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Generalized AOR methods for linear complementarity problem

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Abstract

In this paper, we firstly establish a class of generalized AOR (GAOR) methods for solving a linear complementarity problem LCP(M,q), whose special case reduces to generalized SOR (GSOR) method. Then, some sufficient conditions for convergence of the GAOR and GSOR methods are presented, when the system matrix M is an H-matrix, M-matrix and a strictly or irreducible diagonally dominant matrix. Moreover, when M is an L-matrix, we discuss the monotone convergence of the new methods. Lastly, we report some computational results with the proposed methods.

Keywords: Linear complementarity problem; GAOR method; GSOR method; Convergence; Monotone

1. Introduction

For given matrix $M \in R^{n\times n}$ and vector $q \in R^n$, the linear complementarity problem LCP(M,q) consists of finding a vector $z \in R^n$ which satisfies the conditions

$$z \geqslant 0, \qquad Mz+q \geqslant 0, \quad z^{\mathrm{T}}(Mz+q)=0. \tag{1.1}$$

Because the LCP(M,q) has a variety of applications such as the Nash equilibrium point of a bimatrix game, contact problems, the free boundary problem for journal bearings, etc. (see [1,14]), the research on the numerical methods for solving (1.1) have attracted much attention.

It is known that iterative methods have been found very useful for solving linear complementarity problem (see [11] and references therein). Most of these iterative methods are from the extension of their counterparts for solving systems of linear algebraic equation (cf. [2–5,11]).

In [6], James presented a class of generalized accelerated overrelaxation (GAOR) method for linear system and some convergence conditions were given. While in [8,9], also for linear system, Song proposed some sufficient and/or necessary conditions of convergence when coefficient matrix of the linear system is positive definite, an H-, L-, or M-matrix, or strictly or irreducible diagonally dominant matrix.

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In this paper, a class of generalized AOR (GAOR) and generalized SOR (GSOR) methods for LCP(M,q) will be established, which is a generalization of modified AOR (MAOR) and modified SOR (MSOR) methods in [7], based on the models in [5]. Some sufficient conditions for convergence of the GAOR and GSOR methods will be proposed, when M is an H-, M-matrix and a strictly or irreducible diagonally dominant matrix. Moreover, when M is an L-matrix, we shall discuss the monotone convergence of the new methods, and give a comparison theorem, which describes the influences of the parameters upon the monotone convergence rates of the new methods.

We shall use the following notation. Let $C=(c_{kj})\in R^{n\times n}$ be an $n\times n$ matrix. By diag(C) we denote the $n\times n$ diagonal matrix coinciding in its diagonal with C. For $A=(a_{kj})$, $B=(b_{kj})\in R^{n\times n}$, we write $A\geqslant B$ if $a_{kj}\geqslant b_{kj}$ holds for all $k,j=1,2,\ldots,n$. Calling A nonnegative if $A\geqslant 0$, we say that $B\leqslant C$ if and only if $-B\geqslant -C$. These definitions carry immediately over to vectors by identifying them with $n\times 1$ matrices. By $|A|=(|a_{kj}|)$ we define the absolute value of $A\in R^{n\times n}$. We denote by $\langle A\rangle=(\langle a_{kj}\rangle)$ the comparison matrix of $A\in R^{n\times n}$ where $\langle a_{kj}\rangle=|a_{kj}|$ for $k=j$ and $\langle a_{kj}\rangle=-|a_{kj}|$ for $k\neq j$, $k,j=1,2,\ldots,n$. Spectral radius of a matrix A is denoted by $\rho(A)$.

Definition 1 [15]. Let $A\in R^{n\times n}$. It is called an

1. L-matrix if $a_{kk}>0$ for $k=1,2,\ldots,n$, and $a_{kj}\leqslant 0$ for $k\neq j$, $k,j=1,2,\ldots,n$;
2. M-matrix if it is a nonsingular L-matrix satisfying $A^{-1}\geqslant 0$;
3. H-matrix if $\langle A\rangle$ is an M-matrix.

Definition 2. For $x\in R^n$, vector x_+ is defined such that $(x_+)_j=\max\{0,x_j\}$, $j=1,2,\ldots,n$. Then, for any $x,y\in R^n$, the following facts hold:

1. $(x+y)_+\leqslant x_+ + y_+$;
2. $x_+ - y_+\leqslant (x-y)_+$;
3. $|x|=x_+ + (-x)_+$;
4. $x\leqslant y$ implies $x_+\leqslant y_+$.

Linear complementarity problems can be transformed to equivalent fixed point system of equations (see [10,12]). If $D=\mathrm{diag}(M)$ is a nonsingular matrix, then, solving LCP(M,q) is equivalent to finding a solution of the system:

$$z=(z-D^{-1}(Mz+q))_+. \tag{1.2}$$

So, in order to solve LCP(M,q), a class of iterative methods for solving linear systems has been developed.

