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Nontrivial harmonic waves with positive instantaneous frequency

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Abstract

The concept of time-varying frequency is fundamental in communications and nature. One description of a time-varying frequency is the instantaneous frequency of a signal. Our concern is the design of analytic signals with nonlinear phase and positive instantaneous frequency. In this paper, a kind of nontrivial orthonormal harmonic waves are derived by applying the Gram–Schmidt procedure to the Blaschke products. Furthermore, we show the orthonormal harmonic waves satisfy the analytic condition and have positive instantaneous frequencies which are tightly associated with the average frequencies of the two mono-component analytic signals at each time. As a consequence, non-stationary signals can be decomposed into the superposition of nontrivial harmonic waves. Related conclusions are established for the orthonormal analytic signals on the whole time range.

Keywords: Analytic signal; Nonlinear phase; Instantaneous frequency; Blaschke product; Gram–Schmidt procedure

1. Introduction

The amplitude and frequency of non-stationary signals vary with time. The traditional Fourier analysis, however, cannot expose the time-varying property of non-stationary signals, due to the fact that the classical Fourier analysis is that of decomposing a general signal in terms of orthonormal harmonic waves $\mathrm{e}^{\mathrm{j}nt}$, $\mathrm{j} = \sqrt{-1}$, $n \in \mathbb{Z}$ of which each has a constant amplitude and constant frequency that is the derivative of the linear phase $\phi(t) = nt$. It is well known that a purely monochromatic signal such as $a\mathrm{e}^{\mathrm{j}\omega t}$ cannot transmit any information.

Can we find nontrivial orthonormal harmonic waves with time-varying amplitude and frequency? For this purpose amplitude–phase modulation waves are required and the orthonormal property should be maintained by the modulation waves. The amplitude–phase modulation waves are commonly introduced as the complex signals

$$f(t) = \rho(t)\mathrm{e}^{\mathrm{j}\phi(t)} \tag{1.1}$$

with the time-varying amplitude $\rho(t) \geq 0$ and nonlinear phase $\phi(t)$. By analytic signal approach [10], functions satisfying the equation

$$\mathcal{H}(f)(t) = -\mathrm{j}f(t) \tag{1.2}$$

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are called analytic signals, where $\mathcal{H}(f)$ is the Hilbert transform of $f(t)$ on the real line, defined by

$$\mathcal{H}(f)(t) := \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty}\frac{f(s)}{t-s}\mathrm{d}s. \tag{1.3}$$

Through representation (1.1) the instantaneous frequency may be defined as the derivative of phase $\phi(t)$. This is a generalization of the trivial case where the phase is linear in time. Signals should have particular properties to ensure that their instantaneous frequencies obtained by analytic signal are physically meaningful. Our concern is the instantaneous frequency should be nonnegative [8,9,13,15,22].

The ideal method of time–frequency analysis for nonlinear and non-stationary signals would be to adaptively decompose a signal into a sum of simple signals. In [17,19], the authors point out that the Blaschke product is analytic and mono-component, which can be regarded as the simple signals. Thus we think it is suitable to approximate a signal by a linear combination of the Blaschke products. However, the restriction is imposed to obtain the simple signals as an orthonormal system. Note that the amplitude–phase modulation waves $f(t) = \rho(t)\mathrm{e}^{\mathrm{j}\phi(t)}$ should have the orthonormal property as the trivial harmonic waves $\mathrm{e}^{\mathrm{j}nt}$. In recognition of this, in this paper, nontrivial orthonormal harmonic waves are derived by the Gram–Schmidt procedure from the Blaschke product. Furthermore, we show that the nontrivial harmonic waves are analytic signals and have positive instantaneous frequencies. The rest of the paper is organized as follows: Section 2 is devoted to reviewing unit circular analytic signals [7,16,17]. In Section 3 we provide some results on Blaschke bases [1,2,4,14,21]. In Section 4, by applying the Gram–Schmidt procedure to the Blaschke product, we get a kind of nontrivial harmonic waves, analogous to the Blaschke bases. In Section 5, we depict that the nontrivial harmonic waves satisfy the analytic condition (1.2) and have positive instantaneous frequencies which are tightly associated with the average frequencies of the two mono-component analytic signals at each time. Section 6 is devoted to orthonormal analytic signals on the whole time range. Some conclusions are drawn in Section 7.

2. Unit circular analytic signals

A time-limited signal $f(t) \in L^2[-\pi, \pi]$ can be considered as a periodic signal restricted on one period. Therefore it can be expanded into a $L^2[-\pi, \pi]$-convergent Fourier series [23]

$$f(t) = \sum_{n\in\mathbb{Z}} c_n(f)\mathrm{e}^{\mathrm{j}nt} \tag{2.1}$$

with the Fourier coefficients $c_n(f)$ defined by

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-\mathrm{j}nt}\mathrm{d}t.$$

Substituting (2.1) into (1.3) leads to

$$\begin{aligned}
\mathcal{H}(f)(t) &= \frac{1}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty}\sum_{n\in\mathbb{Z}} c_n(f)\mathrm{e}^{\mathrm{j}ns}\frac{1}{t-s}\mathrm{d}s \\
&= \sum_{n\in\mathbb{Z}} c_n(f)\frac{1}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{j}ns}}{t-s}\mathrm{d}s \\
&= \sum_{n\in\mathbb{Z}} c_n(f)\mathcal{H}(\mathrm{e}^{\mathrm{j}ns})(t) \\
&= \sum_{n\in\mathbb{Z}} -\mathrm{j}\ \mathrm{sgn}(n)c_n(f)\mathrm{e}^{\mathrm{j}nt}
\end{aligned}$$

in the distribution sense. We call the Hilbert transform of the time-limited signal $f(t)$, $t \in [-\pi, \pi]$ as circular Hilbert transform

$$\mathcal{H}_c f(t) := \sum_{n=-\infty}^{\infty} -\mathrm{j}\ \mathrm{sgn}(n)c_n(f)\mathrm{e}^{\mathrm{j}nt},$$

where $\mathrm{sgn}(n)$ is the signum function

$$\mathrm{sgn}(n) = \begin{cases} 1, & n = 1, 2, \ldots \\ -1, & n = -1, -2, \ldots \\ 0, & n = 0. \end{cases}$$

It is known that the above defined circular Hilbert transform has an alternative form as a singular integral:

$$\mathcal{H}_c f(t) = \frac{1}{\pi} \text{ p.v.} \int_{-\pi}^{\pi} f(x) \cot\left(\frac{t-x}{2}\right) \mathrm{d}x.$$

Using the definition of circular Hilbert transform, one easily gets,

$$\mathcal{H}_c \mathrm{e}^{\mathrm{j}nt} = -\mathrm{j}\, \mathrm{sgn}(n) \mathrm{e}^{\mathrm{j}nt}.$$

Hence, for positive integer n, we have

$$\mathcal{H}_c(\cos nt) = \sin nt \quad \text{and} \quad \mathcal{H}_c(\sin nt) = -\cos nt,$$

which mean that the transform $\mathcal{H}_c$ does not change the shape and corresponds to the property of $\frac{\pi}{2}$-phase-shift of trivial harmonic waves.

