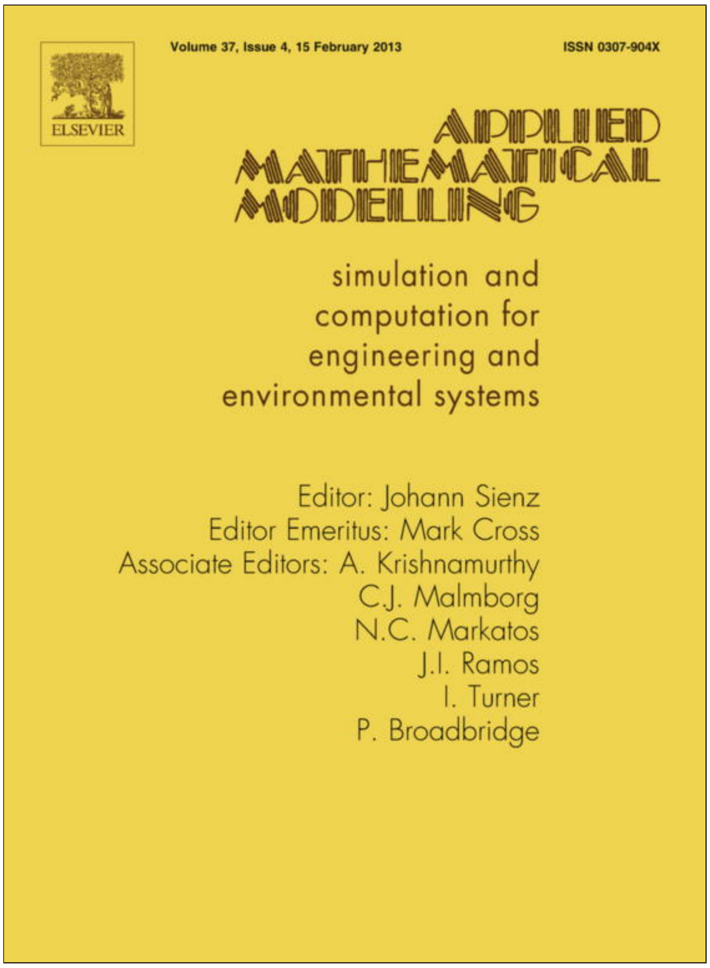
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Almost periodic solution for Nicholson's blowflies model with patch structure and linear harvesting terms ☆

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ABSTRACT

In this paper, we study the existence and exponential convergence of positive almost periodic solutions for a class of Nicholson's blowflies model with patch structure and multiple linear harvesting terms. Under appropriate conditions, we establish some criteria to ensure that the solutions of this system converge locally exponentially to a positive almost periodic solution. Moreover, we give some examples and numerical simulations to illustrate our main results.

1. Introduction

To describe the dynamics of Nicholson's blowflies model in [1], Gurney et al. [2] presented a mathematical model

$$N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}. \tag{1.1}$$

Here, $N(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time. The model and its modifications have been extensively and intensively studied and numerous results about its stability, persistence, attractivity, periodic solution and so on (see [3–8]) have been obtained. However, the main focus of Nicholson's blowflies model is on the scalar equation and results about patch structure of this model are rarely gained. Due to the real world application of patch structure in population dynamics, some population dynamic models with patch structure and delays have been studied by several authors. We refer the reader to [9–12] and the references cited therein. In particular, Faria [13] considered the global dynamics for a Nicholson's blowflies model with patch structure and multiple discrete delays:

$$x_i'(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{m} f_{ij} x_i(t-\tau_{ij}) e^{-x_i(t-\tau_{ij})}, \tag{1.2}$$

where $d_i > 0,\ a_{ij} \geqslant 0$ for $j \neq i,\ \tau_{ik} > 0,\ f_{ik} \geqslant 0$ with $f_i = \sum_{k=1}^{m} f_{ik} > 0$ for all $i, j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m$ and always assume $a_{ii} = 0$ for all $1 \leqslant i \leqslant n$.

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On the other hand, according to the exploitation of biological resources and the harvest of population species are commonly practiced in fishery, forestry and wildlife management, the study of population dynamics with harvesting is an important subject in mathematical bioeconomics, which is related to the optimal management of renewable resources (see [14–16]). Recently, assuming that a harvesting function is a function of the delayed estimate of the true population, Berezansky et al. [17] gave the Nicholson's blowflies model with a linear harvesting term:

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-ax(t-\tau)} - Hx(t-\sigma), \quad \delta, p, \tau, a, H, \sigma \in (0, +\infty), \tag{1.3}$$

where $Hx(t-\sigma)$ is a linear harvesting term, $x(t)$ is the size of the population at time $t, p,\ a,\ \delta$ and τ have the same meaning as in the Eq. (1.1). Moreover, Berezansky et al. [17] put forward an open problem: Find the dynamic behaviors of the Nicholson's blowflies model with a linear harvesting term.

Furthermore, Liu and Meng [18] proposed a class of non-autonomous Nicholson-type delay systems with linear harvesting terms

$$\begin{cases} x_1'(t) & = -\alpha_1(t)x_1(t) + \beta_1(t)x_2(t) + \sum_{j=1}^{m} c_{1j}(t)x_1(t-\tau_{1j}(t))e^{-\gamma_{1j}(t)x_1(t-\tau_{1j}(t))} - H_1(t)x_1(t-\sigma_1(t)), \\ x_2'(t) & = -\alpha_2(t)x_2(t) + \beta_2(t)x_1(t) + \sum_{j=1}^{m} c_{2j}(t)x_2(t-\tau_{2j}(t))e^{-\gamma_{2j}(t)x_2(t-\tau_{2j}(t))} - H_2(t)x_2(t-\sigma_2(t)), \end{cases} \tag{1.4}$$

where $\alpha_i, \beta_i, H_i, \sigma_i, c_{ij}, \gamma_{ij}, \tau_{ij} : R^1 \to R^1_+ = [0, +\infty)$ are almost periodic functions, and $i = 1, 2,\ j = 1, 2, \ldots, m$. Some criteria are established to ensure the existence and exponential convergence of positive almost periodic solutions of this systems, which partly answer the above open problem proposed by Berezansky et al. [17].

Motivated by [13,17,18], a corresponding question arises: Discover the existence and convergence of positive almost periodic solutions of Nicholson's blowflies model with patch structure and multiple linear harvesting terms. The main purpose of this paper is to give the conditions to ensure the existence and convergence of positive almost periodic solutions of the following non-autonomous Nicholson's blowflies model with patch structure and multiple linear harvesting terms:

$$x_i'(t) = -\alpha_i(t)x_i(t) + \sum_{j=1}^{n} \beta_{ij}(t)x_j(t) + \sum_{j=1}^{m} c_{ij}(t)x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))} - \sum_{j=1}^{l} H_{ij}(t)x_i(t-\sigma_{ij}(t)), \quad i = 1, 2, \ldots, n, \tag{1.5}$$

where $\alpha_i, \beta_{ij}, c_{ik_1}, \gamma_{ik_1}, \tau_{ik_1}, \sigma_{ik_2}, H_{ik_2} : R^1 \to R^1_+$ are almost periodic functions, and $i,j = 1, 2, \ldots, n,\ k_1 = 1, 2, \ldots, m,\ k_2 = 1, 2, \ldots, l$. To simplify the notation and without loss of generality, we will always assume $\beta_{ii}(t) = 0$ for all $t \in R^1, i = 1, 2, \ldots, n$.