2. GAOR method for LCP(M,q)

Let

$$M=D+L_1+U_1, \tag{2.1}$$

where $D=\mathrm{diag}(M)$, L_1 and U_1 are strictly lower and strictly upper triangular matrices, respectively. Then, from (1.2), we definite the GAOR method for LCP(M,q) as follows:

$$z^{k+1}=\left(z^k-D^{-1}[\alpha\Omega L_1 z^{k+1}+(\Omega M-\alpha\Omega L_1)z^k+\Omega q]\right)_+. \tag{2.2}$$

For $\alpha=1$, the GAOR method reduces to GSOR method:

$$z^{k+1}=\left(z^k-D^{-1}[\Omega L_1 z^{k+1}+\Omega(M-L_1)z^k+\Omega q]\right)_+.$$

For $\alpha=\gamma/\omega$ and $\Omega=\omega I$, the GAOR method reduces to the AOR method:

$$z^{k+1}=\left(z^k-D^{-1}[\gamma L_1 z^{k+1}+(\omega M-\gamma L_1)z^k+\omega q]\right)_+.$$

For $\alpha = \gamma/\omega_2$ and $\Omega = (\omega_1 I, \omega_2 I)$, the GAOR method reduces to the MAOR method for 2-cyclic matrix, given in [7]:

$$z^{k+1} = \left(z^k - D^{-1}[\gamma L_1 z^{k+1} + (\Omega M - \gamma L_1)z^k + \Omega q]\right)_+.$$

Let us denote

$$J = D^{-1}(L_1 + U_1).$$

Then, for $\Omega = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_n)$ with $\omega_i \in R_+$ and α be a real number, the generalized AOR (GAOR) method for LCP(M, q) can be described as follows:

GAOR method

Step 1: Choose an initial vector $z^0 \in R^n$ and set $k = 0$.

Step 2: Calculate

$$z^{k+1} = \left(z^k - D^{-1}[\alpha\Omega L_1 z^{k+1} + (\Omega M - \alpha\Omega L_1)z^k + \Omega q]\right)_+. \tag{2.3}$$

Step 3: If $z^{k+1} = z^k$ then stop, otherwise set $k := k + 1$ and return to Step 2.

When $\alpha = 1$, the GAOR method reduces to GSOR method. So, we can define the GSOR method in the following:

GSOR method

Step 1: Choose an initial vector $z^0 \in R^n$ and set $k = 0$.

Step 2: Calculate

$$z^{k+1} = (z^k - D^{-1}[\Omega L_1 z^{k+1} + \Omega(M - L_1)z^k + \Omega q])_+. \tag{2.4}$$

Step 3: If $z^{k+1} = z^k$ then stop, otherwise set $k := k + 1$ and return to Step 2.

3. Convergence analysis for *H*-matrices

The operator $f: R^n \to R^n$, is defined such that $f(z) = \xi$, where ξ is the fixed point of the system

$$\xi = \left(z - D^{-1}(\alpha\Omega L_1 \xi + (\Omega M - \alpha\Omega L_1)z + \Omega q)\right)_+. \tag{3.1}$$

Let

$$G = I - \alpha\Omega D^{-1}|L_1|, \quad F = |I - D^{-1}(\Omega M - \alpha\Omega L_1)|.$$

At present, we review an important result.

Lemma 1 [5]. *Let M be an H-matrix with positive diagonal elements. Then the* LCP(M, q) *has a unique solution* $z^* \in R^n$.

With Lemma 1, we can prove the following convergence theorem for the GAOR method.

Theorem 3.1. *Let* $M = (m_{kj}) \in R^{n\times n}$ *be an H-matrix with positive diagonal elements. Then, for any initial vector* $z^0 \in R^n$, *the iterative sequence* $\{z^k\}$ *generated by the GAOR method converges to the unique solution* z^* *of the* LCP(M, q) *and*

$$\rho(G^{-1}F) \leqslant \max_{1\leqslant i\leqslant n}\{|1 - \omega_i| + \omega_i \rho(|J|)\} < 1,$$

whenever

$$0 < \omega_i < \frac{2}{1 + \rho(|J|)}, \quad 0 \leqslant \alpha \leqslant 1.$$

Proof. Let $\psi = f(y)$, i.e.,

$$\psi = \left(y - D^{-1}[\alpha\Omega L_1\psi + (\Omega M - \alpha\Omega L_1)y + \Omega q]\right)_+. \tag{3.2}$$

Then, by subtracting (3.2) from (3.1) we have

$$\begin{aligned}\xi - \psi &= \left(z - D^{-1}(\alpha\Omega L_1\xi + (\Omega M - \alpha\Omega L_1)z + \Omega q)\right)_+ - \left(y - D^{-1}[\alpha\Omega L_1\psi + (\Omega M - \alpha\Omega L_1)y + \Omega q]\right)_+ \\ &\leqslant \left((z-y) - D^{-1}[\alpha\Omega L_1(\xi - \psi) + (\Omega M - \alpha\Omega L_1)(z-y)]\right)_+.\end{aligned}$$

Therefore,

$$(\xi - \psi)_+ \leqslant (-\alpha\Omega D^{-1}L_1(\xi - \psi))_+ + ([I - D^{-1}(\Omega M - \alpha\Omega L_1)](z-y))_+. \tag{3.3}$$

