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Application of Exp-function method to a KdV equation with variable coefficients

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Abstract

In this Letter, the Exp-function method is used to obtain generalized solitonary solutions and periodic solutions of a KdV equation with variable coefficients. It is shown that the Exp-function method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving nonlinear evolution equations with variable coefficients in mathematical physics.

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Keywords: Exp-function method; Generalized solitonary solutions; Periodic solutions; Nonlinear evolution equations; KdV equation with variable coefficients

1. Introduction

The investigation of exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented, such as inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], sine–cosine method [5], homogenous balance method [6], homotopy perturbation method [7–9], variational method [10–13], asymptotic methods [14], non-perturbative methods [15], Adomian Pade approximation [16], tanh-function method [17–21], algebraic method [22–25], Jacobi elliptic function expansion method [26–28], F-expansion method [29–32] and so on.

Recently, He and Wu [33] proposed a straightforward and concise method, called Exp-function method, to obtain generalized solitonary solutions and periodic solutions of NLEEs. The solution procedure of this method, by the help of Matlab or Mathematica, is of utter simplicity and this method can be easily extended to all kinds of NLEEs.

The present Letter is motivated by the desire to extend the Exp-function method to a KdV equation with variable coefficients, which reads

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} = 0, \tag{1}$$

where $\alpha(t)$ and $\beta(t)$ are arbitrary functions of t. Eq. (1) is well known as a model equation describing the propagation of weakly nonlinear and weakly dispersive waves in inhomogenous media. Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions of Eq. (1) can be found in [34–36].

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2. Application to the KdV equation with variable coefficients

Using the transformation

$$u = U(\eta), \quad \eta = kx + \int \tau(t)\,\mathrm{d}t, \tag{2}$$

where k is a constant, $\tau(t)$ is an integrable function of t to be determined later, Eq. (1) becomes

$$\tau(t)U' + k\alpha(t)UU' + k^3\beta(t)U''' = 0, \tag{3}$$

where prime denotes the differential with respect to η.

According to the Exp-function method [33], we assume that the solution of Eq. (3) can be expressed in the following form

$$U(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{4}$$

where c, d, p and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. Eq. (4) can be re-written in an alternative form [39] as follows

$$U(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d}\exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q}\exp(-q\eta)}. \tag{5}$$

In order to determine values of c and p, we balance the linear term of highest order in Eq. (3) with the highest order nonlinear term [33,39]. By simple calculation, we have

$$U''' = \frac{c_1 \exp[(7p + c)\eta] + \cdots}{c_2 \exp[8p\eta] + \cdots} \tag{6}$$

and

$$UU' = \frac{c_3 \exp[(p + 2c)\eta] + \cdots}{c_4 \exp[3p\eta] + \cdots} = \frac{c_3 \exp[2(3p + c)\eta] + \cdots}{c_4 \exp[8p\eta] + \cdots}, \tag{7}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs. (6) and (7), we have

$$7p + c = 2(3p + c), \tag{8}$$

which leads to the result

$$p = c. \tag{9}$$

Similarly to determine values of d and q, we balance the linear term of lowest order in Eq. (3)

$$U''' = \frac{\cdots + d_1 \exp[-(7q + d)\eta]}{\cdots + d_2 \exp[(-8q)\eta]} \tag{10}$$

and

$$UU' = \frac{\cdots + d_3 \exp[-(q + 2d)\eta]}{\cdots + d_4 \exp[(-3q)\eta]} = \frac{\cdots + d_3 \exp[-2(3q + d)\eta]}{\cdots + d_4 \exp[(-8q)\eta]}, \tag{11}$$

where d_i are determined coefficients only for simplicity. Balancing lowest order of Exp-function in Eqs. (10) and (11), we have

$$-(7q + d) = -2(3q + d), \tag{12}$$

which leads to the result

$$q = d. \tag{13}$$

2.1. Case 1: $p = c = 1$, $d = q = 1$

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d [33,39]. For simplicity, we set $p = c = 1$ and $d = q = 1$, the trial function, Eq. (5) becomes

$$U(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{14}$$

Substituting Eq. (14) into Eq. (3), and using Mathematica, equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for a_1, a_0, a_{-1}, b_1, b_0, b_{-1} and $\tau(t)$ as follows:

