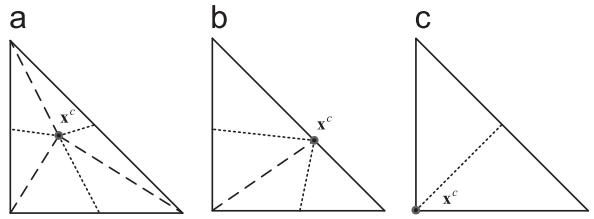

Fig. 4. Subdivision into sub-triangles when the nearly singular point is located (a) inside the element; (b) on one side; (c) on one vertex.

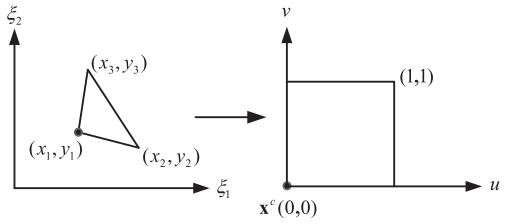

Fig. 5. Mapping each sub-triangle into a quadrangle.

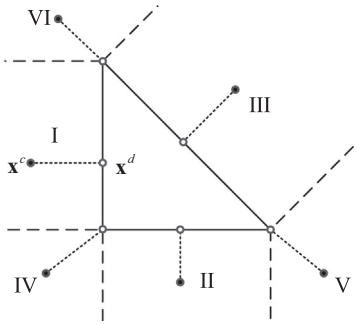

Fig. 6. The position of the approximate nearly singular point $\mathbf{x}^d$.

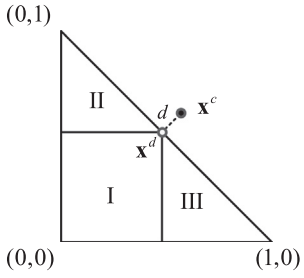

Fig. 7. The parametric plane of triangular elements.

With the relation of Eq. (6), the real distance between the source and field points can be written in the following asymptotic form:

$$r^2(\rho,\theta) = r_0^2 + \rho^2 A^2(\theta) + 2r_0 A_k(\theta) n_k(c_1,c_2) + O(\rho^3)$$
$$= A^2(\theta) g^2(\rho,\theta) + O(\rho^3) \tag{9}$$

where $g(\rho,\theta)$ is the distance function in the local parametric system defined as

$$g(\rho,\theta) = \sqrt{\alpha^2(\theta) + \rho^2} \tag{10}$$

with

$$A(\theta) = \sqrt{A_k(\theta) A_k(\theta)} \tag{11}$$

$$\alpha(\theta) = \frac{r_0}{A(\theta)} \tag{12}$$

In the polar coordinate system, the near singularities appear only in the radial direction and the corresponding transformation pairs are expressed as follows:

$$\eta(\rho,\theta) = \log[g(\rho,\theta)] \tag{13}$$

$$\rho(\eta) = \sqrt{\exp(2\eta) - \alpha^2} \tag{14}$$

The distance function can also be defined in the local Cartesian coordinate system as follows:

$$g_i(\xi_i) = \sqrt{\alpha_i^2 + (\xi_i - c_i)^2} \quad (i = 1,2 \text{ with no summation}) \tag{15}$$

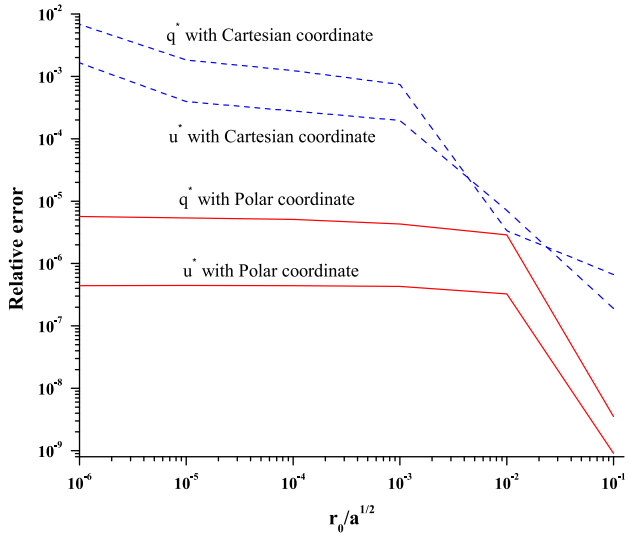

Fig. 8. Comparisons for the results at point (0.2,0.2) between the polar and Cartesian coordinate systems with kernels u^* and q^*.