For convenience, we introduce some notations. Throughout this paper, given a bounded continuous function g defined on R^1, let g^+ and g^- be defined as

$$g^- = \inf_{t \in R^1} g(t), \quad g^+ = \sup_{t \in R^1} g(t).$$

It will be assumed that

$$\alpha_i^- > 0, \quad c_{ik}^- > 0, \quad \beta_{ij}^- > 0 \quad (i \neq j), \quad r_i = \max\left\{ \max_{1 \leqslant j \leqslant m}\{\tau_{ij}^+\}, \max_{1 \leqslant j \leqslant l} \sigma_{ij}^+ \right\} > 0 \quad i,j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m. \tag{1.6}$$

Let $R^n\,(R^n_+)$ be the set of all (nonnegative) real vectors, we will use $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. we let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$ and define $||x|| = \max_{1 \leqslant i \leqslant n}|x_i|$. For matrix $A = (a_{ij})_{n \times n},\ A^T$ denotes the transpose of A, A^{-1} denotes the inverse of $A,\ |A|$ denotes the absolute-value matrix given by $|A| = (|a_{ij}|)_{n \times n}$ and $\rho(A)$ denotes the spectral radius of A. A matrix or vector $A \geqslant 0$ means that all entries of A are greater than or equal to zero. $A > 0$ can be defined similarly. For matrices or vectors A and $B,\ A \geqslant B$ (resp. $A > B$) means that $A - B \geqslant 0$ (resp. $A - B > 0$). For $V(t) \in C((a, +\infty), R^1)$, let

$$\begin{aligned} D^- V(t) &= \lim_{h \to 0^-} \sup \frac{V(t+h) - V(t)}{h}, \\ D_- V(t) &= \lim_{h \to 0^-} \inf \frac{V(t+h) - V(t)}{h}, \quad \forall t \in (a, +\infty). \end{aligned} \tag{1.7}$$

'Denote $C = \prod_{i=1}^{n} C([-r_i, 0], R^1)$ and $C_+ = \prod_{i=1}^{n} C\left([-r_i, 0], R^1_+\right)$ as Banach space equipped with the supremum norm defined by

$$||\varphi|| = \max_{1 \leqslant i \leqslant n}\left\{ \sup_{-r_i \leqslant t \leqslant 0} |\varphi_i(t)| \right\} \quad \text{for all } \varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T \in C (\text{or } \in C_+).$$

If $x_i(t)$ is defined on $[t_0 - r_i, \nu)$ with $t_0, \nu \in R^1$ and $i = 1, 2, \ldots, n$, then we define $x_t \in C$ as $x_t = \left(x_t^1, x_t^2, \ldots, x_t^n\right)^T$ where $x_t^i(\theta) = x_i(t + \theta)$ for all $\theta \in [-r_i, 0]$ and $i = 1, 2, \ldots, n$.

The initial conditions associated with system (1.5) are of the form:

$$x_{t_0} = \varphi, \quad \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T \in C_+ \quad \text{and} \quad \varphi_i(0) > 0, \quad i = 1, 2, \ldots, n. \tag{1.8}$$

We write $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ for a solution of the initial value problem (1.5) and (1.8) . Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of existence of $x_t(t_0, \varphi)$.

The remaining part of this paper is organized as follows. In Section 2, we shall give some notations and preliminary results. In Section 3, we shall derive new sufficient conditions for checking the existence, uniqueness and local exponential convergence of the positive almost periodic solution of (1.5). In Section 4, we shall give some examples and numerical simulations to illustrate our results obtained in the previous section.

2. Preliminary results

In this section, some lemmas and definitions will be presented, which are of importance in proving our main results in Section 3.

Definition 2.1 (*see* [19,20]). Let $u(t) : R^1 \to R^n$ be continuous in t. $u(t)$ is said to be almost periodic on R^1, if for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : |u(t + \delta) - u(t)| < \varepsilon \text{ for all } t \in R^1\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, such that for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \varepsilon$, for all $t \in R^1$.

Definition 2.2 (*see [19,20]*). Let $x \in R^n$ and $Q(t)$ be a $n \times n$ continuous matrix defined on R^1. The linear system

$$x'(t) = Q(t)x(t) \tag{2.1}$$

is said to admit an exponential dichotomy on R^1 if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$\|X(t)PX^{-1}(s)\| \leqslant ke^{-\alpha(t-s)} \quad \text{for all } t \geqslant s,$$
$$\|X(t)(I - P)X^{-1}(s)\| \leqslant ke^{-\alpha(s-t)} \quad \text{for all } t \leqslant s.$$

Definition 2.3. A real $n \times n$ matrix $K = (k_{ij})$ is said to be an M-matrix if $k_{ij} \leqslant 0,\ i, j = 1, \ldots, n,\ i \neq j$ and $K^{-1} \geqslant 0$.

Set

$$B = \{\varphi | \varphi = (\varphi_1(t), \ldots, \varphi_n(t))^T \text{ is an almost periodic vector function on } R^1\}.$$

For any $\varphi \in B$, we define induced module $\|\varphi\|_B = \sup_{t \in R^1} \max_{1 \leqslant i \leqslant n} |\varphi_i(t)|$, then B is a Banach space.

Lemma 2.1 (see [19,20]). *If the linear system (2.1) admits an exponential dichotomy, then almost periodic system*

$$x'(t) = Q(t)x + g(t) \tag{2.2}$$

has a unique almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)ds - \int_{t}^{+\infty} X(t)(I - P)X^{-1}(s)g(s)ds. \tag{2.3}$$

Lemma 2.2 (see [19,20]). *Let $c_i(t)$ be an almost periodic function on R^1 and*

$$M[c_i] = \lim_{T \to +\infty} \frac{1}{T} \int_{t}^{t+T} c_i(s)ds > 0, \quad i = 1, 2, \ldots, n.$$

Then the linear system

$$x'(t) = \operatorname{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t),$$

admits an exponential dichotomy on R^1.

Lemma 2.3 (see [21,22]). *Let $A \geqslant 0$ be an $n \times n$ matrix and $\rho(A) < 1$, then $(I_n - A)^{-1} \geqslant 0$, where I_n denotes the identity matrix of size n.*

Lemma 2.4. *Suppose that there exist positive constants E_{i1} and E_{i2} such that*

$$E_{i1} > E_{i2}, \quad \sum_{j=1}^{n} \frac{\beta_{ij}^{+} E_{j1}}{\alpha_i^{-}} + \sum_{j=1}^{m} \frac{c_{ij}^{+}}{\alpha_i^{-} \gamma_{ij}^{-} e} < E_{i1}, \tag{2.4}$$

$$\sum_{j=1}^{n} \frac{\beta_{ij}^{-} E_{j2}}{\alpha_i^{+}} + \sum_{j=1}^{m} \frac{c_{ij}^{-}}{\alpha_i^{+}} E_{i1} e^{-\gamma_{ij}^{+} E_{i1}} - \sum_{j=1}^{l} \frac{H_{ij}^{+} E_{i1}}{\alpha_i^{+}} > E_{i2} \geqslant \frac{1}{\min_{1\leqslant j\leqslant m} \gamma_{ij}^{-}}, \tag{2.5}$$

where $i = 1, 2, \ldots, n$. *Let*

$$C^0 := \{\varphi | \varphi \in C, E_{i2} < \varphi_i(t) < E_{i1}, \quad \text{for all } t \in [-r_i, 0], \ i = 1, 2, \ldots, n\}.$$

Moreover, assume that $x(t; t_0, \varphi)$ *is the solution of* (1.5) *with* $\varphi \in C^0$. *Then,*

$$E_{i2} < x_i(t; t_0, \varphi) < E_{i1}, \quad \text{for all } t \in [t_0, \eta(\varphi)), \ i = 1, 2, \ldots, n \tag{2.6}$$

and $\eta(\varphi) = +\infty$.

Proof. Set $x(t) = x(t; t_0, \varphi)$ for all $t \in [t_0, \eta(\varphi))$. Let $[t_0, T) \subseteq [t_0, \eta(\varphi))$ be an interval such that

$$0 < x_i(t) \quad \text{for all } t \in [t_0, T), \ i = 1, 2, \ldots, n, \tag{2.7}$$

we claim that

$$0 < x_i(t) < E_{i1} \quad \text{for all } t \in [t_0, T), \ i = 1, 2, \ldots, n. \tag{2.8}$$

Assume, by way of contradiction, that (2.8) does not hold. Then, it exist $t^* \in (t_0, T)$ and $k \in \{1, 2, \ldots, n\}$ such that

$$x_k(t^*) = E_{k1} \quad \text{and} \quad 0 < x_i(t) < E_{i1} \quad \text{for all } t \in [t_0 - r_i, t^*), \ i = 1, 2, \ldots, n. \tag{2.9}$$

Calculating the derivative of $x_k(t)$, together with (2.4) and the fact that $\sup_{u\geqslant 0} u e^{-u} = \frac{1}{e}$, (1.5) and (2.9) imply that

$$\begin{aligned}
0 \leqslant x_k'(t^*) &= -\alpha_k(t^*) x_k(t^*) + \sum_{j=1}^{n} \beta_{kj}(t^*) x_j(t^*) + \sum_{j=1}^{m} \frac{c_{kj}(t^*)}{\gamma_{kj}(t^*)} \gamma_{kj}(t^*) x_k(t^* - \tau_{kj}(t^*)) e^{-\gamma_{kj}(t^*) x_k(t^* - \tau_{kj}(t^*))} \\
&\quad - \sum_{j=1}^{l} H_{kj}(t^*) x_k(t^* - \sigma_{kj}(t^*)) \leqslant -\alpha_k^{-} x_k(t^*) + \sum_{j=1}^{n} \beta_{kj}^{+} E_{j1} + \sum_{j=1}^{m} \frac{c_{kj}^{+}}{\gamma_{kj}^{-} e} \\
&= \alpha_k^{-} \left(-E_{k1} + \sum_{j=1}^{n} \frac{\beta_{kj}^{+} E_{j1}}{\alpha_k^{-}} + \sum_{j=1}^{m} \frac{c_{kj}^{+}}{\alpha_k^{-} \gamma_{kj}^{-} e} \right) < 0,
\end{aligned}$$

which is a contradiction and implies that (2.8) holds.