$$\begin{aligned}
&b_1(a_1b_0-a_0b_1)\{a_1k\alpha(t)+b_1[k^3\beta(t)+\tau(t)]\}=0,\\
&[-a_0^2b_1^2-2a_1a_{-1}b_1^2+a_1^2(b_0^2+2b_1b_{-1})]k\alpha(t)+2b_1\{2(-a_1b_0^2+a_0b_0b_1-2a_{-1}b_1^2+2a_1b_1b_{-1})k^3\beta(t)\\
&\quad+[-b_1(a_0b_0+a_{-1}b_1)+a_1(b_0^2+b_1b_{-1})]\tau(t)\}=0,\\
&[-a_0^2b_0b_1+a_1b_0(-2a_{-1}b_1+3a_1b_{-1})+a_0(a_1b_0^2-3a_{-1}b_1^2+2a_1b_1b_{-1})]k\alpha(t)+[b_1(-a_0b_0^2-5a_{-1}b_0b_1+23a_0b_1b_{-1})\\
&\quad+a_1b_0(b_0^2-18b_1b_{-1})]k^3\beta(t)+[-b_1(a_0b_0^2+5a_{-1}b_0b_1+a_0b_1b_{-1})+a_1b_0(b_0^2+6b_1b_{-1})]\tau(t)=0,\\
&2(a_1b_{-1}-a_{-1}b_1)\{2(a_0b_0+a_1b_{-1}+a_{-1}b_1)k\alpha(t)+2[(b_0^2-8b_1b_{-1})k^3\beta(t)+(b_0^2+b_1b_{-1})\tau(t)]\}=0,\\
&[a_0^2b_0b_{-1}+a_{-1}b_0(2a_1b_{-1}-3a_{-1}b_1)+a_0(3a_1b_{-1}^2-a_{-1}b_0^2-2a_{-1}b_1b_{-1})]k\alpha(t)+[b_{-1}(a_0b_0^2+5a_1b_0b_{-1}-23a_0b_1b_{-1})\\
&\quad-a_{-1}b_0(b_0^2-18b_1b_{-1})]k^3\beta(t)+[b_{-1}(a_0b_0^2+5a_1b_0b_{-1}+a_0b_1b_{-1})-a_{-1}b_0(b_0^2+6b_1b_{-1})]\tau(t)=0,\\
&[a_0^2b_{-1}^2+2a_1a_{-1}b_{-1}^2-a_{-1}^2(b_0^2+2b_1b_{-1})]k\alpha(t)-2b_{-1}\{2(-a_{-1}b_0^2+a_0b_0b_{-1}-2a_1b_{-1}^2+2a_{-1}b_1b_{-1})k^3\beta(t)\\
&\quad+[-b_{-1}(a_0b_0+a_1b_{-1})+a_{-1}(b_0^2+b_1b_{-1})]\tau(t)\}=0,\\
&b_{-1}(a_0b_{-1}-a_{-1}b_0)\{a_{-1}k\alpha(t)+b_{-1}[k^3\beta(t)+\tau(t)]\}=0.
\end{aligned}$$

Solving the system of algebraic equations with the aid of Mathematica, we obtain

$$a_1=a_1,\qquad a_0=\frac{b_0(a_1+6b_1\delta_1k^2)}{b_1},\qquad a_{-1}=\frac{a_1b_0^2}{4b_1^2},\qquad b_1=b_1,\qquad b_0=b_0, \tag{15}$$

$$b_{-1}=\frac{b_0^2}{4b_1},\qquad \beta(t)=\delta_1\alpha(t),\qquad \tau(t)=-\frac{k(a_1+b_1\delta_1k^2)}{b_1}\alpha(t),\qquad \delta_1=\text{const.} \tag{16}$$

Inserting Eqs. (15) and (16) into (14) yields the following generalized solitonary solution (see Figs. 1–3) of Eq. (1)

$$\begin{aligned}
u&=\frac{a_1e^{[kx-\frac{k(a_1+b_1\delta_1k^2)}{b_1}\int\alpha(t)\,dt]}+\frac{b_0(a_1+6b_1\delta_1k^2)}{b_1}+\frac{a_1b_0^2}{4b_1^2}e^{[-kx+\frac{k(a_1+b_1\delta_1k^2)}{b_1}\int\alpha(t)\,dt]}}{b_1e^{[kx-\frac{k(a_1+b_1\delta_1k^2)}{b_1}\int\alpha(t)\,dt]}+b_0+\frac{b_0^2}{4b_1}e^{[-kx+\frac{k(a_1+b_1\delta_1k^2)}{b_1}\int\alpha(t)\,dt]}}\\
&=\frac{a_1}{b_1}+\frac{6b_0\delta_1k^2}{b_1e^{[kx-\frac{k(a_1+b_1\delta_1k^2)}{b_1}\int\alpha(t)\,dt]}+b_0+\frac{b_0^2}{4b_1}e^{[-kx+\frac{k(a_1+b_1\delta_1k^2)}{b_1}\int\alpha(t)\,dt]}}.
\end{aligned} \tag{17}$$