They are the cases with linear phase. For a signal with nonlinear phase, in [16], the authors discuss the existence of circular unit analytic signals with nonlinear phase and construct the unit circular analytic signals

$$\mathrm{e}^{\mathrm{j}\theta_a(t)} = \frac{\mathrm{e}^{\mathrm{j}t} - a}{1 - \bar{a}\mathrm{e}^{\mathrm{j}t}}, \quad |a| < 1,\ t \in [-\pi, \pi],$$

through the Möbius transform

$$\tau_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

For any complex number $a = |a|\mathrm{e}^{\mathrm{i}t_a}$ with $|a| < 1$, we can get the phase $\theta_a(t)$ has the unique decomposition: the sum of a linear part and a periodic part, so-called nonlinear Fourier atom, that is

$$\theta_a(t) := t + 2 \arctan \frac{|a| \sin(t - t_a)}{1 - |a| \cos(t - t_a)}, \quad t \in [-\pi, \pi]. \tag{2.2}$$

We know that the real part and imaginary part of $\mathrm{e}^{\mathrm{j}\theta_a(t)}$ satisfy the Hilbert relation

$$\mathcal{H}_c \cos\theta_a(t) = \sin\theta_a(t).$$

Consequently, the complex signal $\mathrm{e}^{\mathrm{j}\theta_a(t)}$, $t \in [-\pi, \pi]$ is a circular analytic signal with the nonlinear phase $\theta_a(t)$ and the instantaneous frequency $\theta_a'(t)$ is the Poisson kernel $p_a(t) > 0$ for the point a [12].

3. Blaschke bases on the circle

Let $\mathbb{R}$ be the field of real numbers and let $\mathbb{C}$ be the field of complex numbers. We denote the open upper-half complex plane by $\mathbb{C}^+$ and the open unit disc in $\mathbb{C}$ by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the boundary of $\mathbb{D}$. We also use $\mathbb{Z}$ to denote the set of integral numbers and $\mathbb{N}$ the set of positive integer numbers and $\mathbb{N}_0$ the set of nonnegative integer numbers, respectively.

We denote $A(\mathbb{D})$ and $A(\mathbb{C}^+)$ the set of the analytic functions in $\mathbb{D}$ and in $\mathbb{C}^+$, respectively. We deal with Hardy spaces H^p of the unit disk $\mathbb{D}$ or the upper half plane $\mathbb{C}^+$. H^∞ is the Banach algebra of bounded analytic functions. The norm of an element $f \in H^\infty$ is given as $\|f\|_\infty = \sup_z |f(z)| < \infty$, with z ranging over the disk. For $0 < p < \infty$, $f \in H^p(\mathbb{D})$ means that f is in $A(\mathbb{D})$ and that

$$\|f\|_p^p = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\mathrm{e}^{\mathrm{j}t})|^p \mathrm{d}t < \infty;$$

$f \in H^p(\mathbb{C}^+)$ means that f is in $A(\mathbb{C}^+)$ and that

$$\|f\|_p^p = \sup_{0<y<\infty} \int_{\mathbb{R}} |f(t+\mathrm{i}y)|^p \mathrm{d}t < \infty.$$

Suppose that $\{a_k\}_{k=1}^{\infty} \subset \mathbb{D}$, denote the Blaschke product as

$$b_n(z) := \prod_{k=1}^{n} \frac{z-a_k}{1-\bar{a}_k z}. \tag{3.1}$$

Set the corresponding Blaschke bases as

$$B_0(z) := 1, \qquad B_n(z) := \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z} b_{n-1}(z), \quad n = 1, 2, \ldots \tag{3.2}$$

where the Blaschke product $b_n(z)$, $n \in \mathbb{N}$ is defined as in (3.1) and $b_0(z) := 1$. They are orthonormal in $H^2(\mathbb{T})$ with respect to the inner-product

$$\langle f, g\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{j}\omega})\overline{g(\mathrm{e}^{\mathrm{j}\omega})}\mathrm{d}\omega, \quad f, g \in H^2(\mathbb{T}). \tag{3.3}$$

That is,

$$\langle B_n, B_m\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} B_n(\mathrm{e}^{\mathrm{j}\omega})\overline{B_m(\mathrm{e}^{\mathrm{j}\omega})}\mathrm{d}\omega = \delta_{nm},$$

where δ_{nm} is the Kronecker delta function.

The following results [1,2,4,14,21] tell us that the orthonormal set $\{B_n : n \in \mathbb{N}_0\}$ is complete in Hardy space $H^2(\mathbb{T})$ for a suitable choice of the zeros $\{a_n : n \in \mathbb{N}\}$ in $\mathbb{D}$. Completeness means that any function in $H^2(\mathbb{T})$ can be arbitrarily approximated by linear combinations of the basis functions. In that case the closure of span$\{B_n : n \in \mathbb{N}_0\}$ equals $H^2(\mathbb{T})$.

Proposition 1. *The orthonormal set* $\{B_n : n \in \mathbb{N}_0\}$ *is complete in Hardy space* $H^2(\mathbb{T})$ *if and only if*

$$\sum_{n=1}^{\infty}(1-|a_n|) = \infty.$$

Remark. The basis $\{B_n : n \in \mathbb{N}_0\}$ is uniformly bounded if

$$\sup_{n\in\mathbb{N}} |a_n| = r < 1.$$

Obviously, the completeness condition $\sum_{n=1}^{\infty}(1-|a_n|) = \infty$ is satisfied by the uniformly bounded basis.

A simple calculation yields that

$$B_n(z) = \frac{\overline{a}_n b_n(z) + b_{n-1}(z)}{\sqrt{1-|a_n|^2}}. \tag{3.4}$$

Therefore, the set $\{b_n(\mathrm{e}^{\mathrm{j}t}) : n \in \mathbb{N}_0\}$ is complete in Hardy space $H^2(\mathbb{T})$ if and only if $\sum_{n=1}^{\infty}(1-|a_n|) = \infty$ is satisfied.

4. Orthonormal harmonic waves on the circle

It is worth pointing out that the Gram–Schmidt procedure of the Blaschke product system $\{b_n(\mathrm{e}^{\mathrm{j}t}), n \in \mathbb{N}_0\}$ will lead to the nontrivial orthonormal harmonic waves

$$\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t}) := \frac{\mathrm{e}^{\mathrm{j}t}\sqrt{1-|a_n|^2}}{1-\bar{a}_n\mathrm{e}^{\mathrm{j}t}}\, b_{n-1}(\mathrm{e}^{\mathrm{j}t}), \qquad \mathcal{B}_0(\mathrm{e}^{\mathrm{j}t}) := 1, \quad n = 1, 2, \ldots \tag{4.1}$$

with respect to the inner product defined as in (3.3). Consequently, we would like to explore the links between the orthonormal construction and bases formed from the classical orthonormal polynomials, and to try to apply the Gram–Schmidt procedure to the Blaschke-products system. The already known Laguerre basis and two-parameter Kautz basis are the special cases of our general construction where all the zeros $\{a_n : n \in \mathbb{N}\}$ are chosen to be the same and real (Laguerre) or complex (Kautz).