We next show that

$$x_i(t) > E_{i2}, \quad \text{for all } t \in (t_0, \eta(\varphi)), \ i = 1, 2, \ldots, n. \tag{2.10}$$

Suppose, for the sake of contradiction, that (2.10) does not hold. Then, there exist $s^* \in (t_0, \eta(\varphi))$ and $k \in \{1, 2, \ldots, n\}$ such that

$$x_k(s^*) = E_{k2} \quad \text{and} \quad x_i(t) > E_{i2} \quad \text{for all } t \in [t_0 - r_i, s^*), \ i = 1, 2, \ldots, n. \tag{2.11}$$

From (2.5), (2.8) and (2.11), we get

$$E_{i2} < x_i(t) < E_{i1}, \quad \gamma_{ij}^{+} x_i(t) \geqslant \gamma_{ij}^{+} E_{i2} \geqslant \gamma_{ij}^{+} \frac{1}{\min_{1\leqslant j\leqslant m} \gamma_{ij}^{-}} \geqslant 1, \tag{2.12}$$

for all $t \in [t_0 - r_i, s^*)$, $i = 1, 2, \ldots, n$, $j = 1, 2 \ldots, m$. Calculating the derivative of $x_k(t)$, together with (2.5) and the fact that $\min_{1\leqslant u\leqslant \kappa} u e^{-u} = \kappa e^{-\kappa}$, (1.5), (2.11) and (2.12) imply that

$$\begin{aligned}
0 \geqslant x_k'(s^*) &= -\alpha_k(s^*) x_k(s^*) + \sum_{j=1}^{n} \beta_{kj}(s^*) x_j(s^*) + \sum_{j=1}^{m} c_{kj}(s^*) x_k(s^* - \tau_{kj}(s^*)) e^{-\gamma_{kj}(s^*) x_k(s^* - \tau_{kj}(s^*))} - \sum_{j=1}^{l} H_{kj}(s^*) x_k(s^* - \sigma_{kj}(s^*)) \\
&= -\alpha_k(s^*) x_k(s^*) + \sum_{j=1}^{n} \beta_{kj}(s^*) x_j(s^*) + \sum_{j=1}^{m} \frac{c_{kj}(s^*)}{\gamma_{kj}^{+}} \gamma_{kj}^{+} x_k(s^* - \tau_{kj}(s^*)) e^{-\gamma_{kj}(s^*) x_k(s^* - \tau_{kj}(s^*))} - \sum_{j=1}^{l} H_{kj}(s^*) x_k(s^* - \sigma_{kj}(s^*)) \\
&\geqslant -\alpha_k(s^*) x_k(s^*) + \sum_{j=1}^{n} \beta_{kj}^{-} E_{j2} + \sum_{j=1}^{m} \frac{c_{kj}^{-}}{\gamma_{kj}^{+}} \gamma_{kj}^{+} x_k(s^* - \tau_{kj}(s^*)) e^{-\gamma_{kj}^{+} x_k(s^* - \tau_{kj}(s^*))} - \sum_{j=1}^{l} H_{kj}^{+} E_{k1} \\
&\geqslant -\alpha_k^{+} x_k(s^*) + \sum_{j=1}^{n} \beta_{kj}^{-} E_{j2} + \sum_{j=1}^{m} \frac{c_{kj}^{-}}{\gamma_{kj}^{+}} \gamma_{kj}^{+} E_{k1} e^{-\gamma_{kj}^{+} E_{k1}} - \sum_{j=1}^{l} H_{kj}^{+} E_{k1} \\
&= \alpha_k^{+} \left(-E_{k2} + \sum_{j=1}^{n} \frac{\beta_{kj}^{-} E_{j2}}{\alpha_k^{+}} + \sum_{j=1}^{m} \frac{c_{kj}^{-}}{\alpha_k^{+}} E_{k1} e^{-\gamma_{kj}^{+} E_{k1}} - \sum_{j=1}^{l} \frac{H_{kj}^{+} E_{k1}}{\alpha_k^{+}} \right) > 0,
\end{aligned}$$

which is a contradiction and implies that (2.10) holds.

It follows from (2.8) and (2.10) that (2.6) is true. From Theorem 2.3.1 in [23], we easily obtain $\eta(\varphi) = +\infty$. This ends the proof of Lemma 2.4. □

3. Main results

Theorem 3.1. *Let (2.4) and (2.5) hold. Moreover, suppose that*

$$\rho(A^{-1}(B + C + H)) < 1, \tag{3.1}$$

where

$$A = \mathrm{diag}(\alpha_1^-, \alpha_2^-, \ldots, \alpha_n^-), \quad C = \mathrm{diag}\left(\sum_{j=1}^m \frac{c_{1j}^+}{e^2}, \sum_{j=1}^m \frac{c_{2j}^+}{e^2}, \cdots, \sum_{j=1}^m \frac{c_{nj}^+}{e^2}\right),$$

$$H = \mathrm{diag}\left(\sum_{j=1}^l H_{1j}^+, \sum_{j=1}^l H_{2j}^+, \cdots, \sum_{j=1}^l H_{nj}^+\right), \quad B = \left(\beta_{ij}^+\right)_{n\times n}.$$

Then, there exists a unique positive almost periodic solution of system (1.5) in the region $B^* = \{\varphi | \varphi \in B, E_{i2} \leqslant \varphi_i(t) \leqslant E_{i1}$, for all $t \in R^1,\ i = 1, 2, \ldots, n\}$.

Proof. For any $\phi \in B$, we consider an auxiliary system

$$x_i'(t) = -\alpha_i(t)x_i(t) + \sum_{j=1}^n \beta_{ij}(t)\phi_j(t) + \sum_{j=1}^m c_{ij}(t)\phi_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)\phi_i(t-\tau_{ij}(t))} - \sum_{j=1}^l H_{ij}(t)\phi_i(t-\sigma_{ij}(t)), \quad i = 1, 2, \ldots, n. \tag{3.2}$$

Notice that $M[\alpha_i] > 0 (i = 1, 2, \ldots, n)$, it follows from Lemma 2.2 that the linear system

$$x_i'(t) = -\alpha_i(t)x_i(t), \quad i = 1, 2, \ldots, n, \tag{3.3}$$

admits an exponential dichotomy on R^1. Thus, by Lemma 2.1, we obtain that the system (3.2) has exactly one almost periodic solution $x^\phi(t) = (x_1^\phi(t), \ldots, x_n^\phi(t))^T$:

$$x_i^\phi(t) = \int_{-\infty}^t e^{-\int_s^t \alpha_i(u)du}\left[\sum_{j=1}^n \beta_{ij}(s)\phi_j(s) + \sum_{j=1}^m c_{ij}(s)\phi_i(s-\tau_{ij}(s))e^{-\gamma_{ij}(s)\phi_i(s-\tau_{ij}(s))} - \sum_{j=1}^l H_{ij}(s)\phi_i(s-\sigma_{ij}(s))\right]ds,$$
$$i = 1, 2, \ldots, n. \tag{3.4}$$