Table 1
The relative errors of various nearly singular integrals for different values of $r_0/a^{1/2}$ based on the transformations in the polar coordinate system.

$\mathbf{x}^c$	Kernel	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
(0.2,0.2)	u^*	9.0531E−10	3.2445E−07	4.3159E−07	4.4262E−07	4.4369E−07	4.4332E−07
	q^*	3.5335E−09	2.8839E−06	4.2896E−06	5.0969E−06	5.3766E−06	5.6708E−06
	$1/r^3$	1.4933E−09	6.4514E−06	6.1072E−06	7.3583E−06	6.8012E−06	6.2472E−06
(0.2,0.6)	u^*	1.3095E−07	1.1051E−07	1.0846E−07	1.0827E−07	1.0828E−07	1.0880E−07
	q^*	1.1672E−08	6.9344E−09	5.5024E−09	4.8001E−09	4.9304E−09	4.2103E−09
	$1/r^3$	1.1076E−08	9.3607E−11	7.3690E−09	2.7917E−10	7.1086E−09	1.1189E−09
(0.6,0.2)	u^*	1.2863E−07	1.0691E−07	1.0506E−07	1.0487E−07	1.0488E−07	1.0540E−07
	q^*	1.9467E−09	1.9320E−09	2.1197E−09	3.6542E−09	2.7778E−09	3.8972E−09
	$1/r^3$	6.6010E−09	6.2672E−09	2.0479E−09	1.3018E−08	8.7797E−10	8.8676E−09

where

$$\alpha_1 = r_0\left\{\frac{\partial x_k}{\partial \xi_1}\frac{\partial x_k}{\partial \xi_1}|_{\xi_1=c_1}\right\}^{-1/2}, \quad \alpha_2 = r_0\left\{\frac{\partial x_k}{\partial \xi_2}\frac{\partial x_k}{\partial \xi_2}|_{\xi_2=c_2}\right\}^{-1/2} \quad (k=1,2,3) \tag{16}$$

and the corresponding transformation pairs in each direction are written as follows:

$$\eta_i(\xi_i) = \log[g_i(\xi_i) + (\xi_i - c_i)] \tag{17}$$

$$\xi_i(\eta_i) = \frac{1}{2}\left[\exp(\eta_i) - \alpha_i^2 \exp(-\eta_i)\right] + c_i \tag{18}$$

After the two kinds of distance transformation, the nearly singularities of the integrand resulted from r can be damped out by the distance function g and the Jacobian of the nonlinear transformation.

4. Subdivision into sub-triangles

Before performing the nonlinear transformations, the triangle should be split into sub-triangles depending on the position of the nearly singular point in the parametric plane. We categorise into two cases: case (i) the nearly singular point is located inside the element or on the sides/vertices of the element; case (ii) the nearly singular point is located outside the element. The subdivision method is described as follows.

4.1. Subdivision for case (i)

When the nearly singular point is located inside the element, we initially split the triangle into three sub-triangles at the nearly singular point by drawing lines from the nearly singular point to each vertex of the triangle. If the sub-triangles contain angles greater than $2\pi/3$, poor result may be obtained [27]. Therefore, we split these sub-triangles again by drawing lines from the nearly singular point to the midpoints of each side to ensure that every angle is less than $2\pi/3$. Similar results can be obtained when the nearly singular point is located on one of the sides or vertices as shown in Fig. 4.