The nontrivial harmonic waves defined by (4.1) are analogous to the Blaschke bases discussed in the above section; we will find that the modified parameter e^{jt} can guarantee that the nontrivial harmonic waves have the positive instantaneous frequencies.

Theorem 4.1. *The nontrivial harmonic waves $\{\mathcal{B}_n(e^{jt}) : n \in \mathbb{N}_0\}$ are orthonormal in $H^2(\mathbb{T})$ with respect to the inner-product*

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{jt})\overline{g(e^{jt})}dt, \quad f, g \in H^2(\mathbb{T}).$$

Furthermore the waves $\{\mathcal{B}_n(e^{jt}) : n \in \mathbb{N}_0\}$ can be derived by Gram–Schmidt procedure from the Blaschke products $b_n(e^{jt})$.

Proof. For convenience we denote by $b_n = b_n(e^{jt})$. Starting from the Blaschke products $\{b_0, b_1, \ldots\}$, the Gram–Schmidt procedure can generate the nontrivial orthonormal harmonic waves $\{\mathcal{B}_n(e^{jt})\}_{n=0}^{\infty}$ by the recursion

$$\mathcal{G}_n := b_n - \sum_{k=0}^{n-1}\langle b_n, \mathcal{B}_k\rangle \mathcal{B}_k,$$

and the normalization

$$\mathcal{B}_n := \frac{\mathcal{G}_n}{\|\mathcal{G}_n\|}.$$

We will show that $\mathcal{B}_n$ is just $\mathcal{B}_n(e^{jt})$ defined as in (4.1). We adopt induction. Obviously $\mathcal{B}_0 = \mathcal{G}_0 = 1$ since $b_0 = 1$. Thus

$$\begin{aligned}\mathcal{G}_1 &= b_1 - \langle b_1, \mathcal{B}_0\rangle \mathcal{B}_0\\ &= b_1(e^{jt}) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{jt}-a_1}{1-\bar{a}_1 e^{jt}}dt\\ &= \frac{e^{jt}-a_1}{1-\bar{a}_1 e^{jt}} + a_1\\ &= (1-|a_1|^2)\frac{e^{jt}}{1-\bar{a}_1 e^{jt}}.\end{aligned}$$

Note that the norm of $\mathcal{G}_1$ can be obtained by the following calculation

$$\begin{aligned}\|\mathcal{G}_1\|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-|a_1|^2)e^{jt}}{1-\bar{a}_1 e^{jt}}\frac{(1-|a_1|^2)e^{-jt}}{1-a_1 e^{-jt}}dt\\ &= \frac{1}{2\pi j}\oint_{|z|=1}\frac{(1-|a_1|^2)^2}{(1-\bar{a}_1 z)(z-a_1)}dz\\ &= 1-|a_1|^2.\end{aligned}$$

Therefore, $\mathcal{B}_1 = \mathcal{B}_1(e^{jt})$. Now we consider the case with $n = 2$, in this case, we have

$$\mathcal{G}_2 = b_2 - \langle b_2, \mathcal{B}_0\rangle \mathcal{B}_0 - \langle b_2, \mathcal{B}_1\rangle \mathcal{B}_1.$$

We can easily calculate the inner product $\langle b_2, \mathcal{B}_k\rangle$ with $k = 0, 1$ as follows

$$\begin{aligned}\langle b_2, \mathcal{B}_0\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{jt}-a_1}{1-\bar{a}_1 e^{jt}}\frac{e^{jt}-a_2}{1-\bar{a}_2 e^{jt}}dt = \frac{1}{2\pi j}\oint_{|z|=1}\frac{z-a_1}{1-\bar{a}_1 z}\frac{z-a_2}{1-\bar{a}_2 z}\frac{dz}{z}\\ &= a_1 a_2\end{aligned}$$

and

$$\langle b_2, \mathcal{B}_1\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{j}t}-a_1}{1-\bar{a}_1\mathrm{e}^{\mathrm{j}t}}\frac{\mathrm{e}^{\mathrm{j}t}-a_2}{1-\bar{a}_2\mathrm{e}^{\mathrm{j}t}}\frac{\sqrt{1-|a_1|^2}\mathrm{e}^{-\mathrm{j}t}}{1-a_1\mathrm{e}^{-\mathrm{j}t}}\mathrm{d}t = -a_2\sqrt{1-|a_1|^2}.$$

Thus we get the explicit representation of $\mathcal{G}_2$

$$\begin{aligned}\mathcal{G}_2 &= \frac{\mathrm{e}^{\mathrm{j}t}-a_1}{1-\bar{a}_1\mathrm{e}^{\mathrm{j}t}}\frac{\mathrm{e}^{\mathrm{j}t}-a_2}{1-\bar{a}_2\mathrm{e}^{\mathrm{j}t}} - a_1a_2 + a_2\sqrt{1-|a_1|^2}\frac{\sqrt{1-|a_1|^2}\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_1\mathrm{e}^{\mathrm{j}t}} \\ &= \frac{1}{(1-\bar{a}_1\mathrm{e}^{\mathrm{j}t})(1-\bar{a}_2\mathrm{e}^{\mathrm{j}t})}(\mathrm{e}^{2\mathrm{j}t} - a_1\mathrm{e}^{\mathrm{j}t} + a_1|a_2|^2\mathrm{e}^{\mathrm{j}t} - |a_2|^2\mathrm{e}^{2\mathrm{j}t}) \\ &= \frac{(1-|a_2|^2)\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_2\mathrm{e}^{\mathrm{j}t}} b_1(\mathrm{e}^{\mathrm{j}t})\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{G}_2\|^2 &= \frac{1}{2\pi\mathrm{j}}\oint_{|z|=1}\frac{(1-|a_2|^2)z}{1-\bar{a}_2 z}b_1(z)\frac{(1-|a_2|^2)\bar{z}}{1-a_2\bar{z}}\overline{b_1(z)}\frac{\mathrm{d}z}{z} \\ &= 1-|a_2|^2.\end{aligned}$$

This implies that $\mathcal{B}_2 = \mathcal{B}_2(\mathrm{e}^{\mathrm{j}t})$. Now suppose that $\mathcal{B}_n = \mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})$ for $n > 1$. It follows that

$$\mathcal{G}_n := b_n - \langle b_n, \mathcal{B}_0\rangle\mathcal{B}_0 - \sum_{k=1}^{n-1}\langle b_n, \mathcal{B}_k\rangle\mathcal{B}_k = \frac{(1-|a_n|^2)\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_n\mathrm{e}^{\mathrm{j}t}}b_{n-1}(\mathrm{e}^{\mathrm{j}t})$$

with

$$\langle b_n, \mathcal{B}_0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} b_n(\mathrm{e}^{\mathrm{j}t})\mathrm{d}t = \frac{1}{2\pi\mathrm{j}}\oint_{|z|=1} b_n(\mathrm{e}^{\mathrm{j}t})\frac{\mathrm{d}z}{z} = (-1)^n a_1a_2\cdots a_n,$$

and

$$\begin{aligned}\langle b_n, \mathcal{B}_k\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b_n(\mathrm{e}^{\mathrm{j}t})\frac{\sqrt{1-|a_k|^2}\mathrm{e}^{-\mathrm{j}t}}{1-a_k\mathrm{e}^{-\mathrm{j}t}}\overline{b_{k-1}(\mathrm{e}^{\mathrm{j}t})}\mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{j}}\oint_{|z|=1}\frac{b_n(z)}{b_{k-1}(z)}\frac{\sqrt{1-|a_k|^2}}{z-a_k}\frac{\mathrm{d}z}{z} \\ &= (-1)^{n-k}a_{k+1}a_{k+2}\cdots a_n\sqrt{1-|a_k|^2}, \quad k = 1, 2, \ldots, n-1.\end{aligned}$$