To compare our result, Eq. (17), with that in open literature, we write down Taogetusang and Sirendaoerji's solution [34], which reads

$$u=g_0+g_1\,\text{sech}^2\Big[px-\int\big(g_0p\alpha(t)+4p^3\beta(t)\big)\,dt\Big]\quad\Big(\alpha(t)=\frac{12p^2\beta(t)}{g_1}\Big). \tag{18}$$

We re-write Eq. (18) in the form

$$u=g_0+\frac{4g_1}{e^{[2px-\frac{2p(3g_0+g_1)}{3}\int\alpha(t)\,dt]}+2+e^{[-2px+\frac{2p(3g_0+g_1)}{3}\int\alpha(t)\,dt]}}. \tag{19}$$

If we choose $a_1=g_0$, $b_1=1$, $b_0=2$, $k=2p$ and $\delta_1=\frac{g_1}{12p^2}$, our solution, Eq. (17), turns out to be Taogetusang and Sirendaoerji's solution as expressed in Eq. (19). In addition, the solution (32) in [35] can be easily recovered from Eq. (17).

When k is an imaginary number, the obtained solitonary solution can be converted into periodic solution [33,39]. We write $k=iK$. Using the transformation

$$e^{[kx-\frac{k(a_1+b_1\delta_1k^2)}{b_1}\int\alpha(t)\,dt]}=\cos\Big[Kx-\frac{K(a_1-b_1\delta_1K^2)}{b_1}\int\alpha(t)\,dt\Big]+i\sin\Big[Kx-\frac{K(a_1-b_1\delta_1K^2)}{b_1}\int\alpha(t)\,dt\Big]$$

and

$$e^{[-kx+\frac{k(a_1+b_1\delta_1k^2)}{b_1}\int\alpha(t)\,dt]}=\cos\Big[Kx-\frac{K(a_1-b_1\delta_1K^2)}{b_1}\int\alpha(t)\,dt\Big]-i\sin\Big[Kx-\frac{K(a_1-b_1\delta_1K^2)}{b_1}\int\alpha(t)\,dt\Big],$$

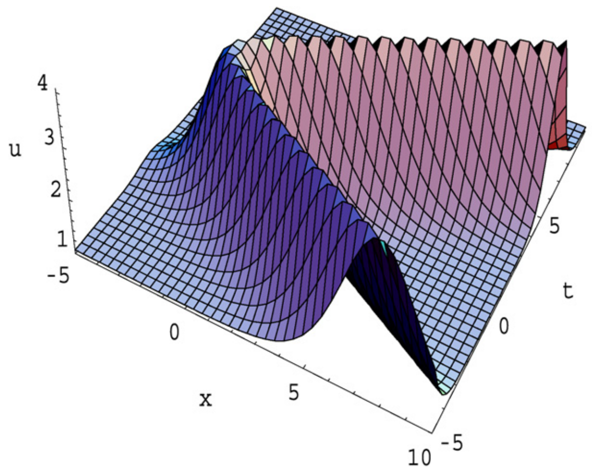

Fig. 1. Solution (17) is shown at $a_1 = b_1 = b_0 = k = \delta_1 = 1$, $\alpha(t) = \tanh(t)$.

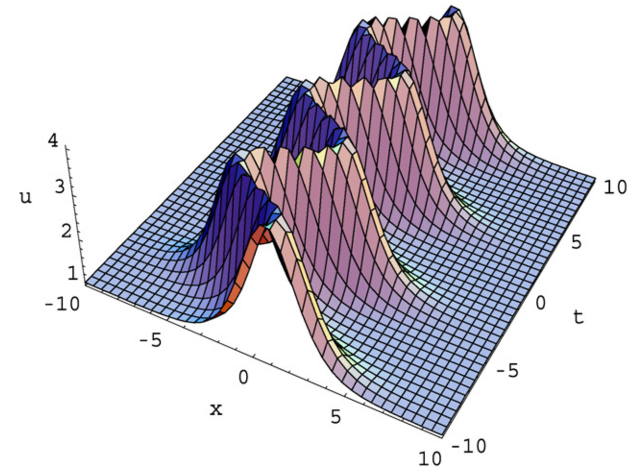

Fig. 2. Solution (17) is shown at $a_1 = b_1 = b_0 = k = \delta_1 = 1$, $\alpha(t) = \sin(t)$.