Define a mapping $T : B \to B$ by setting

$$T(\phi(t)) = x^\phi(t), \quad \forall \phi \in B.$$

Since $B^* = \{\varphi | \varphi \in B, E_{i2} \leqslant \varphi_i(t) \leqslant E_{i1},$ for all $t \in R^1, i = 1, 2, \ldots, n\}$, it is easy to see that B^* is a closed subset of B. For $i = 1, 2, \ldots, n$ and any $\phi \in B^*$, from (2.4) and (3.4) and the fact that $\sup_{u \geqslant 0} ue^{-u} = \frac{1}{e}$, we have

$$x_i^\phi(t) \leqslant \int_{-\infty}^t e^{-\int_s^t \alpha_i(u)du}\left[\sum_{j=1}^n \beta_{ij}^+ E_{j1} + \sum_{j=1}^m c_{ij}(s)\frac{1}{\gamma_{ij}(s)e}\right]ds \leqslant \frac{1}{\alpha_i^-}\left[\sum_{j=1}^n \beta_{ij}^+ E_{j1} + \sum_{j=1}^m \frac{c_{ij}^+}{\gamma_{ij}^- e}\right]$$
$$= \sum_{j=1}^n \frac{\beta_{ij}^+ E_{j1}}{\alpha_i^-} + \sum_{j=1}^m \frac{c_{ij}^+}{\alpha_i^-\gamma_{ij}^- e} < E_{i1}, \quad \text{for all } t \in R^1. \tag{3.5}$$

In view of the fact that $\min_{1 \leqslant u \leqslant \kappa} ue^{-u} = \kappa e^{-\kappa}$, from (2.5), (2.6) and (3.4), we obtain

$$x_i^\phi(t) \geqslant \int_{-\infty}^t e^{-\int_s^t \alpha_i(u)du}\left[\sum_{j=1}^n \beta_{ij}^- E_{j2} + \sum_{j=1}^m c_{ij}(s)\frac{1}{\gamma_{ij}^+}\gamma_{ij}^+\phi_i(s-\tau_{ij}(s))e^{-\gamma_{ij}^+\phi_i(s-\tau_{ij}(s))} - \sum_{j=1}^l H_{ij}(t)\phi_i(t-\sigma_{ij}(t))\right]ds$$
$$\geqslant \frac{1}{\alpha_i^+}\left[\sum_{j=1}^n \beta_{ij}^- E_{j2} + \sum_{j=1}^m c_{ij}^- E_{i1}e^{-\gamma_{ij}^+ E_{i1}} - \sum_{j=1}^l H_{ij}^+ E_{i1}\right] = \sum_{j=1}^n \frac{\beta_{ij}^- E_{j2}}{\alpha_i^+} + \sum_{j=1}^m \frac{c_{ij}^-}{\alpha_i^+}E_{i1}e^{-\gamma_{ij}^+ E_{i1}} - \sum_{j=1}^l \frac{H_{ij}^+ E_{i1}}{\alpha_i^+} > E_{i2}, \quad \text{for all } t \in R^1. \tag{3.6}$$

This implies that the mapping T is a self-mapping from B^* to B^*.

Let $\varphi,\psi\in B^*$, for $i=1,2,\ldots,n$, we get

$$\begin{aligned}\sup_{t\in R^1}|(T(\varphi(t))-T(\psi(t)))_i| &= \sup_{t\in R^1}\left|\int_{-\infty}^{t}e^{-\int_s^t\alpha_i(u)du}\left[\sum_{j=1}^{n}\beta_{ij}(s)(\varphi_j(s)-\psi_j(s))+\sum_{j=1}^{m}c_{ij}(s)(\varphi_i(s-\tau_{ij}(s))e^{-\gamma_{ij}(s)\varphi_i(s-\tau_{ij}(s))}\right.\right.\\ &\quad\left.\left.-\psi_i(s-\tau_{ij}(s))e^{-\gamma_{ij}(s)\psi_i(s-\tau_{ij}(s))})-\sum_{j=1}^{l}H_{ij}(s)(\varphi_i(s-\sigma_{ij}(s))-\psi_i(s-\sigma_{ij}(s)))\right]ds\right|\\ &= \sup_{t\in R^1}\left|\int_{-\infty}^{t}e^{-\int_s^t\alpha_i(u)du}\left[\sum_{j=1}^{n}\beta_{ij}(s)(\varphi_j(s)-\psi_j(s))+\sum_{j=1}^{m}\frac{c_{ij}(s)}{\gamma_{ij}(s)}(\gamma_{ij}(s)\varphi_i(s-\tau_{ij}(s))e^{-\gamma_{ij}(s)\varphi_i(s-\tau_{ij}(s))}\right.\right.\\ &\quad\left.\left.-\gamma_{ij}(s)\psi_i(s-\tau_{ij}(s))e^{-\gamma_{ij}(s)\psi_i(s-\tau_{ij}(s))})-\sum_{j=1}^{l}H_{ij}(s)(\varphi_i(s-\sigma_{ij}(s))-\psi_i(s-\sigma_{ij}(s)))\right]ds\right|,\quad i=1,2,\ldots,n.\end{aligned}\tag{3.7}$$

Since

$$\gamma_{ij}(s)\varphi_i(s-\tau_{ij}(s))\ \geqslant\ \gamma_{ij}^-E_{i2}\ \geqslant\ \gamma_{ij}^-\frac{1}{\min_{1\leqslant j\leqslant m}\gamma_{ij}^-}\geqslant 1,\quad \text{for all } s\in R^1,\ i=1,2,\ldots,n,\ j=1,2,\ldots,m,$$

and

$$\gamma_{ij}(s)\psi_i(s-\tau_{ij}(s))\ \geqslant\ \gamma_{ij}^-E_{i2}\ \geqslant\ \gamma_{ij}^-\frac{1}{\min_{1\leqslant j\leqslant m}\gamma_{ij}^-}\geqslant 1,\quad \text{for all } s\in R^1,\ i=1,2,\ldots,n,\ j=1,2,\ldots,m.$$

According to (1.6), (2.5), (3.5)–(3.7), from $\sup_{u\geqslant1}|\frac{1-u}{e^u}|=\frac{1}{e^2}$ and the inequality

$$|xe^{-x}-ye^{-y}|=\left|\frac{1-(x+\theta(y-x))}{e^{x+\theta(y-x)}}\right|\left|x-y\right|\leqslant\frac{1}{e^2}|x-y|\quad \text{where } x,y\in[1,+\infty),\quad 0<\theta<1,\tag{3.8}$$

we have

$$\begin{aligned}\sup_{t\in R^1}\left|(T(\varphi(t))-T(\psi(t)))_i\right| &\leqslant \sum_{j=1}^{n}\frac{\beta_{ij}^+}{\alpha_i^-}\sup_{t\in R^1}|\varphi_j(t)-\psi_j(t)|+\sup_{t\in R^1}\int_{-\infty}^{t}e^{-\int_s^t\alpha_i(u)du}\sum_{j=1}^{m}c_{ij}^+\frac{1}{e^2}|\varphi_i(s-\tau_{ij}(s))-\psi_i(s-\tau_{ij}(s))|ds\\ &\quad+\sum_{j=1}^{l}\frac{H_{ij}^+}{\alpha_i^-}\sup_{t\in R^1}|\varphi_i(t)-\psi_i(t)|\\ &\leqslant \sum_{j=1}^{n}\frac{\beta_{ij}^+}{\alpha_i^-}\sup_{t\in R^1}|\varphi_j(t)-\psi_j(t)|+\left(\sum_{j=1}^{m}\frac{c_{ij}^+}{\alpha_i^-e^2}+\sum_{j=1}^{l}\frac{H_{ij}^+}{\alpha_i^-}\right)\sup_{t\in R^1}|\varphi_i(t)-\psi_i(t)|,\end{aligned}\tag{3.9}$$