Thus we get

$$\begin{aligned}\sum_{k=1}^{n-1}\langle b_n, \mathcal{B}_k\rangle\mathcal{B}_k &= \sum_{k=1}^{n-1}(-1)^{n-k}a_{k+1}a_{k+2}\cdots a_n\sqrt{1-|a_k|^2}\frac{\sqrt{1-|a_k|^2}\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_k\mathrm{e}^{\mathrm{j}t}}b_{k-1}(\mathrm{e}^{\mathrm{j}t}) \\ &= b_n(\mathrm{e}^{\mathrm{j}t}) - (-1)^n a_1a_2\cdots a_n - \frac{(1-|a_n|^2)\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_n\mathrm{e}^{\mathrm{j}t}}b_{n-1}(\mathrm{e}^{\mathrm{j}t}).\end{aligned} \tag{4.2}$$

Then we will get the representation of $\mathcal{G}_{n+1}$. Note that

$$\mathcal{G}_{n+1} := b_{n+1} - \langle b_{n+1}, \mathcal{B}_0\rangle\mathcal{B}_0 - \sum_{k=1}^{n}\langle b_{n+1}, \mathcal{B}_k\rangle\mathcal{B}_k,$$

with

$$\begin{aligned}\langle b_{n+1}, \mathcal{B}_0\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b_{n+1}(\mathrm{e}^{\mathrm{j}t})\mathrm{d}t = \frac{1}{2\pi\mathrm{j}}\oint_{|z|=1} b_{n+1}(\mathrm{e}^{\mathrm{j}t})\frac{\mathrm{d}z}{z} \\ &= (-1)^{n+1}a_1a_2\cdots a_{n+1},\end{aligned}$$

and

$$\begin{aligned}\langle b_{n+1}, \mathcal{B}_k\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b_{n+1}(\mathrm{e}^{\mathrm{j}t})\frac{\sqrt{1-|a_k|^2}\mathrm{e}^{-\mathrm{j}t}}{1-a_k\mathrm{e}^{-\mathrm{j}t}}\overline{b_{k-1}(\mathrm{e}^{\mathrm{j}t})}\mathrm{d}t\\ &= \frac{1}{2\pi\mathrm{j}}\oint_{|z|=1}\frac{b_{n+1}(z)}{b_{k-1}(z)}\frac{\sqrt{1-|a_k|^2}}{z-a_k}\frac{\mathrm{d}z}{z}\\ &= (-1)^{n+1-k}a_{k+1}a_{k+2}\cdots a_{n+1}\sqrt{1-|a_k|^2},\quad k=1,2,\ldots,n.\end{aligned}$$

Therefore, $\mathcal{G}_{n+1}$ equals

$$\mathcal{G}_{n+1} = b_{n+1}(\mathrm{e}^{\mathrm{j}t}) - (-1)^{n+1}a_1a_2\cdots a_{n+1} - \Sigma_n,$$

where

$$\Sigma_n = \sum_{k=1}^{n}(-1)^{n+1-k}a_{k+1}a_{k+2}\cdots a_{n+1}\frac{(1-|a_k|^2)\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_k\mathrm{e}^{\mathrm{j}t}}b_{k-1}(\mathrm{e}^{\mathrm{j}t}).$$

We need to calculate the summation of the right side of above equation from the relation (4.2),

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{n+1-k}a_{k+1}a_{k+2}\cdots a_{n+1}\frac{(1-|a_k|^2)\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_k\mathrm{e}^{\mathrm{j}t}}b_{k-1}(\mathrm{e}^{\mathrm{j}t})\\ &\quad= (-1)a_{n+1}\left\{\sum_{k=1}^{n-1}\langle b_n, \mathcal{B}_k\rangle\mathcal{B}_k + \frac{(1-|a_n|^2)\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_n\mathrm{e}^{\mathrm{j}t}}b_{n-1}(\mathrm{e}^{\mathrm{j}t})\right\}\\ &\quad= (-1)a_{n+1}b_n(\mathrm{e}^{\mathrm{j}t}) - (-1)^{n+1}a_1a_2\cdots a_{n+1}.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{G}_{n+1} = b_{n+1}(\mathrm{e}^{\mathrm{j}t}) + a_{n+1}b_n(\mathrm{e}^{\mathrm{j}t}) &= \left(\frac{\mathrm{e}^{\mathrm{j}t}-a_{n+1}}{1-\bar{a}_{n+1}\mathrm{e}^{\mathrm{j}t}} + a_{n+1}\right)b_n(\mathrm{e}^{\mathrm{j}t})\\ &= \frac{(1-|a_{n+1}|^2)\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_{n+1}\mathrm{e}^{\mathrm{j}t}}b_n(\mathrm{e}^{\mathrm{j}t}),\end{aligned}$$

and the norm of $\mathcal{G}_{n+1}$ is easy to get

$$\begin{aligned}\|\mathcal{G}_{n+1}\|^2 &= \frac{1}{2\pi j}\oint_{|z|=1}\frac{(1-|a_{n+1}|^2)z}{1-\bar{a}_{n+1}z}b_n(z)\frac{(1-|a_{n+1}|^2)\bar{z}}{1-a_{n+1}\bar{z}}\overline{b_z(z)}\frac{\mathrm{d}z}{z}\\ &= 1-|a_{n+1}|^2.\end{aligned}$$

This implies that $\mathcal{B}_{n+1} = \mathcal{B}_{n+1}(\mathrm{e}^{\mathrm{j}t}) = \frac{\sqrt{1-|a_{n+1}|^2}\mathrm{e}^{\mathrm{j}t}}{1-\bar{a}_{n+1}\mathrm{e}^{\mathrm{j}t}}b_n(\mathrm{e}^{\mathrm{j}t})$. □

Remark. It is easy to see that the case $a_k = 0,\ k = 0, 1, 2, \ldots, n$ corresponds to the trivial orthonormal harmonic waves $\{\mathrm{e}^{jnt}, n \in \mathbb{N}_0\}$.

Remark. For the case $a_k = a,\ k = 0, 1, 2, \ldots, n$, the nontrivial orthonormal harmonic waves reduce to the Laguerre bases discussed in [6,11].