When using transformations in the local Cartesian coordinate system, each sub-triangle should be mapped into a quadrangle as shown in Fig. 5. Here, we use the mapping method proposed by Qin [23]. The relation between ξ_1, ξ_2 and u, v is defined as

$$\begin{cases} \xi_1 = x_1 + (x_2 - x_1)u + (x_3 - x_2)uv \\ \xi_2 = y_1 + (y_2 - y_1)u + (y_3 - y_2)uv \end{cases} \tag{19}$$

The Jacobian of the transformation is uS_Δ, and

$$S_\Delta = |x_2y_3 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 - x_2y_1| \tag{20}$$

The advantage of this mapping method is that the nearly singular point is always located at the origin of the (u, v) coordinate system and no more split is necessary. Therefore, the transformations can be applied directly in u and v direction.

4.2. Subdivision for case (ii)

When the nearly singular point is located outside the element, the result obtained by the conventional subdivision method is not quite accurate. An alternative method is proposed here using an approximate nearly singular point $\mathbf{x}^d$. The point $\mathbf{x}^d$ is the nearest point from the nearly singular point $\mathbf{x}^c$ to the element in the parametric plane. When $\mathbf{x}^c$ is located in regions I–III, $\mathbf{x}^d$ is on the sides of the triangle, and on the vertices with regions IV–VI as shown in Fig. 6. Then we use $\mathbf{x}^d$ to subdivide the triangle element as mentioned in Section 4.1 instead of the nearly singular point $\mathbf{x}^c$, and the corresponding transformations can be applied in a straightforward fashion without introducing extra subdivision method. Therefore, this method is quite simple for programming and accurate results can be obtained as demonstrated in Section 5.

5. Numerical examples

In this section, numerical examples for planar and curved triangular elements are presented to verify the accuracy and efficiency of presented method. The relative distance from the nearly singular point to the element is given in terms of $r_0/a^{1/2}$, where r_0 is the minimum distance as shown in Fig. 2 and a stands for the area of the element. $r_0/a^{1/2}$ varies from 10^{-1} to 10^{-6},

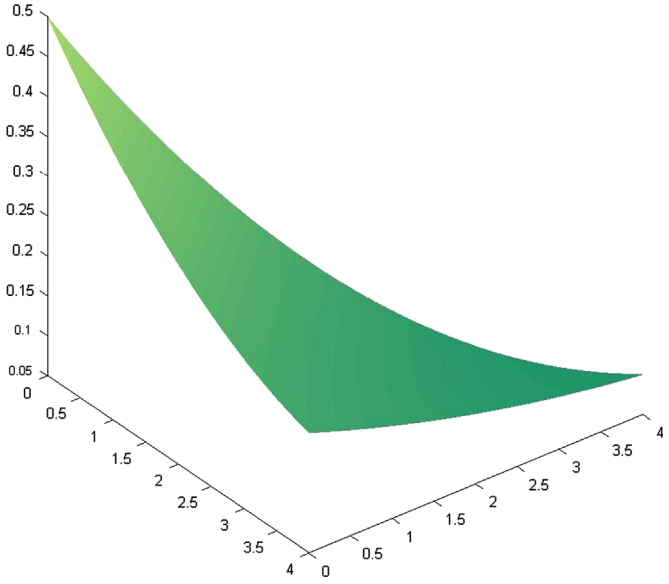

Fig. 9. Curved triangular element.

Table 2
The relative errors of various nearly singular integrals for different values of $r_0/a^{1/2}$ based on the transformations in the polar and Cartesian coordinate systems.

Kernel	Method	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
u^*	Refer	0.08999601	0.09179458	0.09181360	0.09181379	0.09181379	0.09181379
	Polar	8.0828E−06	6.4172E−10	1.5775E−06	1.2544E−05	5.2014E−05	5.8150E−05
	Cartesian	1.1258E−08	7.7396E−09	3.7503E−06	9.8371E−06	9.0062E−05	7.0457E−05
q^*	Refer	0.01636314	0.00172724	0.00017282	0.00001728	0.00000173	0.00000017
	Polar	3.9633E−05	2.2238E−07	1.4959E−05	1.0585E−04	9.5053E−05	9.3695E−04
	Cartesian	1.1964E−07	2.6722E−06	3.2466E−06	1.3233E−04	5.8362E−04	8.6919E−04
$1/r^3$	Refer	8.65886976	9.64996493	9.66135890	9.66147302	9.66147416	9.66147417
	Polar	1.2571E−04	1.7554E−06	3.5059E−05	2.1446E−04	7.2377E−04	2.8336E−03
	Cartesian	6.1719E−07	1.6065E−05	−1.4080E−04	8.2862E−04	7.6092E−04	4.7337E−03