Hence

$$\begin{aligned}&\left(\sup_{t\in R^1}|(T(\varphi(t))-T(\psi(t)))_1|,\ldots,\sup_{t\in R^1}|(T(\varphi(t))-T(\psi(t)))_n|\right)^T\\ &\leqslant\left(\sum_{j=1}^{n}\frac{\beta_{1j}^+}{\alpha_1^-}\sup_{t\in R^1}|\varphi_j(t)-\psi_j(t)|+\left(\sum_{j=1}^{m}\frac{c_{1j}^+}{\alpha_1^-e^2}+\sum_{j=1}^{l}\frac{H_{1j}^+}{\alpha_1^-}\right)\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|,\cdots,\sum\nolimits_{j=1}^{n}\frac{\beta_{nj}^+}{\alpha_n^-}\sup_{t\in R^1}|\varphi_j(t)-\psi_j(t)|\right.\\ &\quad\left.+\left(\sum_{j=1}^{m}\frac{c_{nj}^+}{\alpha_n^-e^2}+\sum_{j=1}^{l}\frac{H_{nj}^+}{\alpha_n^-}\right)\sup_{t\in R^1}|\varphi_n(t)-\psi_n(t)|\right)^T=F\left(\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|,\ldots,\sup_{t\in R^1}|\varphi_n(t)-\psi_n(t)|\right)^T\\ &=F\left(\sup_{t\in R^1}|(\varphi(t)-\psi(t))_1|,\cdots,\sup_{t\in R^1}|(\varphi(t)-\psi(t))_n|\right)^T,\end{aligned}\tag{3.10}$$

where $F = A^{-1}(B + C + H)$. Let μ be a positive integer. Then, from (3.10) we get

$$\begin{aligned}
&\left(\sup_{t\in R^1}|(T^{\mu}(\varphi(t)) - T^{\mu}(\psi(t)))_1|,\cdots,\sup_{t\in R^1}|(T^{\mu}(\varphi(t)) - T^{\mu}(\psi(t)))_n|\right)^T\\
&= \left(\sup_{t\in R^1}|(T(T^{\mu-1}(\varphi(t))) - T(T^{\mu-1}(\psi(t))))_1|,\cdots,\sup_{t\in R^1}|(T(T^{\mu-1}(\varphi(t))) - T(T^{\mu-1}(\psi(t))))_n|\right)^T\\
&\leqslant F\left(\sup_{t\in R^1}|(T^{\mu-1}(\varphi(t)) - T^{\mu-1}(\psi(t)))_1|,\cdots,\sup_{t\in R^1}|(T^{\mu-1}(\varphi(t)) - T^{\mu-1}(\psi(t)))_n|\right)^T\\
&\ \vdots\\
&\leqslant F^{\mu}\left(\sup_{t\in R^1}|(\varphi(t) - \psi(t))_1|,\cdots,\sup_{t\in R^1}|(\varphi(t) - \psi(t))_n|\right)^T\\
&= F^{\mu}\left(\sup_{t\in R^1}|\varphi_1(t) - \psi_1(t)|,\cdots,\sup_{t\in R^1}|\varphi_n(t) - \psi_n(t)|\right)^T. \qquad (3.11)
\end{aligned}$$

Since $\rho(F) < 1$, we obtain

$$\lim_{\mu\to+\infty} F^{\mu} = 0,$$

which implies that there exist a positive integer and N and a positive constant $r < 1$ such that

$$F^N = (A^{-1}(B + C + H))^N = (g_{ij})_{n\times n} \quad \text{and} \quad \sum_{j=1}^{n} g_{ij} \leqslant r, \quad i = 1,\ldots,n. \qquad (3.12)$$

In view of (3.11) and (3.12), we have

$$\begin{aligned}
\left|(T^N(\varphi(t)) - T^N(\psi(t)))_i\right| &\leqslant \sup_{t\in R^1}|(T^N(\varphi(t)) - T^N(\psi(t)))_i| \leqslant \sum_{j=1}^{n} g_{ij}\sup_{t\in R^1}|\varphi_j(t) - \psi_j(t)| \leqslant \sup_{t\in R^1}\max_{1\leqslant j\leqslant n}|\varphi_j(t) - \psi_j(t)|\sum_{j=1}^{n} g_{ij}\\
&\leqslant r\|\varphi(t) - \psi(t)\|_B
\end{aligned}$$

for all $t \in R,\ i = 1, 2, \ldots, n$. It follows that

$$\|T^N(\varphi(t)) - T^N(\psi(t))\|_B = \sup_{t\in R^1}\max_{1\leqslant i\leqslant n}\left|(T^N(\varphi(t)) - T^N(\psi(t)))_i\right| \leqslant r\|\varphi(t) - \psi(t)\|_B. \qquad (3.13)$$

This implies that the mapping $T^N : B^* \to B^*$ is a contraction mapping.

By the fixed point theorem of Banach space, T possesses a unique fixed point $\varphi^* \in B^*$ such that $T\varphi^* = \varphi^*$. By (3.2), φ^* satisfies (1.5). So φ^* is an almost periodic solution of (1.5) in B^*. The proof of Theorem 3.1 is now complete. $\square$

Theorem 3.2. *Let $x^*(t)$ be the positive almost periodic solution of Eq. (1.5) in the region B^*. Suppose that (2.4), (2.5) and (3.1) hold. Then, the solution $x(t; t_0, \varphi)$ of (1.5) with $\varphi \in C^0$ converges exponentially to $x^*(t)$ as $t \to +\infty$.*

Proof. Since $\rho(A^{-1}(B + C + H)) < 1$, it follows Theorem 3.1 that system (1.5) has a unique almost periodic solution $x^*(t) = (x_1^*(t),\ldots,x_n^*(t))^T$ in the region B^*. Set $x(t) = x(t; t_0, \varphi)$ and $y_i(t) = x_i(t) - x_i^*(t)$, where $\varphi \in C^0$, $t \in [t_0 - r_i, +\infty)$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}
y_i'(t) = &-\alpha_i(t)y_i(t) + \sum_{j=1}^{n}\beta_{ij}(t)y_j(t) + \sum_{j=1}^{m}c_{ij}(t)(x_i(t - \tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))} - x_i^*(t - \tau_{ij}(t))e^{-\gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))})\\
&- \sum_{j=1}^{l}H_{ij}(t)y_i(t - \sigma_{ij}(t)), \quad i = 1, 2, \ldots, n. \qquad (3.14)
\end{aligned}$$

In view of (2.5) and (2.6), for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, we obtain

$$\gamma_{ij}(t)x_i(t - \tau_{ij}(t))) \geqslant \gamma_{ij}^- E_{i2} \geqslant \gamma_{ij}^-\frac{1}{\min_{1\leqslant j\leqslant m}\gamma_{ij}^-} \geqslant 1, \quad \text{for all } t \in [t_0 - r_i, +\infty)$$

and

$$\gamma_{ij}(t)x_i^*(t - \tau_{ij}(t))) \geqslant \gamma_{ij}^- E_{i2} \geqslant \gamma_{ij}^-\frac{1}{\min_{1\leqslant j\leqslant m}\gamma_{ij}^-} \geqslant 1, \quad \text{for all } t \in R^1,$$

which, together with (3.8) and (3.14), imply that

$$\begin{aligned}
D^-|y_i(t)| &\leqslant -\alpha_i(t)|y_i(t)| + \sum_{j=1}^{n}\beta_{ij}(t)|y_j(t)| + \sum_{j=1}^{m}c_{ij}(t)|x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))} - x_i^*(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))}| \\
&\quad + \sum_{j=1}^{l}H_{ij}(t)|y_i(t-\sigma_{ij}(t))| \\
&= -\alpha_i(t)|y_i(t)| + \sum_{j=1}^{n}\beta_{ij}(t)|y_j(t)| + \sum_{j=1}^{m}\frac{c_{ij}(t)}{\gamma_{ij}(t)}|\gamma_{ij}(t)x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))} \\
&\quad - \gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))}| + \sum_{j=1}^{l}H_{ij}(t)|y_i(t-\sigma_{ij}(t))| \leqslant -\alpha_i^-|y_i(t)| + \sum_{j=1}^{n}\beta_{ij}^+|y_j(t)| \\
&\quad + \sum_{j=1}^{m}\frac{c_{ij}(t)}{e^2}|y_i(t-\tau_{ij}(t))| + \sum_{j=1}^{l}H_{ij}^+|y_i(t-\sigma_{ij}(t))| \\
&\leqslant -\alpha_i^-|y_i(t)| + \sum_{j=1}^{n}\beta_{ij}^+|y_j(t)| + \left(\sum_{j=1}^{m}\frac{c_{ij}^+}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)|\widetilde{y}_i(t)|, \qquad (3.15)
\end{aligned}$$

where $\widetilde{y}_i(t) = \sup_{t-r_i\leqslant s\leqslant t}|y_i(s)|,\ i = 1,2,\ldots,n.$