We will denote $\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})$ in the polar form $\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}$ with amplitude and phase as follows. For complex numbers $a_n = |a_n|\mathrm{e}^{\mathrm{j}t_{a_n}}$, $n = 1, 2, \ldots$ with $|a_n| < 1$, simple calculation offers us that

$$\frac{\mathrm{e}^{\mathrm{j}t}\sqrt{1-|a_n|^2}}{1-\bar{a}_n\mathrm{e}^{\mathrm{j}t}} = \sqrt{\frac{1-|a_n|^2}{1-2|a_n|\cos(t-t_{a_n})+|a_n|^2}}\,\mathrm{e}^{\mathrm{j}\left(t+\arctan\frac{|a_n|\sin(t-t_{a_n})}{1-|a_n|\cos(t-t_{a_n})}\right)}$$

and the Blaschke product $b_{n-1}(\mathrm{e}^{\mathrm{j}t})$ defined as in (3.1) can be denoted by the nonlinear Fourier atoms $\theta_{a_k}(t)$ defined as in (2.2)

$$b_{n-1}(\mathrm{e}^{\mathrm{j}t}) = \mathrm{e}^{\mathrm{j}\sum_{k=1}^{n-1}\theta_{a_k}(t)},$$

thus we have

$$\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t}) = \sqrt{p_{a_n}(t)}\mathrm{e}^{\mathrm{j}\psi_n(t)} := \rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}, \tag{4.3}$$

where the amplitude $\rho_n(t)$ is the positive square root of Poisson kernel $p_{a_n}(t)$ for the point a_n [12], satisfying

$$\frac{\mathrm{d}\theta_a(t)}{\mathrm{d}t} = \frac{1-|a|^2}{1-2|a|\cos(t-t_a)+|a|^2} := p_a(t) > 0, \tag{4.4}$$

and the phase is

$$\begin{aligned}\psi_n(t) &= \sum_{k=1}^{n-1}\theta_{a_k}(t) + t + \arctan\frac{|a_n|\sin(t-t_{a_n})}{1-|a_n|\cos(t-t_{a_n})}\\ &= \sum_{k=1}^{n-1}\theta_{a_k}(t) + \frac{1}{2}(\theta_{a_n}(t)+t).\end{aligned} \tag{4.5}$$

We should point out this kind of generalization is not only in the sense of mathematics, but emphasizes much more on physical aspects. We will offer a brief explanation for it. Firstly, we will point out the system

$$\{1\} \cup \{\rho_n(t)\cos\psi_n(t) : n \geq 1\} \cup \{\rho_n(t)\sin\psi_n(t) : n \geq 1\}$$

is orthonormal in the space of $L^2([-\pi,\pi])$. In fact, simple calculation offers us that

$$\int_{-\pi}^{\pi}\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})\mathcal{B}_m(\mathrm{e}^{\mathrm{j}t})\mathrm{d}t = 0, \qquad \int_{-\pi}^{\pi}\overline{\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})}\,\overline{\mathcal{B}_m(\mathrm{e}^{\mathrm{j}t})}\mathrm{d}t = 0, \quad \forall m, n$$

and

$$\int_{-\pi}^{\pi}\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})\overline{\mathcal{B}_m(\mathrm{e}^{\mathrm{j}t})}\mathrm{d}t = 0, \quad m \neq n.$$

From the relation (4.3) and $\mathrm{e}^{\mathrm{j}\psi_n(t)} = \cos\psi_n(t) + j\sin\psi_n(t)$, thus we have

$$\int_{-\pi}^{\pi}\rho_n(t)\cos\psi_n(t)\rho_m(t)\sin\psi_m(t)\mathrm{d}t = 0, \quad \forall m, n;$$
$$\int_{-\pi}^{\pi}\rho_n(t)\cos\psi_n(t)\rho_m(t)\cos\psi_m(t)\mathrm{d}t = 0, \quad m \neq n;$$
$$\int_{-\pi}^{\pi}\rho_n(t)\sin\psi_n(t)\rho_m(t)\sin\psi_m(t)\mathrm{d}t = 0, \quad m \neq n.$$

Secondly, since the system

$$\{1\} \cup \{\rho_n(t)\cos\psi_n(t) : n \geq 1\} \cup \{\rho_n(t)\sin\psi_n(t) : n \geq 1\}$$

is orthonormal in the space of $L^2([-\pi,\pi])$, any complicated signal can be expanded into different components with time-varying amplitude and phase, that is

$$f(t) = \sum_{n=-\infty}^{\infty}c_n(f)\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}$$

with $c_n(f) = \langle f, \rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}\rangle$. Compared with the classical case, the advantage of nontrivial harmonic waves can expose the time-varying property for time–frequency analysis of nonlinear and non-stationary signals.

5. Circular harmonic waves

Generally, when we discuss whether or not the product of functions $\rho(t)\cos\theta(t)$ are analytic signals, we always correspond to the Bedrosian theorem [5]. The Bedrosian theorem shows that if $f(t) = A(t)e^{j\varphi(t)}$, the amplitude $A(t)$ has low frequency and $e^{j\varphi(t)}$ has high frequency such that the ranges of two frequencies are disjoint, then

$$\mathcal{H}[A(x)e^{j\varphi(x)}](t) = A(t)\mathcal{H}[e^{j\varphi(x)}](t).$$

If $e^{j\varphi(x)}$ is just the unit analytic signal we have discussed, in such a case, we furthermore get that the signal $A(t)e^{j\varphi(t)}$ is an analytic signal satisfying the analytic condition

$$\mathcal{H}[A(x)e^{j\varphi(x)}](t) = -jA(t)e^{j\varphi(t)}.$$

As we all know, the spectral condition in the Bedrosian theorem is just a sufficient condition, not necessary. The following theorem deals with whether the product of functions $\rho_n(t)\cos\psi_n(t)$ is analytic without considering the spectral condition in the Bedrosian theorem.

Theorem 5.1. *Suppose that $\rho_n(t)$ and $\psi_n(t)$ are defined as in (4.3) and (4.5) respectively. For any positive integer n, the time-limited signals $\rho_n(t)\cos\psi_n(t)$ and $\rho_n(t)\sin\psi_n(t)$ form a pair of circular Hilbert transform, viz.*

$$\mathcal{H}_c(\rho_n(t)\cos\psi_n(t)) = \rho_n(t)\sin\psi_n(t), \quad \text{or}$$
$$\mathcal{H}_c(\rho_n(t)e^{j\psi_n(t)}) = -j\rho_n(t)e^{j\psi_n(t)}, \quad t \in [-\pi, \pi].$$

Proof. We first consider the Möbius transform function $\tau_a(z) = \frac{z-a}{1-\bar{a}z}$ can be expanded into the series

$$\frac{z-a}{1-\bar{a}z} = -a + (1-|a|^2)z + (1-|a|^2)\sum_{k=2}^{\infty}\bar{a}^{k-1}z^k$$

with the convergence domain $\{z : |z| < |a|^{-1}\}$. Noting that $e^{j\theta_a(t)}$ is the boundary value on the unit circle of Möbius transform $\tau_a(z)$, we get that

$$e^{j\theta_a(t)} = -a + (1-|a|^2)e^{jt} + (1-|a|^2)\sum_{k=2}^{\infty}\bar{a}^{k-1}e^{jkt}.$$

For positive integer n, we note that the Blaschke product $b_n(z)$ of Möbius transform equals to

$$\prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_k z} = \prod_{k=1}^{n}\left(-a_k + (1-|a_k|^2)z + (1-|a_k|^2)\sum_{k=2}^{\infty}\bar{a}_k^{k-1}z^k\right).$$

