Quantum Information and Computation, Vol. 1, No. 0 (2012) 000–000

A NOTE ON LOCALLY UNEXTENDIBLE NON-MAXIMALLY ENTANGLED BASIS
— Comment on "Quant. Inf. Comput. 12, 0271(2012)"

BIN CHEN * HALQEM NIZAMIDIN † SHAO-MING FEI ‡
School of Mathematical Sciences, Capital Normal University
Beijing, 100048, China

Received (received date)
Revised (revised date)

We study the locally unextendible non-maximally entangled basis (LUNMEB) in $H^d \bigotimes H^d$. We point out that there exists an error in the proof of the main result of LUNMEB [Quant. Inf. Comput. 12, 0271(2012)], which claims that there are at most d orthogonal vectors in a LUNMEB, constructed from a given non-maximally entangled state. We show that both the proof and the main result are not correct in general. We present a counter example for $d = 4$, in which five orthogonal vectors from a specific non-maximally entangled state are constructed. Besides, we completely solve the problem of LUNMEB for the case of $d = 2$.

Keywords: Unextendible basis, Non-maximally entangled state, LUNMEB

Communicated by: to be filled by the Editorial

1. Introduction

The unextendible product basis (UPB) has been extensively investigated. Considerable elegant results have been obtained with interesting applications to the theory of quantum information [1,2,3]. Recently S. Bravyi and J. A. Smolin generalized the notion of the UPB to unextendible entangled basis. They studied the special case – unextendible maximally entangled basis (UMEB) in $H^d \bigotimes H^d$ for $d = 2, 3, 4$ [4]. After that I. Chakrabarty, P. Agrawal and A.K. Paty introduced the concept of locally unextendible non-maximally entangled basis (LUNMEB) in $H^d \bigotimes H^d$ [5]: A set of states $\{|\phi_a\rangle \in H^d \bigotimes H^d,\ a = 1, 2, \cdots, n\}$ is called LUNMEB if and only if

(i) all states $|\phi_a\rangle$ are non-maximally entangled;

(ii) $\langle\phi_a|\phi_b\rangle = \delta_{a,b}$;

(iii) $\forall\ a, b$, there exits a unitary operator U_{ab} such that $|\phi_a\rangle = (U_{ab} \bigotimes I)|\phi_b\rangle$;

(iv) if $\langle\Phi|\phi_a\rangle = 0$ for all a, then there is no unitary operator V such that $(V \bigotimes I)|\phi_a\rangle = |\Phi\rangle$ for some a.

As the local unitary transformations do not change the degree of entanglement, all the basic vectors $|\phi_a\rangle$ in LUNMEB have the same entanglement. The main theorem in [5] claims

*Email:chenbin5134@163.com
†Email:nhalqem@gmail.com
‡Email:feishm@mail.cnu.edu.cn

that starting from a general non-maximally entangled state, one can get at most d orthogonal vectors to form a LUNMEB by applying a set of specific local unitary transformations. However, the proof of this theorem has a flaw, and the main result is also not correct in general. We can give a counter example to show the theorem is wrong for $d = 4$.

In addition, we study thoroughly the case of $d = 2$. In [5] special unitary operators are used to show that there are no more than two orthogonal vectors in LUNMEB for $d = 2$. We will show that for $d = 2$, any unitary transformation applying to one party of a given non-maximally entangled state can ensure that there does not exist the third vector in constructing a LUNMEB. Thus we give a complete result of LUNMEB for $H^2 \bigotimes H^2$.

2. Counter example in $H^4 \bigotimes H^4$

We first review the proof of the main theorem in [5] which claims that a LUNMEB in $H^d \bigotimes H^d$ consists of at most d orthogonal vectors of non-maximally entangled states.

The unitary basic operators on H^d used in [5] are given by

$$U_{nm} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d} nk} |k \oplus m\rangle\langle k|, \tag{1}$$

where $i = \sqrt{-1}$. Thus any unitary transformation V can be represented as

$$V = \sum_{p,q} f_{pq} U_{pq}, \tag{2}$$

where the coefficients f_{pq} should satisfy some conditions such that V is unitary. Let $|\phi\rangle$ be a non-maximally entangled state in Schmidt form:

$$|\phi\rangle = \sum_{k=0}^{d-1} C_k |kk\rangle, \qquad \sum_{k=0}^{d-1} C_k^2 = 1, \tag{3}$$

where $C_k \geq 0$, and the Schmidt rank of $|\phi\rangle$ is larger than one (at least two non-zero Schmidt numbers C_k). Start with $|\phi\rangle$ one can get d states which are mutually orthogonal:

$$|\phi_{0m}\rangle = (U_{0m} \bigotimes I)|\phi\rangle, \qquad m = 0, 1, \cdots, d-1. \tag{4}$$