Again from $\rho(A^{-1}(B+C+H)) < 1$, it follows from Lemma 2.3 that $I_n - A^{-1}(B+C+H)$ is an M-matrix, we obtain that there exist a constant $\bar{\mu} > 0$ and a vector $\xi = (\xi_1,\xi_2,\ldots\xi_n)^T > (0,\ldots,0)^T$ such that

$$\left(I_n - A^{-1}(B+C+H)\right)\xi > (\bar{\mu},\ldots,\bar{\mu})^T.$$

Therefore,

$$\xi_i - \frac{1}{\alpha_i^-}\sum_{j=1}^{n}\beta_{ij}^+\xi_j - \frac{1}{\alpha_i^-}\left(\sum_{j=1}^{m}\frac{c_{ij}^+}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)\xi_i > \bar{\mu}, \quad i = 1,2,\ldots,n,$$

which implies that

$$-\alpha_i^-\xi_i + \sum_{j=1}^{n}\beta_{ij}^+\xi_j + \left(\sum_{j=1}^{m}\frac{c_{ij}^+}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)\xi_i < -\alpha_i^-\bar{\mu}, \quad i = 1,2,\ldots,n. \qquad (3.16)$$

We can choose a positive constant $\eta < 1$ such that

$$\eta\xi_i + \left[-\alpha_i^-\xi_i + \sum_{j=1}^{n}\beta_{ij}^+\xi_j + \left(\sum_{j=1}^{m}\frac{c_{ij}^+}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)\xi_i e^{\eta r_i}\right] < 0, \quad i = 1,\ldots,n. \qquad (3.17)$$

Let $\varsigma > 1$ be a positive constant such that

$$\varsigma\xi_i e^{-\eta(t-t_0)} > 1, \quad \text{for all } t \in [t_0 - r_i, t_0],\ i = 1,2,\ldots,n. \qquad (3.18)$$

For all $\varepsilon > 0$, let

$$Z_i(t) = \varsigma\xi_i\left[\sum_{j=1}^{n}\widetilde{y}_j(t_0) + \varepsilon\right]e^{-\eta(t-t_0)}, \quad \text{for all } t \in R^1,\ i = 1,2,\ldots,n. \qquad (3.19)$$

From (3.17) and (3.19), we obtain

$$\begin{aligned}
D_-Z_i(t) &= (-\eta\xi_i)\varsigma\left[\sum_{j=1}^{n}\widetilde{y}_j(t_0) + \varepsilon\right]e^{-\eta(t-t_0)} > \left[-\alpha_i^-\xi_i + \sum_{j=1}^{n}\beta_{ij}^+\xi_j + \left(\sum_{j=1}^{m}\frac{c_{ij}^+}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)\xi_i e^{\eta r_i}\right]\varsigma\left[\sum_{j=1}^{n}\widetilde{y}_j(t_0) + \varepsilon\right]e^{-\eta(t-t_0)} \\
&= -\alpha_i^-\varsigma\xi_i\left[\sum_{j=1}^{n}\widetilde{y}_j(t_0) + \varepsilon\right]e^{-\eta(t-t_0)} + \left(\sum_{j=1}^{n}\beta_{ij}^+\xi_j\right)\varsigma\left[\sum_{j=1}^{n}\widetilde{y}_j(t_0) + \varepsilon\right]e^{-\eta(t-t_0)} \\
&\quad + \left(\sum_{j=1}^{m}\frac{c_{ij}^+}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)\varsigma\xi_i\left[\sum_{j=1}^{n}\widetilde{y}_j(t_0) + \varepsilon\right]e^{\eta r_i}e^{-\eta(t-t_0)} \\
&= -\alpha_i^-Z_i(t) + \sum_{j=1}^{n}\beta_{ij}^+Z_j(t) + \left(\sum_{j=1}^{m}\frac{c_{ij}^+}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)\widetilde{Z}_i(t), \qquad (3.20)
\end{aligned}$$

where $\widetilde{Z}_i(t) = \sup_{t-r_i \leqslant s \leqslant t} Z_i(s),\ t \in R^1,\ i = 1, 2, \ldots, n.$

In view of (3.18) and (3.19), for $i = 1, 2, \ldots, n$, we have

$$Z_i(t) = \varsigma\xi_i\left[\sum_{j=1}^{n}\widetilde{y}_j(t_0) + \varepsilon\right]e^{-\eta(t-t_0)} > \sum_{j=1}^{n}\widetilde{y}_j(t_0) + \varepsilon > |y_i(t)|, \tag{3.21}$$

where $t \in [t_0 - r_i, t_0],\ i = 1, 2, \ldots, n.$

We claim that

$$|y_i(t)| < Z_i(t) \quad \text{for all } t > t_0,\ i = 1, 2, \ldots, n. \tag{3.22}$$

Contrarily, there must exist $i \in \{1, 2, \ldots, n\}$ and $r^* > t_0$ such that

$$|y_i(r^*)| = Z_i(r^*) \quad \text{and} \quad |y_j(t)| < Z_j(t), \quad \text{for all } t \in [t_0 - r_j, r^*),\ j = 1, 2, \ldots, n. \tag{3.23}$$

which implies that

$$|y_i(r^*)| - Z_i(r^*) = 0 \quad \text{and} \quad |y_j(t)| - Z_j(t) < 0, \quad \text{for all } t \in [t_0 - r_j, r^*),\ j = 1, 2, \ldots, n. \tag{3.24}$$

It follows that

$$\begin{aligned} 0 \leqslant D^-(|y_i(r^*)| - Z_i(r^*)) &= \limsup_{h \to 0^-} \frac{[|y_i(r^*+h)| - Z_i(r^*+h)] - [|y_i(r^*)| - Z_i(r^*)]}{h} \\ &\leqslant \limsup_{h \to 0^-} \frac{|y_i(r^*+h)| - |y_i(r^*)|}{h} - \liminf_{h \to 0^-} \frac{Z_i(r^*+h) - Z_i(r^*)}{h} = D^-|y_i(r^*)| - D_-Z_i(r^*). \end{aligned} \tag{3.25}$$

From (3.15), (3.20) and (3.23), we obtain

$$\begin{aligned} D^-|y_i(r^*)| &\leqslant -\alpha_i^-|y_i(r^*)| + \sum_{j=1}^{n}\beta_{ij}^+|y_j(r^*)| + \left(\sum_{j=1}^{m}\frac{c_{ij}(r^*)}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)|\widetilde{y}_i(r^*)| \\ &= -\alpha_i^- Z_i(r^*) + \sum_{j=1}^{n}\beta_{ij}^+|y_j(r^*)| + \left(\sum_{j=1}^{m}\frac{c_{ij}(r^*)}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)|\widetilde{y}_i(r^*)| \\ &\leqslant -\alpha_i^- Z_i(r^*) + \sum_{j=1}^{n}\beta_{ij}^+ Z_j(r^*) + \left(\sum_{j=1}^{m}\frac{c_{ij}(r^*)}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)|\widetilde{Z}_i(r^*)| < D_-Z_i(r^*), \end{aligned} \tag{3.26}$$

which contradicts (3.25). Hence, (3.22) holds. Letting $\varepsilon \to 0^+$ and $M = n\max_{1 \leqslant i \leqslant n}\{\varsigma\xi_i + 1\}$, we obtain from (3.19) and (3.22) that

$$|x_i(t) - x_i^*(t)| = |y_i(t)| \leqslant \varsigma\xi_i\sum_{j=1}^{n}\widetilde{y}_j(t_0)e^{-\eta(t-t_0)} \leqslant \varsigma\xi_i n\|\varphi - x^*\|e^{-\eta(t-t_0)} \leqslant M\|\varphi - x^*\|e^{-\eta(t-t_0)}, \quad \text{for all } t > t_0,\ i = 1, 2, \ldots, n. \tag{3.27}$$

This completes the proof of Theorem 3.2. □

Remark 3.1. When $n = 2$ and $l = 1$ in system (1.5), the existing results on almost periodic solutions for (1.5) have been obtained under the assumption that the row norm of matrix $A^{-1}(B + C + H)$ is less than 1. Therefore, the related results in [18] are direct corollaries of Theorems 3.1 and 3.2 of this paper.