Similarly, we get

$$(\psi - \xi)_+ \leqslant (-\alpha\Omega D^{-1}L_1(\psi - \xi))_+ + \left([I - D^{-1}(\Omega M - \alpha\Omega L_1)](y-z)\right)_+. \tag{3.4}$$

From (3.3) and (3.4), we obtain the following estimates:

$$\begin{aligned}|\xi - \psi| &= (\xi - \psi)_+ + (\psi - \xi)_+ \leqslant (-\alpha\Omega D^{-1}L_1(\xi - \psi))_+ + (-\alpha\Omega D^{-1}L_1(\psi - \xi))_+ \\ &\quad + ([I - D^{-1}(\Omega M - \alpha\Omega L_1)](z-y))_+ + ([I - D^{-1}(\Omega M - \alpha\Omega L_1)](y-z))_+ = |\alpha\Omega D^{-1}L_1(\xi - \psi)| \\ &\quad + |[I - D^{-1}(\Omega M - \alpha\Omega L_1)](z-y)| \leqslant \alpha\Omega D^{-1}|L_1||\xi - \psi| + |I - D^{-1}(\Omega M - \alpha\Omega L_1)||z-y|.\end{aligned}$$

So,

$$|\xi - \psi| \leqslant G^{-1}F|z-y|. \tag{3.5}$$

We know from Lemma 1 that the LCP(M,q) has a unique solution $z^* \in R^n$ under the hypotheses of the theorem. That is to say, $z^* = f(z^*)$. According to the definition of the GAOR method and (3.5) we have

$$|z^{k+1} - z^*| = |f(z^k) - f(z^*)| \leqslant G^{-1}F|z^k - z^*|.$$

Hence, the iterative sequence $\{z^k\}$, $k = 0,1,2,\ldots$, converges to z^* if $\rho(G^{-1}F) < 1$. Because $M \in R^{n\times n}$ is an H-matrix, we know that $\rho(|J|) < 1$. With the proving process of [8, Theorem 3.1], we can get

$$\rho(T) \leqslant \max_{1\leqslant i\leqslant n}\{|1-\omega_i| + \omega_i\rho(|J|)\},$$

where

$$T = (I - \alpha\Omega D^{-1}|L_1|)^{-1}[|I - \Omega| + (1-\alpha)\Omega D^{-1}|L_1| + \Omega D^{-1}|U_1|]. \tag{3.6}$$

It is easy to see that $|G^{-1}F| \leqslant T$ and [16, Theorem 2.8] ensures

$$\rho(G^{-1}F) \leqslant \rho(T) \leqslant \max_{1\leqslant i\leqslant n}\{|1-\omega_i| + \omega_i\rho(|J|)\}.$$

Notice that if $\omega_i \leqslant 1$, we obtain

$$|1-\omega_i| + \omega_i\rho(|J|) = 1 - \omega_i + \omega_i\rho(|J|) < 1.$$

While if $\omega_i > 1$, we also have

$$|1-\omega_i| + \omega_i\rho(|J|) = -1 + \omega_i + \omega_i\rho(|J|) = \omega_i[1 + \rho(|J|)] - 1 < \frac{2}{[1+\rho(|J|)]}[1+\rho(|J|)] - 1 = 1.$$

The proof is completed. □

As a special case, for the GSOR method, we have the following convergence result.

Corollary 3.2. *Let $M = (m_{kj}) \in R^{n\times n}$ be an H-matrix with positive diagonal elements. Then, for any initial vector $z^0 \in R^n$, the iterative sequence $\{z^k\}$ generated by the GSOR method converges to the unique solution z^* of the* LCP(M,q) *and*

$$\rho(G^{-1}F) \leqslant \max_{1\leqslant i\leqslant n}\{|1-\omega_i| + \omega_i\rho(|J|)\} < 1,$$

whenever

$$0 < \omega_i < \frac{2}{1+\rho(|J|)}.$$

It is known that an M-matrix is also an H-matrix. Therefore, the convergence results in Theorem 3.1 and Corollary 3.2 are valid for M-matrix.

Since a strictly or irreducible diagonally dominant matrix with positive diagonal elements is also satisfying the condition of Theorem 3.1, then Theorem 3.1 and Corollary 3.2 are also valid for these kinds of matrices.

Furthermore, for strictly or irreducible diagonally dominant matrices, $\||J|\|_\infty$ can take the place of $\rho(|J|)$ in Theorem 3.1 and Corollary 3.2.

If M is strictly diagonally dominant by rows, then

$$\rho(|J|) \leqslant \|J\|_\infty < 1.$$

By Theorem 3.1, a convergence theorem follows directly.

Corollary 3.3. *Let $M = (m_{kj}) \in R^{n\times n}$ be strictly diagonally dominant by rows with positive diagonally elements. Then, for any initial vector $z^0 \in R^n$, the iterative sequence $\{z^k\}$ generated by the GAOR method or the GSOR method converges to the unique solution z^* of the* LCP(M, q) *and*

$$\rho(G^{-1}F) \leqslant \max_{1\leqslant i\leqslant n}\{|1-\omega_i| + \omega_i\|J\|_\infty\} < 1,$$

whenever

$$0 < \omega_i < \frac{2}{1+\|J\|_\infty}, \quad 0 \leqslant \alpha \leqslant 1$$

for the GAOR method, and

$$0 < \omega_i < \frac{2}{1+\|J\|_\infty}$$

for the GSOR method.