To show that the set of states $|\phi_{0m}\rangle$ in (4) is unextendible, one has to show that there does not exist unitary operator $V = \sum_{p,q} f_{pq} U_{pq}$ such that $|\Phi\rangle = (V \bigotimes I)|\phi\rangle$ is orthogonal to all $|\phi_{0m}\rangle$. From

$$\langle \phi_{0m}|\Phi\rangle = \sum_{p=0}^{d-1} f_{pm} (C_0^2 + e^{\frac{2\pi i}{d} p} C_1^2 + \cdots + e^{\frac{2\pi i}{d}(d-1)p} C_{d-1}^2) = 0, \qquad \forall m, \tag{5}$$

the authors in [5] derived that

$$\sum_{p=0}^{d-1} f_{pm} = 0, \quad \sum_{p=0}^{d-1} f_{pm} e^{\frac{2\pi i}{d} p} = 0, \quad \cdots, \quad \sum_{p=0}^{d-1} f_{pm} e^{\frac{2\pi i}{d}(d-1)p} = 0, \tag{6}$$

thereby $f_{pq}=0, \forall p,q$. However, this conclusion is not correct. Since $\sum_{p=0}^{d-1} f_{pm}$, $\sum_{p=0}^{d-1} f_{pm}e^{\frac{2\pi i}{d}p}$, $\cdots$, $\sum_{p=0}^{d-1} f_{pm}e^{\frac{2\pi i}{d}(d-1)p}$ are complex numbers, it is possible that some $f_{pq}\neq 0$ while $\langle\phi_{0m}|\Phi\rangle = 0,\ \forall m$. In the following, we give a simple counter example.

We consider the case of $d=4$. We take $C_0=C_2=1/\sqrt{3}$, $C_1=C_3=1/\sqrt{6}$ and begin with the state

$$|\phi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{6}}|11\rangle + \frac{1}{\sqrt{3}}|22\rangle + \frac{1}{\sqrt{6}}|33\rangle.$$

The first four states which are mutually orthogonal can be obtained from (4) by using the set of special unitary operators given in equation (1):

$$\begin{aligned} |\phi_{0m}\rangle &= (U_{0m}\bigotimes I)\,|\phi\rangle \\ &= \frac{1}{\sqrt{3}}|m\rangle|0\rangle + \frac{1}{\sqrt{6}}|1\oplus m\rangle|1\rangle + \frac{1}{\sqrt{3}}|2\oplus m\rangle|2\rangle + \frac{1}{\sqrt{6}}|3\oplus m\rangle|3\rangle, \end{aligned} \tag{7}$$

$m=0,1,2,3.$

To find a unitary operator $V=\sum_{p,q} f_{pq}U_{pq}$ such that $|\Phi\rangle = (V\bigotimes I)|\phi\rangle$ is orthogonal to all vectors $|\phi_{0m}\rangle$, we take

$$f_{10}=\frac{1-i}{2\sqrt{2}},\quad f_{30}=\frac{1+i}{2\sqrt{2}},\quad f_{11}=f_{31}=\frac{1}{2\sqrt{2}},\quad f_{13}=-f_{33}=\frac{i}{2\sqrt{2}}, \tag{8}$$

and other $f_{pq}=0$. The unitary operator V has a matrix form, under the ordered basis $|0\rangle$, $|1\rangle$, $|2\rangle$ and $|3\rangle$:

$$V=\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ & & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ & & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Therefore we have

$$\langle\phi_{0m}|\Phi\rangle = \frac{1}{3}\langle m|V|0\rangle + \frac{1}{6}\langle m\oplus 1|V|1\rangle + \frac{1}{3}\langle m\oplus 2|V|2\rangle + \frac{1}{6}\langle m\oplus 3|V|3\rangle. \tag{9}$$

It is easily checked that $\langle\phi_{0m}|\Phi\rangle = 0,\ \forall\, m$. Hence we can get at least five orthogonal vectors which are local unitary equivalent. And the theorem in [5] does not hold in this case.