Corollary 3.1. *Let (2.4) and (2.5) hold. Suppose that $I_n - A^{-1}(B + C + H)$ is an M-matrix. Then system (1.5) has exactly one almost periodic solution $x^*(t)$. Moreover,the solution $x(t; t_0, \varphi)$ of (1.5) with $\varphi \in C^0$ converges exponentially to $x^*(t)$ as $t \to +\infty$.*

Proof. Notice that $I_n - A^{-1}(B + C + H)$ is an M-matrix, it follows that there exists a vector $d = (d_1, \ldots, d_n)^T > (0, \ldots, 0)^T$ such that

$$(I_n - A^{-1}(B + C + H))d > 0, \tag{3.28}$$

this is,

$$-\alpha_i^- d_i + \sum_{j=1}^{n}\beta_{ij}^+ d_j + \left(\sum_{j=1}^{m}\frac{c_{ij}^+}{e^2} + \sum_{j=1}^{l}H_{ij}^+\right)d_i < 0,\ i = 1, 2, \ldots, n. \tag{3.29}$$

For any matrix norm $\|\cdot\|$ and nonsingular matrix D. $\|A\|_D = \|D^{-1}AD\|$ also defines a matrix norm. Let $D = diag(d_1, \ldots, d_n)$. Then (3.29) implies that the row norm of matrix $D^{-1}A^{-1}(B + C + H)D$ is less than 1. Hence $\rho(A^{-1}(B + C + H)) < 1$. Corollary 3.1 follows immediately from Theorems 3.1 and 3.2. □

4. Examples and numerical simulation

In this section, we give two examples and numerical simulation to demonstrate the results obtained in the previous section.

Example 4.1. Consider the following Nicholson's blowflies model with patch structure and multiple linear harvesting terms:

$$\begin{cases} x_1'(t) &= -(18+\cos^2 t)x_1(t) + (1+0.7\sin^2 t)e^{e-2}x_2(t) \\ & +e^{e-1}(9.5+0.005|\sin\sqrt{2}t|)x_1(t-e^{2|\sin t|})e^{-x_1(t-e^{2|\sin t|})}, \\ & +e^{e-1}(9.5+0.005|\sin\sqrt{5}t|)x_1(t-e^{2|\cos\sqrt{3}t|})e^{-x_1(t-e^{2|\cos\sqrt{3}t|})} \\ & -0.1e^{e-2}\cos^2 t x_1(t-e^{2|\cos\sqrt{3}t|}) \\ x_2'(t) &= -(18+\sin^2 t)x_2(t) + (1+0.7\cos^2 t)e^{e-2}x_1(t) \\ & +e^{e-1}(9.5+0.005|\cos\sqrt{2}t|)x_2(t-e^{2|\cos t|})e^{-x_2(t-e^{2|\cos t|})}, \\ & +e^{e-1}(9.5+0.005|\sin\sqrt{6}t|)x_2(t-e^{2|\cos\sqrt{7}t|})e^{-x_2(t-e^{2|\cos\sqrt{7}t|})} \\ & -0.05e^{e-2}\sin^2 t x_2(t-e^{2|\cos\sqrt{3}t|}), \end{cases} \tag{4.1}$$

Obviously, $\alpha_i^- = 18,\ \alpha_i^+ = 19,\ \gamma_{ij}^+ = \gamma_{ij}^- = 1,\ c_{ij}^- = 9.5e^{e-1},\ c_{ij}^+ = 9.505e^{e-1}(i,j=1,2);\ \beta_{12}^- = \beta_{21}^- = e^{e-2},\ \beta_{12}^+ = \beta_{21}^+ = 1.7e^{e-2},$ $H_{11}^+ = 0.1e^{e-2},\ H_{21}^+ = 0.05e^{e-2},\ r_1 = r_2 = e^2,$

$$A = \begin{pmatrix} 18 & 0 \\ 0 & 18 \end{pmatrix},\quad B = \begin{pmatrix} 0 & 1.7e^{e-2} \\ 1.7e^{e-2} & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 19.01e^{e-3} & 0 \\ 0 & 19.01e^{e-3} \end{pmatrix},\quad H = \begin{pmatrix} 0.1e^{e-2} & 0 \\ 0 & 0.05e^{e-2} \end{pmatrix}.$$

Let $E_{i1} = e$ and $E_{i2} = 1$ for $i = 1,2$, we obtain

$$\sum_{j=1}^{2}\frac{\beta_{ij}^+ E_{j1}}{\alpha_i^-} + \sum_{j=1}^{2}\frac{c_{ij}^+}{\alpha_i^-\gamma_{ij}^- e} = \frac{1.7e^{e-1} + 19.01e^{e-2}}{18} < e, \quad i = 1,2 \tag{4.2}$$

$$\sum_{j=1}^{2}\frac{\beta_{1j}^- E_{j2}}{\alpha_1^+} + \sum_{j=1}^{2}\frac{c_{1j}^-}{\alpha_1^+}E_{11}e^{-\gamma_{1j}^+ E_{11}} - \frac{H_{11}^+ E_{11}}{\alpha_1^+} = \frac{19 + e^{e-2} - 0.1e^{e-1}}{19} > 1, \tag{4.3}$$

$$\sum_{j=1}^{2}\frac{\beta_{2j}^- E_{j2}}{\alpha_2^+} + \sum_{j=1}^{2}\frac{c_{2j}^-}{\alpha_2^+}E_{21}e^{-\gamma_{2j}^+ E_{21}} - \frac{H_{21}^+ E_{21}}{\alpha_2^+} = \frac{19 + e^{e-2} - 0.05e^{e-1}}{19} > 1, \tag{4.4}$$

$$\rho(A^{-1}(B+C+H)) \approx 0.9991 < 1, \tag{4.5}$$

Then (4.2)–(4.5) imply that the Nicholson's Blowflies model with patch structure and multiple linear harvesting terms system (4.1) satisfies (2.4), (2.5) and (3.1). Hence, from Theorem 3.1 and Theorem 3.2, system (4.1) has a positive almost periodic solution

$$x^*(t) \in B^* = \{\varphi | \varphi \in B, 1 \leqslant \varphi_i(t) \leqslant e, \quad \text{for all } t \in R,\ i = 1,2\}.$$

Moreover, if $\varphi \in C^0 = \{\varphi|\varphi \in C, 1 < \varphi_i(t) < e,\ \text{for all } t \in [-e^2, 0],\ i = 1,2\}$, then $x(t; t_0, \varphi)$ converges exponentially to $x^*(t)$ as $t \to +\infty$.

Remark 4.1. System (4.1) is a very simple form of Nicholson's Blowflies model with patch structure and multiple linear harvesting terms. One can observe that $||A^{-1}(B+C+H)||_1 = \frac{1.8e^{e-2}+19.01e^{e-3}}{18} \approx 1.0019 > 1$, where $||\cdot||_1$ is the row norm of matrix. Therefore, all the results in [18] and the references therein can not be applicable to system (4.1). This implies that the results of this paper are essentially new.

Example 4.2. Consider the following Nicholson's blowflies model with patch structure and multiple linear harvesting terms:

$$\begin{cases} x_1'(t) &= -(19+\cos^2 t)x_1(t) + (0.5+0.05\sin^2 t)e^{e-2}x_2(t) \\ &\quad +(0.5+0.05\cos^2\sqrt{2}t)e^{e-2}x_3(t) \\ &\quad +e^{e-1}(10+0.005|\sin\sqrt{3}t|)x_1(t-e^{2|\sin\sqrt{2}t|})e^{-x_1(t-e^{2|\sin\sqrt{2}t|})} \\ &\quad +e^{e-1}(10+0.005|\sin\sqrt{3}t|)x_1(t-e^{2|\cos\sqrt{5}t|})e^{-x_1(t-e^{2|\cos\sqrt{5}t|})} \\ &\quad -(0.1e^{e-2}\cos^2\sqrt{2}t)x_1(t-e^{2|\cos t|}) - (0.1e^{e-2}\sin^2\sqrt{2}t)x_1(t-e^{2|\sin t|}) \\ x_2'(t) &= -(19+\sin^2 t)x_2(t) + (0.5+0.05\cos^2 t)e^{e-2}x_1(t) \\ &\quad +(0.5+0.05\cos^2\sqrt{3}t)e^{e-2}x_3(t) \\ &\quad +e^{e-1}(10+0.005|\cos 2t|)x_2(t-e^{2|\sin t|})e^{-x_2(t-e^{2|\sin t|})} \\ &\quad +e^{e-1}(10+0.005|\cos 5t|)x_2(t-e^{2|\sin 7t|})e^{-x_2(t-e^{2|\sin 7t|})} \\ &\quad -(0.1e^{e-2}\sin^2\sqrt{2}t)x_2(t-e^{2|\cos\sqrt{3}t|}) - (0.1e^{e-2}\cos^2\sqrt{2}t)x_2(t-e^{2|\sin\sqrt{3}t|}) \\ x_3'(t) &= -(19+\sin^2\sqrt{2}t)x_3(t) + (0.5+0.05\sin^2\sqrt{5}t)e^{e-2}x_1(t) \\ &\quad +(0.5+0.05\cos^2\sqrt{5}t)e^{e-2}x_2(t) \\ &\quad +e^{e-1}(10+0.005|\cos 3t|)x_3(t-e^{2|\sin 3t|})e^{-x_3(t-e^{2|\sin 3t|})} \\ &\quad +e^{e-1}(10+0.005|\sin 5t|)x_3(t-e^{2|\cos 3t|})e^{-x_3(t-e^{2|\cos 3t|})} \\ &\quad -(0.1e^{e-2}\sin^2\sqrt{5}t)x_3(t-e^{2|\sin\sqrt{5}t|}) - (0.1e^{e-2}\cos^2\sqrt{5}t)x_3(t-e^{2|\cos\sqrt{5}t|}), \end{cases} \tag{4.6}$$