For irreducible diagonally dominant matrices ,the parameters ω_i can equal to $2/[1+\|J\|_\infty]$.

Theorem 3.4. *Let $M = (m_{kj}) \in R^{n\times n}$ be irreducible diagonally dominant by rows with positive diagonal elements. Then, for any initial vector $z^0 \in R^n$, the iterative sequence $\{z^k\}$ generated by the GAOR method or the GSOR method converges to the unique solution z^* of the* LCP(M, q) *and*

$$\rho(G^{-1}F) < \max_{1\leqslant i\leqslant n}\{|1-\omega_i| + \omega_i\|J\|_\infty\} \leqslant 1, \tag{3.7}$$

whenever

$$0 < \omega_i \leqslant \frac{2}{1+\|J\|_\infty}, \quad 0 \leqslant \alpha \leqslant 1$$

for the GAOR method, and

$$0 < \omega_i \leqslant \frac{2}{1+\|J\|_\infty}$$

for the GSOR method.

Proof. We only give a proof for the GAOR method. Because M is also an H-matrix, from Theorem 3.1 the iterative sequence $\{z^k\}$, $k = 0, 1, 2, \ldots$, converges to the unique solution z^* of the LCP(M, q). We only need to prove inequality (3.7). Assume that

$$0 < \omega_i \leqslant \frac{2}{1+\|J\|_\infty}, \quad 0 \leqslant \alpha \leqslant 1.$$

With (3.6), we know that T is nonnegative and, so, according to [16, Theorem 2.7], there exists an eigenvector $x \geqslant 0$, $x \neq 0$, such that

$$Tx = \rho(T)x$$

holds, i.e.,

$$[|I - \Omega| + (1-\alpha)\Omega D^{-1}|L_1| + \Omega D^{-1}|U_1|]x = \rho(T)(I - \alpha\Omega D^{-1}|L_1|)x.$$

Multiplying by Ω^{-1}, it holds that

$$[\rho(T)\Omega^{-1} - |I - \Omega^{-1}|]x = \{[1-\alpha+\alpha\rho(T)]D^{-1}|L_1| + D^{-1}|U_1|\}x.$$

As $[1-\alpha+\alpha\rho(T)]D^{-1}|L_1| + D^{-1}|U_1| \geqslant 0$, it follows by [13, Theorem 11] that

$$\min_{1\leqslant i\leqslant n}\{\omega_i^{-1}\rho(T) - |1-\omega_i^{-1}|\} \leqslant \rho([1-\alpha+\alpha\rho(T)]D^{-1}|L_1| + D^{-1}|U_1|). \tag{3.8}$$

With the proof of [8, Theorem 3.1], we get

$$\rho(T) < 1, \quad 0 \leqslant 1 - \alpha + \alpha\rho(T) < 1.$$

Since M, the matrices J and $[1-\alpha+\alpha\rho(T)]D^{-1}|L_1| + D^{-1}|U_1|$are irreducible, by [16, Theorem 2.1], it follows that

$$\rho([1-\alpha+\alpha\rho(T)]D^{-1}|L_1| + D^{-1}|U_1|) < \rho(D^{-1}(|L_1| + |U_1|)) = \rho(|J|) \leqslant \|J\|_\infty. \tag{3.9}$$

With (3.8) and (3.9) and M be irreducible diagonally dominant by rows, we have

$$\rho(T) < \max_{1\leqslant i\leqslant n}\left\{|1-\omega_i| + \omega_i\|J\|_\infty\right\} \leqslant 1.$$

So,

$$\rho(G^{-1}F) < \max_{1\leqslant i\leqslant n}\left\{|1-\omega_i| + \omega_i\|J\|_\infty\right\} \leqslant 1.$$

This completes the proof. □

4. Monotone convergence analysis

In this section, we mainly discuss the monotone convergence properties of the GAOR and GSOR methods when the system matrix $M \in R^{n\times n}$ is an L-matrix. For this purpose, we assume, from now on, that the set

$$\Psi = \{z \in R^n | z \geqslant 0, Mz + q \geqslant 0\}$$

of the LCP is nonempty.

Firstly, we study the monotone properties of the operator $f: R^n \to R^n$ with

$$\begin{aligned} &f(z) = \xi, \\ &\xi = (z - D^{-1}[\alpha\Omega L_1\xi + (\Omega M - \alpha\Omega L_1)z + \Omega q])_+. \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let the operator $f: R^n \to R^n$ be defined in (4.1). Assume that $M \in R^{n\times n}$ is an L-matrix, and it has the splitting (2.1). Also, assume that $0 < \omega_i \leqslant 1$, $0 \leqslant \alpha \leqslant 1$. Then, for any $z \in \Psi$, it holds that*:

(1) $f(z) \leqslant z$;
(2) $y \leqslant z$ *implies* $f(y) \leqslant f(z)$;
(3) $\xi = f(z) \in \Psi$.