Obviously, we can represent $b_n(z) = \prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_k z}$ in series form

$$\prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_k z} = c_0 + c_1 z + \cdots + c_l z^l + \cdots$$

with the convergence domain $\{z : |z| < \min\{|a_k|^{-1}, k = 1, 2, \ldots, n\}\}$ and $c_0 = (-1)^n\prod_{k=1}^{n} a_j$. Since all a_k satisfy $|a_k| < 1$, the boundary value $\prod_{k=1}^{n}\frac{e^{jt}-a_k}{1-\bar{a}_k e^{jt}}$ on the unit circle of $\prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_k z}$ can be written as

$$\prod_{k=1}^{n}\frac{e^{jt}-a_k}{1-\bar{a}_k e^{jt}} = c_0 + \sum_{k=1}^{\infty}c_k e^{jkt}.$$

That is

$$b_n(e^{jt}) = c_0 + \sum_{k=1}^{\infty}c_k e^{jkt},$$

where $b_n(\mathrm{e}^{\mathrm{j}t})$ is defined as in (3.1). Thus we have

$$\mathrm{e}^{\mathrm{j}t} b_n(\mathrm{e}^{\mathrm{j}t}) = \sum_{k=0}^{\infty} c_k \mathrm{e}^{\mathrm{j}(k+1)t}.$$

Acting circular Hilbert transform on both sides of the above equation leads to

$$\mathcal{H}_c\left(\mathrm{e}^{\mathrm{j}t} b_n(\mathrm{e}^{\mathrm{j}t})\right) = -\mathrm{j}\sum_{k=0}^{\infty} c_k \mathrm{e}^{\mathrm{j}(k+1)t} = -\mathrm{j}\mathrm{e}^{\mathrm{j}t} b_n(\mathrm{e}^{\mathrm{j}t}). \tag{5.1}$$

Now we consider $\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})$ defined as in (4.1); from formula (3.4), we have

$$\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t}) = \frac{\mathrm{e}^{\mathrm{j}t} b_{n-1}(\mathrm{e}^{\mathrm{j}t}) + \bar{a}_n \mathrm{e}^{\mathrm{j}t} b_n(\mathrm{e}^{\mathrm{j}t})}{\sqrt{1-|a_n|^2}}.$$

Acting circular Hilbert transform on both sides of the above equation and using (5.1), we get

$$\mathcal{H}_c\left(\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})\right) = -\mathrm{j}\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t}).$$

Since the complex signal $\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})$ can be expressed as the form $\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}$, thus

$$\mathcal{H}_c\left(\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}\right) = -\mathrm{j}\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}.$$

So, we finish the proof of the theorem. □

Consequently, by analytic signal theory, the unique pair $(\rho_n(t), \psi_n(t))$ is the canonical pair associated with the real signal $\rho_n(t)\cos\psi_n(t)$, $t \in [-\pi, \pi]$. The functions $\rho_n(t)$ and $\psi_n(t)$ are called its instantaneous amplitude and instantaneous phase, respectively. From (4.3)–(4.5), the instantaneous frequency $\omega_{\mathrm{ins}}(t)$ [8,9,13,15,22] of the analytic signal $\rho(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}$ can be easily obtained

$$\omega_{\mathrm{ins}}(t) := \psi_n'(t) = \sum_{k=1}^{n-1} p_{a_k}(t) + \frac{1}{2}(p_{a_n}(t)+1). \tag{5.2}$$

As a consequence, the instantaneous frequency is always positive. Thus, the nontrivial orthonormal harmonic waves $\{\mathcal{B}_n : t \in [-\pi, \pi], n \in \mathbb{N}_0\}$ have the form with amplitude and frequency modulation [20] in the space of $L^2([-\pi, \pi])$

$$\{1\} \cup \{\rho_n(t)\cos\psi_n(t) : n \geq 1\} \cup \{\rho_n(t)\sin\psi_n(t) : n \geq 1\}$$

with strictly increasing nonlinear phase $\psi_n(t)$.

As we all know, instantaneous frequency, taken as the derivative of the phase of the signal, is interpreted in the time–frequency literature as the average frequency at each time. The analytic signal $\mathrm{e}^{\mathrm{j}t} b_n(\mathrm{e}^{\mathrm{j}t}) = \mathrm{e}^{\mathrm{j}(t+\sum_{k=1}^{n}\theta_{a_k}(t))}$ is regarded as a mono-component signal [18], where $\theta_{a_k}(t)$ is defined as in (2.2). From formula (3.4), we have

$$\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t}) = \frac{\mathrm{e}^{\mathrm{j}t} b_{n-1}(\mathrm{e}^{\mathrm{j}t}) + \bar{a}_n \mathrm{e}^{\mathrm{j}t} b_n(\mathrm{e}^{\mathrm{j}t})}{\sqrt{1-|a_n|^2}}.$$

It tells us that the orthonormal harmonic waves $\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})$ can be regarded as the linear combination of two mono-component signals. We will point out that the instantaneous frequencies defined as (5.2) are tightly associated with the average frequencies of the mono-component analytic signals $\mathrm{e}^{\mathrm{j}t} b_n(\mathrm{e}^{\mathrm{j}t})$ and $\mathrm{e}^{\mathrm{j}t} b_{n-1}(\mathrm{e}^{\mathrm{j}t})$. In fact, the instantaneous frequency of $\mathrm{e}^{\mathrm{j}t} b_n(\mathrm{e}^{\mathrm{j}t})$ is

$$\omega_{\mathrm{ins}}^{n}(t) := 1 + \sum_{k=1}^{n} p_{a_k}(t)$$

and the instantaneous frequency of $\mathrm{e}^{\mathrm{j}t} b_{n-1}(\mathrm{e}^{\mathrm{j}t})$ is

$$\omega_{\mathrm{ins}}^{n-1}(t) := 1 + \sum_{k=1}^{n-1} p_{a_k}(t).$$

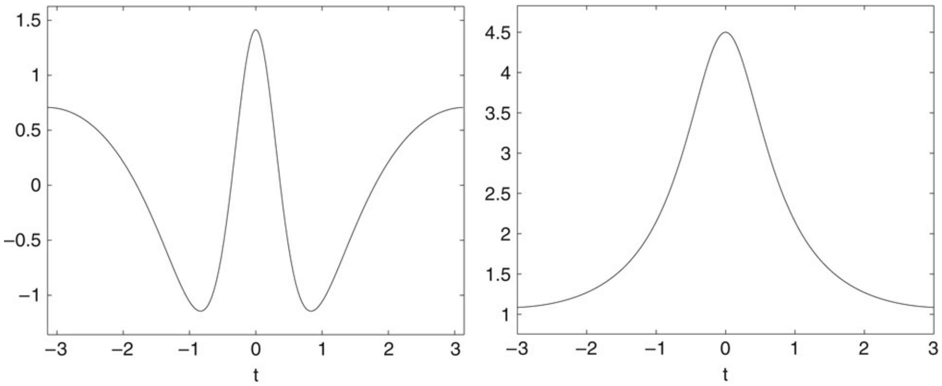

Fig. 1. Left: the signal $f(t) = \rho_2(t)\cos\psi_2(t)$, $t \in [-\pi, \pi]$. Right: the instantaneous frequency of $f(t)$.