3. LUNMEB in $H^2\bigotimes H^2$

We have shown that the basis consisted of d mutually orthogonal states (4) is not unextendible generally. In fact, to investigate thoroughly the existence and the numbers of LUNMEB existed for a given system, it is not enough to use the particular unitary operators (1) in constructing the basis. One should use general unitary operators V to find the second vector $|\Phi\rangle = (V\bigotimes I)|\phi\rangle$ that is orthogonal to the given non-maximally entangled state $|\phi\rangle$, and repeat the procedure to find the basic vectors, until the condition (iv) in the first section applies. However, for general $H^d\bigotimes H^d$ system it is formidable to solve such problems completely. In the following, we give a complete solution to the problem of locally unextendible non-maximally entangled basis for the case of $H^2\bigotimes H^2$.

We start with a non-maximally entangled state in the Schmidt decomposition form:

$$|\phi\rangle = C_0|00\rangle + C_1|11\rangle, \tag{10}$$

where $C_0 \neq C_1$ are non-zero and $C_0^2+C_1^2 = 1$. The number of states in a LUNMEB is relevant to the choice of the second vector. Let the second vector be $|\Phi\rangle = (V \bigotimes I)|\phi\rangle$, with V an arbitrary unitary operator such that, up to a global phase factor,

$$V\left(|0\rangle, |1\rangle\right) = \left(|0\rangle, |1\rangle\right) \begin{pmatrix} \cos\theta & -\sin\theta\, e^{i\theta_1} \\ \sin\theta\, e^{i\theta_2} & \cos\theta\, e^{i\theta_3} \end{pmatrix}.$$

From $\langle\phi|\Phi\rangle = C_0^2\cos\theta + C_1^2\cos\theta\, e^{i\theta_3} = 0$, we have $\cos\theta = 0$, and $V|0\rangle = e^{i\alpha}|1\rangle$, $V|1\rangle = e^{i\beta}|0\rangle$, since if $\cos\theta \neq 0$, then $e^{i\theta_3} = -C_0^2/C_1^2$ is a real number, which gives rise to $C_0 = C_1$, and leads a contradiction.

For the possible construction of the third vector, suppose $|\Psi\rangle = (U \bigotimes I)|\phi\rangle$ is orthogonal to both $|\phi\rangle$ and $|\Phi\rangle$. From the similar reason to the construction of $|\Phi\rangle$, we have $U|0\rangle = e^{i\mu}|1\rangle$, and $U|1\rangle = e^{i\eta}|0\rangle$. From

$$\begin{aligned} \langle\Psi|\Phi\rangle &= C_0^2\langle 0|U^\dagger V|0\rangle + C_1^2\langle 1|U^\dagger V|1\rangle \\ &= C_0^2 e^{i(\alpha-\mu)} + C_1^2 e^{i(\beta-\eta)} = 0, \end{aligned}$$

we obtain $e^{i(\alpha-\mu-\beta+\eta)} = -C_1^2/C_0^2$, which implies $C_0 = C_1$. Therefore the third basis $|\Psi\rangle$ does not exist, and $|\phi\rangle$ and $|\Phi\rangle$ form a LUNMEB.

4. Conclusion and discussion

The unextendible product basis (UPB), the unextendible maximally entangled basis (UMEB) and the locally unextendible non-maximally entangled basis (LUNMEB) in $H^d \bigotimes H^d$ are of significance in the theory of quantum information. However, the results obtained so far are far from being satisfied. Even for UMEB, one can only solve the problem completely for the case of $d = 2$ [4].

We have studied the locally unextendible non-maximally entangled basis. In correcting an error in [5], we have shown that there could be more than d orthogonal vectors in a LUNMEB for $H^d \bigotimes H^d$ systems. Like the UMEB, we completely solved the problem of LUNMEB for the case of $d = 2$. The approach we used can be applied to the study of LUNMEB for high dimensional cases. Nevertheless, the problem becomes complicated as d increases.

References

1. C. H. Bennett, D. P. Divincenzo, T. Mor, P. W. Shor, J. A. Smolin and B. M. Terhal (1999), *Unextendible product bases and bound entanglement*, Phys. Rev. Lett. 82, 5385.
2. D. P. Divincenzo, T. Mor, P. W. Shor, J. A. Smolin and B. M. Terhal (2003), *Unextendible product bases, uncompletable product bases and bound entanglement*, Commun. Math. Phys. 238, pp. 379-410.
3. R. Duan, *Super-activation of zero-error capacity of noise quantum channels*, arXiv:0906.2527.
4. S. Bravyi, J. A. Smolin, *Unextendible maximally entangled bases*, arXiv:0911.4090v2.
5. I. Chakrabarty, P. Agrawal and A. K. Paty (2012), *Locally unextendible non-maximally entangled basis*, Quant. Inf. Comput. 12, 0271.