Proof. We firstly verify (1). We only need to verify that the inequalities

$$\xi_i \leqslant z_i, \quad i = 1, 2, \ldots, n \tag{4.2}$$

hold, with ξ_i satisfying

$$\xi_i = \left(z_i - m_{ii}^{-1}\left[\alpha\omega_i\sum_{j=1}^{i-1}(L_1)_{ij}(\xi_j - z_j) + \omega_i(Mz+q)_i\right]\right)_+, \tag{4.3}$$

where $(\bullet)_{ij}$ represents the (i,j)th element of a matrix, $(\bullet)_i$ the ith element of a vector. Since $z \in \Psi$, it holds that $\xi_1 \leqslant z_1$. Now, suppose (4.2) hold for $i = 1, 2, \ldots, j-1$, then we can easily get $\xi_j \leqslant z_j$ from (4.3). By induction, it is obvious that (4.2) holds for all $i = 1, 2, \ldots, n$.

To verify (2), we denote $\psi = f(y)$, where ψ is the fixed point of the system of equation:

$$\psi = (y - D^{-1}[\alpha\Omega L_1\psi + (\Omega M - \alpha\Omega L_1)y + \Omega q])_+.$$

Therefore, we only need to verify that the inequalities

$$\psi_i \leqslant \xi_i, \quad i = 1, 2, \ldots, n \tag{4.4}$$

hold when $y \leqslant z$. In fact, since

$$\begin{aligned}
\psi_i &= \left(y_i - m_{ii}^{-1}\left[\alpha\omega_i\sum_{j=1}^{i-1}(L_1)_{ij}\psi_j + \omega_i m_{ii}y_i + (1-\alpha)\omega_i\sum_{j=1}^{i-1}(L_1)_{ij}y_j + \omega_i\sum_{\substack{j=1\\ j\neq i}}^{n}(U_1)_{ij}y_j + \omega_i q_i\right]\right)_+ \\
&= \left((1-\omega_i)y_i - m_{ii}^{-1}\left[\alpha\omega_i\sum_{j=1}^{i-1}(L_1)_{ij}\psi_j + (1-\alpha)\omega_i\sum_{j=1}^{i-1}(L_1)_{ij}y_j + \omega_i\sum_{\substack{j=1\\ j\neq i}}^{n}(U_1)_{ij}y_j + \omega_i q_i\right]\right)_+,
\end{aligned}$$

by noticing

$$\begin{aligned}
\xi_1 &= \left((1-\omega_1)z_1 - m_{11}^{-1}\left[\omega_1\sum_{j=2}^{n}(U_1)_{1j}z_j + \omega_1 q_1\right]\right)_+ \\
&\geqslant \left((1-\omega_1)y_1 - m_{11}^{-1}\left[\omega_1\sum_{j=2}^{n}(U_1)_{1j}z_j + \omega_1 q_1\right]\right)_+ = \psi_1
\end{aligned}$$

and considering the hypotheses, we can immediately demonstrate the validity of (4.4) by induction.

Now, we turn to (3). Let $\zeta = f(\xi)$. Then by (1) and $\xi = f(z)$, we easily have $\xi \leqslant z$, and from (2), we immediately get $\zeta \leqslant \xi$. By the definition of operator f, we see that $\xi = f(z) \geqslant 0$, $\zeta = f(\xi) \geqslant 0$. Moreover, we can assert the validity of the inequality $(M\xi + q)_1 \geqslant 0$. Otherwise,

$$\zeta_1 = \left(\xi_1 - m_{11}^{-1}\omega_1(M\xi + q)_1\right)_+ > (\xi_1)_+ = \xi_1,$$

i.e., $\zeta_1 > \xi_1$. This contradicts $\zeta \leqslant \xi$. At present, suppose that we have got $(M\xi + q)_j \geqslant 0, j = 1, 2, \ldots, k-1$. Then we must have $\zeta_j \leqslant \xi_j$, $j = 1, 2, \ldots, k-1$. Besides, we can also assert that it holds that $(M\xi + q)_k \geqslant 0$. Otherwise, by defining

$$v = (\xi - D^{-1}\Omega(M\xi + q))_+,$$

we immediately obtain $v_k \geqslant \xi_k$. On the other hand, the following estimates can by straightforwardly deduced from the definitions of ζ and v:

$$\begin{aligned}
\zeta_k &= \left(\xi_k - m_{kk}^{-1}\left[\alpha\omega_k\sum_{j=1}^{k-1}(L_1)_{kj}(\zeta_j - \xi_j) + \omega_k(M\xi + q)_k\right]\right)_+ \\
&= \left((1-\omega_k)\xi_k - (1-\alpha)\omega_k m_{kk}^{-1}\sum_{j=1}^{k-1}(L_1)_{kj}\xi_j - \omega_k m_{kk}^{-1}\sum_{\substack{j=1\\ j\neq k}}^{n}(U_1)_{kj}\xi_j - \alpha\omega_k m_{kk}^{-1}\sum_{j=1}^{k-1}(L_1)_{kj}\zeta_j - \omega_k m_{kk}^{-1}q_k\right)_+ \\
&\leqslant \left((1-\omega_1)\xi_k - (1-\alpha)\omega_k m_{kk}^{-1}\sum_{j=1}^{k-1}(L_1)_{kj}\xi_j - \omega_k m_{kk}^{-1}\sum_{\substack{j=1\\ j\neq k}}^{n}(U_1)_{kj}\xi_j - \alpha\omega_k m_{kk}^{-1}\sum_{j=1}^{k-1}(L_1)_{kj}\xi_j - \omega_k m_{kk}^{-1}q_k\right)_+ \\
&= \left(\xi_k - \omega_k m_{kk}^{-1}(M\xi + q)_k\right)_+ = v_k.
\end{aligned}$$