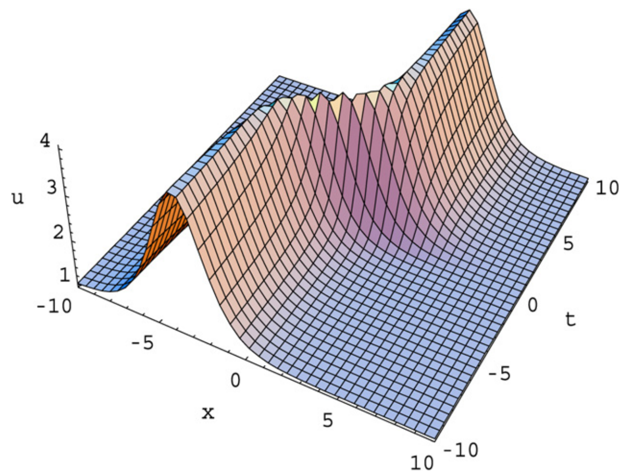

Fig. 3. Solution (17) is shown at $a_1 = b_1 = b_0 = k = \delta_1 = 1$, $\alpha(t) = \frac{1}{1+t^2}$.

then Eq. (17) becomes

$$u = \frac{a_1}{b_1} - \frac{6b_0\delta_1 K^2}{(b_1 + \frac{b_0^2}{4b_1})\cos\left[Kx - \frac{K(a_1 - b_1\delta_1 K^2)}{b_1}\int \alpha(t)\,\mathrm{d}t\right] + b_0 + \mathrm{i}\mu \sin\left[Kx - \frac{K(a_1 - b_1\delta_1 K^2)}{b_1}\int \alpha(t)\,\mathrm{d}t\right]}, \tag{20}$$

where $\mu = b_1 - \frac{b_0^2}{4b_1}$.

If we search for a periodic solution or compact-like solution, the imaginary part in Eq. (20) must be zero [33,39], that requires that

$$\mu = b_1 - \frac{b_0^2}{4b_1} = 0. \tag{21}$$

From Eq. (21) we obtain

$$b_0 = \pm 2b_1. \tag{22}$$

Substituting Eq. (22) into (20) yields two periodic solutions

$$u = \frac{a_1}{b_1} - \frac{6\delta_1 K^2}{\cos\left[Kx - \frac{K(a_1 - b_1\delta_1 K^2)}{b_1}\int \alpha(t)\,\mathrm{d}t\right] + 1} \tag{23}$$

and

$$u = \frac{a_1}{b_1} + \frac{6\delta_1 K^2}{\cos\left[Kx - \frac{K(a_1 - b_1\delta_1 K^2)}{b_1}\int \alpha(t)\,\mathrm{d}t\right] - 1}. \tag{24}$$

2.2. Case 2: $p = c = 2$, $d = q = 2$

As mentioned above the values of c and d can be freely chosen, we set $p = c = 2$ and $d = q = 2$, then the trial function, Eq. (5) becomes

$$U(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{b_2\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\eta)}. \tag{25}$$

There are some free parameters in Eq. (25), we set $b_2 = 1$, $b_1 = 0$ and $b_{-1} = 0$ for simplicity, the trial function, Eq. (25) is simplified as follows

$$U(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-2}\exp(-2\eta)}. \tag{26}$$

By the same manipulation as illustrated above, we obtain

$$a_2 = -\frac{\delta_2(\delta_3 + k^3)}{k}, \qquad a_1 = a_1, \qquad a_0 = \frac{a_1^2(\delta_3 - 11k^3)}{72\delta_2 k^5}, \qquad a_{-1} = \frac{a_1^3}{144\delta_2^2 k^4}, \qquad a_{-2} = -\frac{a_1^4(\delta_3 + k^3)}{20736\delta_2^3 k^9}, \tag{27}$$

$$b_0 = -\frac{a_1^2}{72\delta_2^2 k^4}, \qquad b_{-2} = \frac{a_1^4}{20736\delta_2^4 k^8}, \qquad \beta(t) = \delta_2\alpha(t), \qquad \tau(t) = \delta_2\delta_3\alpha(t), \qquad \delta_i = \text{const} \quad (i = 2, 3). \tag{28}$$