which is thought to be enough for general computational applications. For the purpose of error estimation, the relative error is defined as follows:

$$\text{error} = \left|\frac{I_{num} - I_{ref}}{I_{ref}}\right| \tag{21}$$

where the subscripts *num* and *ref* refer to the numerical and reference solutions, respectively. The reference solutions are obtained by subdivision method with enough sub-elements and denoted as 'Refer' in the tables. The relative errors of the results obtained by transformations in polar or Cartesian coordinate systems are denoted by 'Polar' and 'Cartesian'. For each sub-triangle, 10×10 Gaussian points are always used for the convenience of comparisons.

5.1. Example for planar triangular element

The first example considers nearly singular integrals on a planar triangular boundary element with the node coordinates of (1,0,0), (0,1,0) and (0,0,0). When the source point is located inside the element, the parametric plane is parted into three regions to investigate the influence of the position of the nearly singular point as shown in Fig. 7. Three source point belong to each region are chose to compute the nearly singular integrals with kernels u^*, q^* and $1/r^3$, respectively. Table 1 presents the relative errors of various nearly singular integrals for different values of $r_0/a^{1/2}$ based on the transformations in the polar coordinate system. It can be found that the results for u^* retain the precision of 10^{-7} no matter where the nearly singular point is located. However, for kernels q^* and $1/r^3$, the results when the nearly singular point is placed in regions II and III are more accurate than that in region I by several orders of magnitude. This is because that when the nearly singular point $\mathbf{x}^c$ is located at (0.2,0.2), the angles of sub-triangles may be very close to $2\pi/3$, even with re-splitting. Therefore, there is an obvious different in relative error for region I.

The comparisons for the results at point (0.2,0.2) between the polar and Cartesian coordinate systems with kernels u^* and q^* are plotted in Fig. 8. It can be observed that the results obtained in the polar coordinate system are more accurate and stable than that in the Cartesian coordinate system. The results obtained in the Cartesian coordinate system are sensitive to the values of $r_0/a^{1/2}$.

Now we consider the case when the nearly singular point is located outside the element. The nearly singular point is set at $(c_1, c_2) = (0.6, 0.6)$ and the approximate nearly singular point $\mathbf{x}^d$ used to subdivide the element is (0.5,0.5) as shown in Fig. 7. The results using different transformations are given in Table 2. It can be seen that both the polar and Cartesian coordinate systems can get acceptable results.

5.2. Example for curved triangular element

The second example is computed over a curved triangular element with the node coordinates of (4.0,0.0,0.2), (0.0,4.0,0.0), (0.0,0.0,0.5), (2.0,2.0,0.1), (0.0,2.0,0.2), (2.0,0.0,0.3) as shown in Fig. 9. The same three nearly singular points belong to different regions are chose. The results obtained with transformations in the polar coordinate system for kernels u^*, q^* and $1/r^3$ are presented in Table 3. The same conclusions can be observed as the planar triangular element. Comparisons for results at point (0.2,0.6) between the polar and Cartesian coordinate systems are shown in Fig. 10. It also can be found that the results obtained in the polar coordinate system are superior to that in the Cartesian coordinate system. Besides, the results for q^* in the polar coordinate system are better than that for u^* regardless of the singular order. When the nearly singular point is located at $(c_1, c_2) = (0.6, 0.6)$, The results using different transformations are given in Table 4. Similarly, both the polar and Cartesian coordinate systems can get acceptable results.