Thus, the average frequency of $\omega_{\text{ins}}^{n}(t)$ and $\omega_{\text{ins}}^{n-1}(t)$ is

$$\frac{1}{2}(\omega_{\text{ins}}^{n}(t) + \omega_{\text{ins}}^{n-1}(t)) = \sum_{k=1}^{n-1} p_{a_k}(t) + \frac{1}{2} p_{a_n}(t) + \frac{1}{2}. \tag{5.3}$$

Comparing with (5.2) and (5.3), we get

$$\omega_{\text{ins}}(t) = \frac{1}{2}(\omega_{\text{ins}}^{n}(t) + \omega_{\text{ins}}^{n-1}(t)) + \frac{1}{2}.$$

From this result we are sure that the orthonormal harmonic waves $\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})$ have the physically meaningful instantaneous frequencies.

Corresponding to $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{3}$, Fig. 1 illustrates the real signal $f(t) = \rho_2(t)\cos\psi_2(t)$, $t \in [-\pi, \pi]$ and its instantaneous frequency, respectively, with the instantaneous amplitude $\rho_2(t) = \frac{2}{\sqrt{5-3\cos t}}$ and the instantaneous phase

$$\psi_2(t) = 2t + 2\arctan\frac{\sin t}{2-\cos t} + \arctan\frac{\sin t}{3-\cos t}.$$

6. Orthonormal analytic waves on the whole time range

In the above section, we mainly discuss the orthonormal analytic waves $\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}$ in $[-\pi, \pi]$, which can be periodically extended to the real line. In this section, we will get the orthonormal analytic waves on the real line by two steps. On the one hand, a set of basis functions $\{B_n(s)\}$ are treated which are defined by a choice of numbers $\zeta_k \in \mathbb{C}^+$ as

$$B_n(s) := \frac{\sqrt{2\,\mathrm{Im}\,\zeta_n}}{s - \bar{\zeta}_n}\varphi_{n-1}(s),$$

where $\varphi_n(s)$ is a Blaschke function defined by

$$\varphi_n(s) := \prod_{k=1}^{n} \frac{s-\zeta_k}{s-\bar{\zeta}_k}, \qquad \varphi_0(s) := 1.$$

We set $B_0(s) = 1$. Analogous to the rational basis functions discussed in [3], the basis functions $\{B_n\}$ are orthonormal in $H^2(\mathbb{C}^+)$ with respect to the inner product

$$\langle B_n, B_m \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} B_n(s)\overline{B_m(s)}\mathrm{d}s = \delta_{nm}.$$

Furthermore, the following result [3] tells us that the orthonormal set $\{B_n : n \in \mathbb{N}_0\}$ is complete in Hardy space $H^2(\mathbb{C}^+)$ for a suitable choice of the zeros $\{\zeta_n : n \in \mathbb{N}\}$ in $\mathbb{C}^+$. Completeness means that any function in $\mathbb{C}^+$ can be arbitrarily approximated by linear combinations of the basis functions.

Proposition 2. *The orthonormal set $\{B_n(s) : n \in \mathbb{N}_0\}$ is complete in Hardy space $H^2(\mathbb{C}^+)$ if and only if*

$$\sum_{n=0}^{\infty} \frac{\operatorname{Im}\zeta_n}{1+|\zeta_n|^2} = \infty.$$

On the other hand, we will go to consider the analytic property of the orthonormal basis $B_n(s)$. Simple calculation offers us that

$$B_n(s) = \frac{\varphi_{n-1}(s) - \varphi_n(s)}{\sqrt{2\operatorname{Im}\zeta_n}}, \tag{6.1}$$

which tells us that $B_n(s)$ is the linear combination of the signals $\varphi_n(s)$ and $\varphi_{n-1}(s)$. In [15], it has been asserted that the signals of the form

$$\mathrm{e}^{\mathrm{j}\theta(s)} = e^{\mathrm{j}(\theta_0+\omega s)} \prod_{k=1}^{n} \frac{s-\zeta_k}{s-\bar{\zeta}_k}$$

are unit analytic signals by analyzing the structure of the Fourier transform of $\varphi_n(s)$. In what follows, we give a simple calculation to describe $\varphi_n(s)$ is an analytic signal by the Cayley transform, different from the method in [15].

The Cayley transform $\varpi = \frac{\mathrm{j}-z}{\mathrm{j}+z}$ is a conformed mapping from the upper-half-complex plane $\{z : \operatorname{Im} z > 0\}$ to the unit disc $\{\varpi : |\varpi| < 1\}$, which also extends to their boundaries to become a bijective, bi-continuous, and strictly increasing under their canonical parameterizations $\varpi = \mathrm{e}^{\mathrm{j}t}$, $-\pi \le t \le \pi$, and $z = s$, $-\infty < s < \infty$. It maps the real line to the unit circle via

$$\mathrm{e}^{\mathrm{j}t} = \frac{\mathrm{j}-s}{\mathrm{j}+s} = \frac{1-s^2}{1+s^2} + \mathrm{j}\frac{2s}{1+s^2}.$$

Therefore, noting that the unit circular analytic signal $b_n(\mathrm{e}^{\mathrm{j}t})$ is the boundary value on the unit circle of the Blaschke product $b_n(z)$ defined as in (3.1), the Cayley transform of it will lead to the phase signal on the real line defined as in [15]. In fact, the composition of the phase signal $b_n(\mathrm{e}^{\mathrm{j}t})$ and $\mathrm{e}^{\mathrm{j}t} = \frac{\mathrm{j}-s}{\mathrm{j}+s}$ leads to

$$b_n\left(\frac{\mathrm{j}-s}{\mathrm{j}+s}\right) = \mathrm{e}^{\mathrm{j}\gamma} \prod_{k=1}^{n} \frac{s-\zeta_k}{s-\bar{\zeta}_k} \tag{6.2}$$

with $\zeta_k = \mathrm{j}\frac{1-a_k}{1+a_k}$ and some constant γ satisfying $\mathrm{e}^{\mathrm{j}\gamma} = (-1)^n \prod_{k=1}^{n} \frac{1+a_k}{1+\bar{a}_k}$. Simple calculation shows that

$$\zeta_k = \frac{2\operatorname{Im} a_k}{(1+\operatorname{Re} a_k)^2 + (\operatorname{Im} a_k)^2} + \mathrm{j}\frac{1-|a_k|^2}{(1+\operatorname{Re} a_k)^2 + (\operatorname{Im} a_k)^2},$$

where $\operatorname{Re} a_k$ and $\operatorname{Im} a_k$ denote the real and imaginary part of the complex number a_k, respectively. Then $\operatorname{Im}\zeta_k > 0$ since $|a_k| < 1$. Therefore, the phase signal on the real line defined as in (6.2) is the boundary value of the analytic function on the open upper half plane $\mathbb{C}^+ = \{z : \operatorname{Im} z > 0\}$ and the Blaschke function $\varphi_n(s)$ is an analytic signal on the real line.