Substituting Eqs. (27) and (28) into (26) yields the following solution

$$u = -\frac{\delta_2(\delta_3 + k^3)}{k} + \frac{a_1}{e^{[kx + \delta_2\delta_3\int\alpha(t)\,dt]} + \frac{a_1}{6\delta_2 k^2} + \frac{a_1^2}{144\delta_2^2 k^4}e^{[-kx - \delta_2\delta_3\int\alpha(t)\,dt]}}. \tag{29}$$

It should be noted that if we set $a_1 = -\frac{\delta_2(\delta_3 + k^3)}{k}$, $b_1 = 1$, $b_0 = \frac{a_1}{6\delta_2 k^2}$ and $\delta_1 = \delta_2$ in Eq. (17), we can recover the solution (29).

2.3. Case 3: $p = c = 2$, $d = q = 1$

We consider the case $p = c = 2$ and $d = q = 1$, Eq. (5) can be expressed as

$$U(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_2\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{30}$$

There are some free parameters in Eq. (30), we set $b_2 = 1$ for simplicity. By the same manipulation as illustrated above, we obtain

$$a_2 = -\frac{\delta_4(\delta_5 + k^3)}{k}, \qquad a_{-1} = \frac{(\delta_5 + k^3)(a_1 k + b_1\delta_4\delta_5 + b_1\delta_4 k^3)^2(a_1 k + b_1\delta_4\delta_5 - 5b_1\delta_4 k^3)}{864\delta_4^2 k^{10}}, \tag{31}$$

$$a_1 = a_1, \qquad a_0 = \frac{(a_1 k + b_1\delta_4\delta_5 + b_1\delta_4 k^3)(a_1\delta_5 k + b_1\delta_4\delta_5^2 - 7a_1 k^4 - 14b_1\delta_4\delta_5 k^3 + 33b_1\delta_4 k^6)}{48\delta_4 k^7}, \tag{32}$$

$$b_0 = -\frac{(a_1 k + b_1\delta_4\delta_5 + b_1\delta_4 k^3)(a_1 k + b_1\delta_4\delta_5 - 7b_1\delta_4 k^3)}{48\delta_4^2 k^6}, \qquad \beta(t) = \delta_4\alpha(t), \qquad \tau(t) = \delta_4\delta_5\alpha(t), \tag{33}$$

$$b_{-1} = -\frac{(a_1 k + b_1\delta_4\delta_5 + b_1\delta_4 k^3)^2(a_1 k + b_1\delta_4\delta_5 - 5b_1\delta_4 k^3)}{864\delta_4^3 k^9}, \qquad b_1 = b_1, \qquad \delta_i = \text{const} \quad (i = 4, 5). \tag{34}$$

Substituting Eqs. (31)–(34) along with $b_2 = 1$ into (30) yields the following solution

$$u = -\frac{\delta_4(\delta_5 + k^3)}{k} + \frac{144\delta_4^2 k^5\sigma}{144\delta_4^2 k^6 e^{[kx + \delta_4\delta_5\int\alpha(t)\,dt]} + 24\delta_4 k^3\sigma + \sigma^2 e^{[-kx - \delta_4\delta_5\int\alpha(t)\,dt]}}, \tag{35}$$

where $\sigma = a_1 k + b_1\delta_4\delta_5 + b_1\delta_4 k^3$. It should be noted that if we set $a_1 = -144\delta_4^3 k^5(\delta_5 + k^3)$, $b_1 = 144\delta_4^2 k^6$, $b_0 = 24\delta_4 k^3\sigma$ and $\delta_1 = \delta_4$ in Eq. (17), we can recover the solution (35).

3. Conclusion

The Exp-function method has been used to obtain generalized solitonary solutions and periodic solutions of a KdV equation with variable coefficients. This method can also be extended to other NLEEs with variable coefficients, such as the mKdV equation [37], the $(3 + 1)$-dimensional Burgers equation [38] and so on. The Exp-function method is a promising and powerful new method for NLEEs arising in mathematical physics. Its applications are worth further studying.

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