In addition, since

$$\zeta - v = (\xi - D^{-1}[\alpha\Omega L_1\zeta + (\Omega M - \alpha\Omega L_1)\xi + \Omega q])_+ - (\xi - \Omega D^{-1}(M\xi + q))_+ \leqslant (-\alpha\Omega D^{-1}L_1(\zeta - \xi))_+$$

and analogously,

$$v - \zeta \leqslant (-\alpha\Omega D^{-1}L_1(\zeta - \xi))_+,$$

we get

$$\begin{aligned}|\zeta - v| &= (\zeta - v)_+ + (v - \zeta)_+ \leqslant (-\alpha\Omega D^{-1}L_1(\zeta - \xi))_+ + (-\alpha\Omega D^{-1}L_1(\xi - \zeta))_+ = |\alpha\Omega D^{-1}L_1(\zeta - \xi)|\\ &\leqslant \alpha\Omega D^{-1}|L_1||\zeta - \xi| \leqslant \alpha\Omega D^{-1}|L_1|(I - \alpha\Omega D^{-1}|L_1|)^{-1}\Omega D^{-1}|M\xi + q| \to 0 \quad (\alpha \to 0),\end{aligned}$$

that is, $\lim_{\alpha\to 0}\zeta = v$. Here, the estimate

$$|\zeta - \xi| \leqslant (I - \alpha\Omega D^{-1}|L_1|)^{-1}\Omega D^{-1}|M\xi + q| \tag{4.5}$$

is used. In fact, with the facts of Definition 2 we have

$$f(\xi) = \zeta = (\xi - D^{-1}[\alpha\Omega L_1\zeta + (\Omega M - \alpha\Omega L_1)\xi + \Omega q])_+ \leqslant \xi_+ + (-D^{-1}[\alpha\Omega L_1\zeta + (\Omega M - \alpha\Omega L_1)\xi + \Omega q])_+$$

and

$$f(\xi) = \zeta = (\xi - D^{-1}[\alpha\Omega L_1\zeta + (\Omega M - \alpha\Omega L_1)\xi + \Omega q])_+ \geqslant \xi_+ - (D^{-1}[\alpha\Omega L_1\zeta + (\Omega M - \alpha\Omega L_1)\xi + \Omega q])_+,$$

i.e.,

$$(\zeta - \xi)_+ \leqslant (-\alpha\Omega D^{-1}L_1(\zeta - \xi) - \Omega D^{-1}(M\xi + q))_+$$

and

$$(\xi - \zeta)_+ \leqslant (\alpha\Omega D^{-1}L_1(\zeta - \xi) + \Omega D^{-1}(M\xi + q))_+.$$

So, we obtain

$$|\zeta - \xi| = (\zeta - \xi)_+ + (\xi - \zeta)_+ \leqslant |\alpha\Omega D^{-1}L_1(\zeta - \xi) + \Omega D^{-1}(M\xi + q)| \leqslant \alpha\Omega D^{-1}|L_1||\zeta - \xi| + \Omega D^{-1}|M\xi + q|,$$

i.e.,

$$|\zeta - \xi| \leqslant (I - \alpha\Omega D^{-1}|L_1|)^{-1}\Omega D^{-1}|M\xi + q|.$$

Thus, (4.5) holds. Moreover, observing $v_k \geqslant \xi_k$, we know that $\zeta_k \geqslant \xi_k$ must hold for some sufficiently small $\alpha \in [0,1]$. However, this contracts $\zeta \leqslant \xi$. Therefore, $(M\xi + q)_k \geqslant 0$. By induction, we obtain $\xi \in \Psi$. □

We remark that results in the spirit of this theorem can be found in [4]. Based on Theorem 4.1, we can derive the following monotone convergence theorem about the GAOR and GSOR methods.