Obviously, $\alpha_i^- = 19$, $\alpha_i^+ = 20$, $r_i = \max\{\max_{1\leqslant j\leqslant 2}\{\tau_{ij}^+\}, \max_{1\leqslant j\leqslant 2}\sigma_{ij}^+\} = e^2$ $(i=1,2,3)$; $\gamma_{ij}^+ = \gamma_{ij}^- = 1$, $c_{ij}^- = 10e^{e-1}$, $c_{ij}^+ = 10.005e^{e-1}$, $H_{ij}^+ = 0.1e^{e-2}$ $(i=1,2,3,\ j=1,2)$; $\beta_{ij}^- = 0.5e^{e-2}$, $\beta_{ij}^+ = 0.55e^{e-2}$ $(i,j=1,2,3,\ i\neq j)$.

$$A = \begin{pmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0.55e^{e-2} & 0.55e^{e-2} \\ 0.55e^{e-2} & 0 & 0.55e^{e-2} \\ 0.55e^{e-2} & 0.55e^{e-2} & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 20.01e^{e-3} & 0 & 0 \\ 0 & 20.01e^{e-3} & 0 \\ 0 & 0 & 20.01e^{e-3} \end{pmatrix}, \quad H = \begin{pmatrix} 0.2e^{e-2} & 0 & 0 \\ 0 & 0.2e^{e-2} & 0 \\ 0 & 0 & 0.2e^{e-2} \end{pmatrix}.$$

Let $E_{i1} = e$ and $E_{i2} = 1$ for $i = 1,2,3$, we obtain

$$\sum_{j=1}^{3} \frac{\beta_{ij}^+ E_{j1}}{\alpha_i^-} + \sum_{j=1}^{2} \frac{c_{ij}^+}{\alpha_i^- \gamma_{ij}^- e} = \frac{1.1e^{e-1} + 20.01e^{e-2}}{19} < e, \quad i = 1,2,3, \tag{4.7}$$

$$\sum_{j=1}^{3} \frac{\beta_{ij}^- E_{j2}}{\alpha_i^+} + \sum_{j=1}^{2} \frac{c_{ij}^-}{\alpha_i^+} E_{i1} e^{-\gamma_{ij}^+ E_{i1}} - \sum_{j=1}^{2} \frac{H_{ij}^+ E_{i1}}{\alpha_i^+} = \frac{20 + e^{e-2} - 0.2e^{e-1}}{20} > 1, \quad i = 1,2,3 \tag{4.8}$$

and

$$\rho(A^{-1}(B+C+H)) \approx 0.9349 < 1. \tag{4.9}$$

Then, (4.7)–(4.9) imply that the Nicholson's blowflies model with patch structure and multiple linear harvesting terms system (4.6) satisfies (2.4), (2.5) and (3.1). Hence, from Theorem 3.1 and Theorem 3.2, system (4.6) has a positive almost periodic solution

$$x^*(t) \in B^* = \{\varphi | \varphi \in B, 1 \leqslant \varphi_i(t) \leqslant e, \quad \text{for all } t \in R^1,\ i = 1,2,3\}.$$

Moreover, if $\varphi \in C^0 = \{\varphi | \varphi \in C, 1 < \varphi_i(t) < e,\ \text{for all } t \in [-e^2, 0],\ i = 1,2,3\}$, then $x(t; t_0, \varphi)$ converges exponentially to $x^*(t)$ as $t \to +\infty$. The fact is verified by the numerical simulation in Fig. 1 with numerical solution $x(t) = (x_1(t), x_2(t), x_3(t))^T$ of system (4.7) for initial value $\varphi(t) \equiv (1.5, 1.5, 1.5)^T$.

Remark 4.2. To the best of our knowledge, few authors have considered the problems of positive almost periodic solution of Nicholson's blowflies model with patch structure and multiple linear harvesting terms. Therefore, the main results in [13,18,24–32] and the references therein can not be applicable to prove that all the solutions of (4.6) with initial value $\varphi \in C^0$ converge exponentially to the positive almost periodic solution. This implies that the results of this paper are new and they complement previously known results.

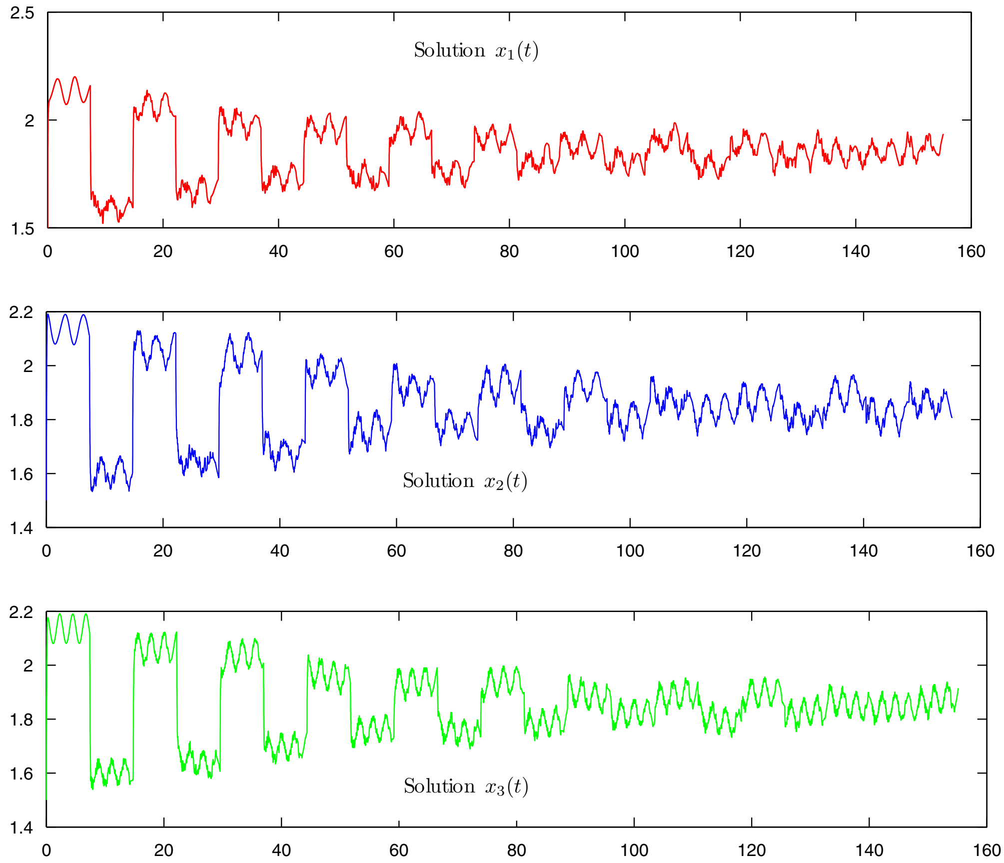

Fig. 1. Numerical solution $x(t) = (x_1(t), x_2(t), x_3(t))^T$ of system (4.7) for initial value $\varphi(t) \equiv (1.5, 1.5, 1.5)^T$.

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