Now we place the nearly singular point at $(0.5 + d/\sqrt{2}, 0.5 + d/\sqrt{2})$, where d is the distance between $\mathbf{x}^c$ and $\mathbf{x}^d$ as shown in Fig. 7. As $d/\sqrt{2}$ varies from 0.01 to 0.1, the relative errors for different transformations with kernel u^* are given in Fig. 11. It can be easily seen that the relative errors increase with decreasing values of d, due to the fact that the nearly singular point is closer to element, and the relative errors with both the polar and Cartesian coordinate systems ground to a 10^{-5} tolerance or less. Therefore,

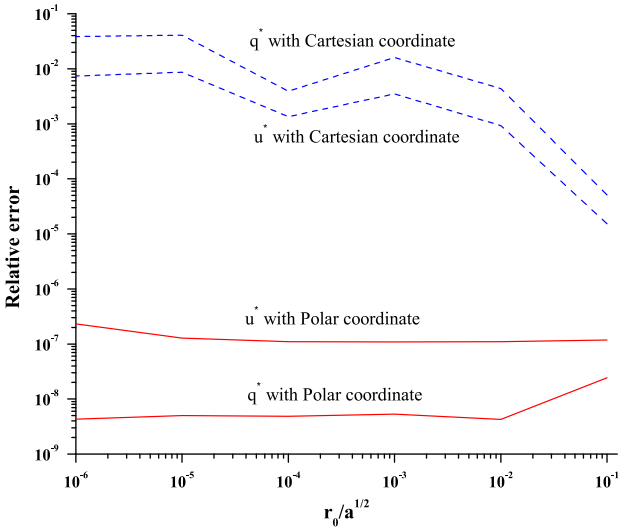

Fig. 10. Comparisons for the results at point (0.2,0.6) between the polar and Cartesian coordinate systems with kernels u^* and q^*.

Table 3
The relative errors of various nearly singular integrals for different values of $r_0/a^{1/2}$ based on the transformations in the polar coordinate system.

$\mathbf{x}^c$	Kernel	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
(0.2,0.2)	u^*	2.8730E−08	3.0421E−07	4.1035E−07	4.1750E−07	3.8300E−07	2.1247E−07
	q^*	8.0638E−08	2.7776E−06	4.1987E−06	4.9982E−06	5.2660E−06	5.5703E−06
	$1/r^3$	1.7789E−07	6.3712E−06	5.9550E−06	7.3926E−06	6.5706E−06	6.2310E−06
(0.2,0.6)	u^*	1.1783E−07	1.1022E−07	1.0840E−07	1.0961E−07	1.2797E−07	2.3062E−07
	q^*	2.4275E−08	4.2521E−09	5.3125E−09	4.8460E−09	5.0046E−09	4.2901E−09
	$1/r^3$	6.2595E−08	1.0753E−08	5.8966E−09	5.2349E−10	7.1564E−09	6.6653E−10
(0.6,0.2)	u^*	1.1526E−07	1.0661E−07	1.0499E−07	1.0622E−07	1.2481E−07	2.2899E−07
	q^*	3.4380E−08	8.8925E−10	1.5949E−09	3.1925E−09	2.3582E−09	3.3780E−09
	$1/r^3$	8.0588E−08	5.0173E−09	3.7773E−09	1.1980E−08	3.4966E−10	8.2967E−09

Table 4
The relative errors of various nearly singular integrals for different values of $r_0/a^{1/2}$ based on the transformations in the polar and Cartesian coordinate systems.

Kernel	Method	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
u^*	Refer	0.36330679	0.36907234	0.36899138	0.36897639	0.36897482	0.36897466
	Polar	8.0582E−06	1.5047E−09	1.5672E−06	1.2463E−05	5.1831E−05	5.8215E−05
	Cartesian	1.1534E−08	1.2117E−08	3.7022E−06	9.9388E−06	9.0108E−05	6.8914E−05
q^*	Refer	0.06629108	0.00694126	0.00069392	0.00006939	0.00000694	0.00000069
	Polar	3.9728E−05	2.2510E−07	1.4892E−05	1.0556E−04	9.5087E−05	9.3539E−04
	Cartesian	1.2137E−07	2.6806E−06	3.3030E−06	1.3167E−04	5.8358E−04	8.6883E−04
$1/r^3$	Refer	2.19352721	2.41466329	2.41415873	2.41385071	2.41381734	2.41381397
	Polar	1.2594E−04	1.7403E−06	3.4815E−05	2.1378E−04	7.2088E−04	2.8282E−03
	Cartesian	6.2054E−07	1.5978E−05	1.4033E−04	8.2524E−04	7.6104E−04	4.7241E−03