Theorem 4.2. *Assume that $M \in R^{n\times n}$ is an L-matrix. Also, assume that $0 < \omega_i \leqslant 1$, $0 \leqslant \alpha \leqslant 1$. Then for any initial vector $z^0 \in \Psi$, the iterative sequence $\{z^k\}$, $k = 0,1,2,\ldots$, generated by the GAOR method or the GSOR method has the following properties*:

(1) $0 \leqslant z^{k+1} \leqslant z^k \leqslant z^0$, $k = 0,1,2,\ldots$;
(2) $\lim_{k\to\infty} z^k = z^*$ *is the unique solution of the LCP(M,q).*

Proof. We only prove the GAOR method. Since $z^0 \in \Psi$, by (1) of Theorem 4.1 we have $z^1 \leqslant z^0$ and $z^1 \in \Psi$. Now, by recursively using Theorem 4.1 again, we obtain (1). From (1), we know that the sequence $\{z^k\}$ is monotone bounded, so that it converges to some vector z^* satisfying

$$z^* = (z^* - D^{-1}[\alpha\Omega L_1 z^* + (\Omega M - \alpha\Omega L_1)z^* + \Omega q])_+,$$

i.e.,

$$z^* = (z^* - D^{-1}[\Omega M z^* + \Omega q])_+.$$

Therefore, z^* is the unique solution of the LCP(M,q). □

The following theorem describes the influences of the parameters ω_i and/α upon the monotone convergence rate the GAOR and GSOR methods.

Theorem 4.3. *Let $M \in R^{n\times n}$ be an L-matrix. Then, for any initial vector $z^0 = \bar{z}^0 \in \Psi$, both the iterative sequences $\{z^k\}$ and $\{\bar{z}^k\}$ generated by the GAOR (or GSOR) method, corresponding to the parameter (ω_i, α) and $(\overline{\omega_i}, \bar{\alpha})$, respectively, converge to the solution $z^* \in R^n$ of the* LCP(M, q)*. Moreover, we have*

$$z^k \leqslant \bar{z}^k, \quad k = 0, 1, 2, \ldots, \tag{4.6}$$

if the parameters satisfy

$$0 < \overline{\omega_i} \leqslant \omega_i \leqslant 1, \quad 0 \leqslant \bar{\alpha} \leqslant \alpha \leqslant 1.$$

Proof. The convergence of sequences $\{z^k\}$ and $\{\bar{z}^k\}$ is proved by Theorem 4.2. Now, we prove (4.6). Similar to the proving process of Theorem 4.1, we can demonstrate that z^k, $\bar{z}^k \in \Psi$ for $k = 0, 1, 2, \ldots$

Let $\overline{\Omega} = \mathrm{diag}(\overline{\omega_1}, \overline{\omega_2}, \ldots, \overline{\omega_n})$. Then

$$\bar{z}^{k+1} = \left(\bar{z}^k - D^{-1}[\bar{\alpha}\overline{\Omega}L_1\bar{z}^{k+1} + (\overline{\Omega}M - \bar{\alpha}\overline{\Omega}L_1)\bar{z}^k + \overline{\Omega}q]\right)_+.$$

Since

$$\bar{\alpha}\overline{\Omega}L_1\bar{z}^{k+1} + (\overline{\Omega}M - \bar{\alpha}\overline{\Omega}L_1)\bar{z}^k + \overline{\Omega}q = \bar{\alpha}\overline{\Omega}L_1(\bar{z}^{k+1} - \bar{z}^k) + \overline{\Omega}(M\bar{z}^k + q) \leqslant \alpha\Omega L_1(\bar{z}^{k+1} - \bar{z}^k) + \Omega(M\bar{z}^k + q),$$

it follows that

$$\bar{z}^{k+1} \geqslant (\bar{z}^k - D^{-1}[\alpha\Omega L_1(\bar{z}^{k+1} - \bar{z}^k) + \Omega(M\bar{z}^k + q)])_+ = f(\bar{z}^k). \tag{4.7}$$

We verify (4.5) by induction. In fact, when $k = 0$, the inequality (4.5) is trivial. Assume that (4.5) holds for some positive integer k. Then, by Theorem 4.1 and the inequality (4.6), we get

$$\bar{z}^{k+1} \geqslant f(\bar{z}^k) \geqslant f(z^k) = z^{k+1}$$

and so $z^k \leqslant \bar{z}^k$ for $k = 0, 1, 2, \ldots$We have completed the proof. □

Remark. Theorems 4.2 and 4.3 show that the parameter collections $\omega_i = \alpha = 1$ can result in faster convergence rate of the GAOR and GSOR methods under the assumptions. This also implies that the optimum parameters, in general, should be ω_i^0, $\alpha^0 \in [1, \infty)$.

5. Numerical results

In this section, we present numerical results of the GAOR and GSOR methods. The codes are written in matlab.

Example 5.1. We consider the LCP(M, q), see [17,18], with the system matrix $M \in R^{n\times n}$ and vector $q \in R^n$:

$$M = \begin{pmatrix} S & -I & -I & 0 & \cdots & 0 & 0 \\ -I & S & -I & -I & \cdots & 0 & 0 \\ I & -I & S & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & -I \\ \vdots & \vdots & & \ddots & \ddots & S & -I \\ 0 & 0 & \cdots & \cdots & I & -I & S \end{pmatrix} \in R^{n\times n}, \quad q = \begin{pmatrix} -1 \\ 1 \\ -1 \\ \vdots \\ (-1)^{n-1} \\ (-1)^n \end{pmatrix} \in R^n,$$

respectively, where $S = \mathrm{tridiag}(-1, 8, -1) \in R^{\bar{n}\times\bar{n}}$, $I \in R^{\bar{n}\times\bar{n}}$ is the identity matrix, and $\bar{n}^2 = n$. It is known that M is a strictly diagonally dominant matrix and thus is an H-matrix. So, the LCP(M, q) has a unique solution as all diagonal elements of M are positive.