Therefore, the orthonormal basis $B_n(s)$ is the analytic signal on the real line. To get the instantaneous amplitude and the instantaneous frequency associated with the analytic signal $B_n(s)$, we write it in the polar form $B_n(s) = \varrho_n(s)\mathrm{e}^{\mathrm{j}\phi_n(s)}$, where the instantaneous amplitude

$$\varrho_n(s) = \sqrt{\frac{2\operatorname{Im}\zeta_n}{(s-\operatorname{Re}\zeta_n)^2 + (\operatorname{Im}\zeta_n)^2}}$$

and the instantaneous phase

$$\phi_n(s) = 2\sum_{k=1}^{n-1} \arctan\frac{\operatorname{Im}\zeta_k}{\operatorname{Re}\zeta_k - s} + \arctan\frac{\operatorname{Im}\zeta_n}{\operatorname{Re}\zeta_n - s}.$$

Thus the instantaneous frequency

$$\phi_n'(s) = 2\sum_{k=1}^{n-1} \frac{\operatorname{Im}\zeta_k}{(s-\operatorname{Re}\zeta_k)^2 + (\operatorname{Im}\zeta_k)^2} + \frac{\operatorname{Im}\zeta_n}{(s-\operatorname{Re}\zeta_n)^2 + (\operatorname{Im}\zeta_n)^2} \tag{6.3}$$

is always positive. From the relation (6.1), we get that the signals $B_n(s)$ can be regarded as the linear combination of two mono-component signals, analogous to the case on the circle. The instantaneous frequencies are physically meaningful because they are the average frequencies of the unit analytic signals $\varphi_{n-1}(s)$ and $\varphi_n(s)$ on the time s. In fact, the instantaneous frequency of $\varphi_n(s)$ is

$$\omega_{\mathrm{ins}}^n(s) := 2\sum_{k=1}^{n} \frac{\operatorname{Im}\zeta_k}{(s-\operatorname{Re}\zeta_k)^2 + (\operatorname{Im}\zeta_k)^2}$$

and the instantaneous frequency of $\varphi_{n-1}(s)$ is

$$\omega_{\mathrm{ins}}^{n-1}(s) := 2\sum_{k=1}^{n-1} \frac{\operatorname{Im}\zeta_k}{(s-\operatorname{Re}\zeta_k)^2 + (\operatorname{Im}\zeta_k)^2}.$$

Thus, the average frequency of $\omega_{\mathrm{ins}}^n(s)$ and $\omega_{\mathrm{ins}}^{n-1}(s)$ is

$$\frac{1}{2}(\omega_{\mathrm{ins}}^n(s) + \omega_{\mathrm{ins}}^{n-1}(s)) = \phi_n'(s), \tag{6.4}$$

where $\phi_n'(s)$ is defined as in (6.3).

7. Conclusions

Instantaneous amplitude, phase and frequency are basic concepts in all the questions dealing with modulation of signals appearing especially in communications or information processing. The nonnegative instantaneous frequency obtained by analytic signal is physically meaningful. To expose the time-varying property of non-stationary signals, in this note, a kind of nontrivial orthonormal harmonic waves $\{\mathcal{B}_n(\mathrm{e}^{\mathrm{j}t})\}_{n=0}^{\infty}$ can be derived by applying Gram–Schmidt procedure to the Blaschke product and denoted as the polar form $\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}$. Thus, we can decompose general signals into the superposition of the nontrivial harmonic waves with time-varying amplitude and phase. Furthermore, we show that the nontrivial harmonic waves $\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}$ satisfy the analytic condition $\mathcal{H}_c(\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}) = -\mathrm{j}\rho_n(t)\mathrm{e}^{\mathrm{j}\psi_n(t)}$ and have positive instantaneous frequencies. We finally get the related orthonormal analytic signals on the whole time range and depict the related instantaneous frequencies are positive. Noting that the orthonormal harmonic waves are regarded as the linear combination of two mono-component signals, we find the instantaneous frequencies are physically meaningful because they are tightly associated with the average frequencies of the mono-component analytic signals at each time.

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References

[1] N.I. Achieser, Theory of Approximation (C.J. Hyman, Trns.), Frederick Ungar, New York, 1956.
[2] H. Akcay, S.M. Islam, B. Ninness, Identification of power transformer models from frequency response data: A case study, Signal Processing 68 (1998) 307–315.
[3] H. Akcay, B. Ninness, Orthonormal basis functions for modelling continuous-time systems, Signal processing 77 (1999) 261–274.
[4] H. Akcay, B. Ninness, Rational basis functions for modelling continuous-time systems, Automatica 34 (1998) 1101–1117.
[5] E. Bedrosian, A product theorem for Hilbert transform, Proceedings of the IEEE 51 (1963) 868–869.
[6] P.W. Broom, Discrete orthonomal sequences, Journal of the Association for Computing Machinery 12 (2) (1965) 151–168.
[7] Q.H. Chen, L.Q. Li, T. Qian, Two families of unit analytic signals with nonlinear phase, Physica D: Nonlinear Phenomena 221 (2006) 1–12.
[8] L. Cohen, Time–Frequency Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1995.
[9] M.I. Doroslovacki, On nontrivial analytic signals with positive instantaneous frequency, Signal Processing 83 (2003) 655–658.
[10] D. Gabor, Theory of communications, Journal of IEEE III (1946) 429–457.
[11] M.J. Gottieb, Concerning some polynomials orthogonal on a finite or enumerable set points, American Journal of Mathematics 60 (1938) 453–458.
[12] J.B. Garnett, Bounded Analytic Functions, Academic Press, 1987.
[13] N.E. Huang, Z. Shen, S.R. Long, M.C. Wu, H.H. Shih, Q. Zheng, N.-C. Yen, C.C. Tung, H.H. Liu, The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis, Proceedings of the Royal Society of London. Series A 454 (1998) 903–995.
[14] B. Ninness, F. Gustafsson, A unifying construction of orthonormal bases for system identification, IEEE Transactions on Automatic Control 42 (1997) 515–521.
[15] B. Picinbono, On instantaneous amplitude and phase of signals, IEEE Transactions on Signal Processing 45 (3) (1997) 552–560.
[16] T. Qian, Q.H. Chen, L.Q. Li, Analytic unit quadrature signals with nonlinear phase, Physica D: Nonlinear Phenomena 203 (2005) 80–87.
[17] T. Qian, Characterization of boundary values of functions in Hardy spaces with applications in signal analysis, Journal of Integral Equations and Applications 17 (2005) 159–198.
[18] T. Qian, Mono-components for decomposition of signals, Mathematical Methods in Applied Sciences 29 (2006) 1187–1198.
[19] T. Qian, Unit analytic signals and harmonic measures, Journal of Mathematical Analysis and Applications 314 (2006) 526–536.
[20] D.E. Vakman, L.A. Vainshtein, Amplitude, phase, frequency-fundamental concepts of oscillation theory, Soviet Physics Uspekhi 20 (1978) 1002–1016.
[21] J.L. Walsh, Interpolation and Approximation, 3rd ed., in: Amer. Math. Soc. Colloq. Pnbl., vol. 20, Amer. Math. Soc., Providence, Rl, 1935.
[22] X.G. Xia, L. Cohen, On analytic signals with nonnegative instantaneous frequency, in: Proceedings of the ICASSP-99, Phoenix, Paper 1483, March 1999.
[23] A. Zgmund, Trigonometric Series, 2nd ed., Cambridge University Press, Cambridge, 1968.