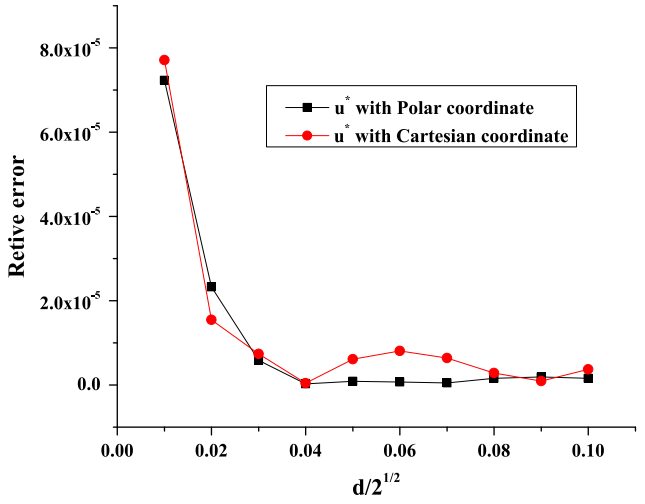

Fig. 11. Relative errors for the polar and Cartesian coordinate systems with kernel u^* corresponding to various values of d.

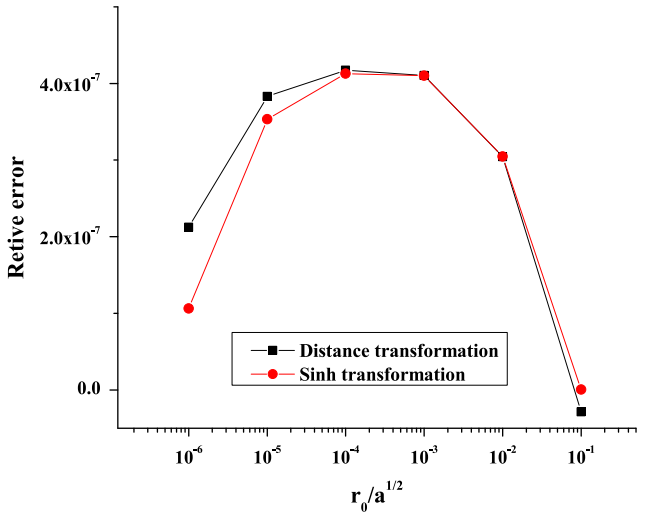

Fig. 12. Comparisons for the results at point (0.2,0.2) between the distance and sinh transformations with kernels u^*.

the new method proposed to deal with the case when the nearly singular point is located outside the element is stable and efficient.

Among the nonlinear variable transformations, the sinh transformation [16,17] is a competitive method. The comparisons with the sinh method are performed for kernel at point (0.2,0.2) and the results based on the polar coordinate system are plotted in Fig. 12. It can be seen that both the distance transformation and the sinh transformation methods can get results with high accuracy, and the results of the sinh transformation is slightly better than the distance transformation.

6. Conclusions

In this paper, the previous distance transformation method is extended into the triangular elements in the polar and Cartesian coordinate system, and a simple and efficient method to deal with the case when the nearly singular point is located outside the element is proposed. Some meaningful conclusions have been obtained as follows:

(1) When the nearly singular point is located inside the element, the results obtained using the transformations in the polar coordinate system is more accurate and stable than that in the Cartesian coordinate.
(2) When the nearly singular point is located inside the element, the locations of the nearly singular point may influence the accuracy of the result due to its special topology, which is different from the quadrilateral element.
(3) The new method proposed to deal with the case when the nearly singular point is located outside the element is proved to be simple and efficient, regardless of the locations of the nearly singular point.

For the accurate numerical evaluation of the nearly hypersingular integrals, the results can be improved by making use of the properties of the derived fundamental solutions, which will be the future work.

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