For the test problems, we take the initial vector $z^0 = (5, 5, \ldots, 5)$. The termination criterion for GAOR and GSOR methods is

$$\varepsilon(z^k) := \|\min(Mz^k + q, z^k)\|_\infty < 0.1,$$

where the minimum is taken component-wise, see [18]. If $\varepsilon(z^k)$ is close to zero, then we can certainly regard z^k as a good approximation to the unique solution of LCP(M, q).

For all tests, we let in.no and cputime denote, respectively, the number of iteration and the executive time of iterative process. The n parameters ω_i, $i = 1, 2, \ldots, n$, are taken from the n equal-partitioned points of the given interval.

The results are summarized in Tables 1 and 2. In Table 1, we test the iterative convergence of GAOR and GSOR methods when $n = 1600$, 2500, 3600 and 4900, respectively. Moreover, it can be seen from Table 1 that, when $0 < \alpha \leqslant 1, 0 < \omega_i < \frac{2}{1+\rho(|J|)}$, the value of α and ω_i is bigger, the iterative speed is faster.

Table 1
Time unit: s $\varepsilon(z^k) < 0.1$

n	$\frac{2}{1+\rho(\lvert J\rvert)}$	Interval	GAOR			GSOR		
			α	cputime	in.no	α	cputime	in. no
1600	1.1457	[1 1.14]	0.9	1.141	6	1	1.265	6
1600	1.1457	[0.5 0.7]	0.7	3.359	16	1	3.219	16
2500	1.1447	[1 1.14]	0.9	2.453	6	1	2.532	6
2500	1.1447	[0.3 0.5]	0.7	11.922	29	1	11.859	29
3600	1.1441	[1 1.14]	0.9	5.187	6	1	4.937	6
3600	1.1441	[0.5 0.7]	0.7	14.234	17	1	13.234	16
4900	1.1438	[1 1.14]	0.9	10.297	6	1	10.484	6
4900	1.1438	[0.3 0.5]	0.7	52.672	30	1	48.031	29

Table 2
GAOR GSOR $n = 1600 \frac{2}{1+\rho(|J|)} = 1.1441 \frac{2}{1+\|J\|_\infty} = 1.1429 \varepsilon(z^k) < 0.1$

Interval	α	$\rho(G^{-1}F)$	$\max_{1\leqslant i\leqslant n}\{\lvert 1-\omega_i\rvert + \omega_i\rho(\lvert J\rvert)\}$	$\max_{1\leqslant i\leqslant n}\{\lvert 1-\omega_i\rvert + \omega_i\|J\|_\infty\}$
[0.3 0.5]	0.4	0.9053	0.9237	0.925
[0.3 0.5]	0.7	0.9014	0.9237	0.925
[0.3 0.5]	1	0.8971	0.9237	0.925
[0.5 0.7]	0.4	0.8443	0.8728	0.875
[0.5 0.7]	0.7	0.8329	0.8728	0.875
[0.5 0.7]	1	0.8193	0.8728	0.875
[0.7 0.9]	0.4	0.7797	0.822	0.825
[0.7 0.9]	0.7	0.7553	0.822	0.825
[0.7 0.9]	1	0.7225	0.822	0.825
[0.9 1.0]	0.4	0.7207	0.7711	0.775
[0.9 1.0]	0.7	0.6777	0.7711	0.775
[0.9 1.0]	1	0.6106	0.7711	0.775
[1 1.14]	0.4	0.9166	0.9901	0.995
[1 1.14]	0.7	0.9018	0.9901	0.995
[1 1.14]	1	0.8800	0.9901	0.995
$\omega_i = \frac{2}{1+\|J\|_\infty}$	0.4	0.9941	0.9951	1
$\omega_i = \frac{2}{1+\|J\|_\infty}$	0.7	0.9930	0.9951	1
$\omega_i = \frac{2}{1+\|J\|_\infty}$	1	0.9914	0.9951	1

Table 3
GAOR $n = 3000$ $\varepsilon(z^k) < 0.001$ time unit: s

α	Interval	Problem # 1		Problem # 2	
		cputime	in.no	cputime	in.no
0.4	[0.3 0.5]	373	6	307.859	5
0.7	[0.5 0.7]	185.547	3	184.672	3
0.9	[0.7 0.9]	123.359	2	123.344	2
1	$\omega_i = 1$	122.066	2	121.868	2

By Table 2, we compare $\rho(G^{-1}F)$ with $\max_{1\leqslant i\leqslant n}\{|1-\omega_i| + \omega_i\rho(|J|)\}$ in according to Theorem 3.1 and Corollary 3.2 as M is an H-matrix with positive diagonal elements. At the same time, since M is also strictly and irreducible diagonally dominant, we compare $\rho(G^{-1}F)$ with $\max_{1\leqslant i\leqslant n}\{|1-\omega_i| + \omega_i\|J\|_\infty\}$ through Corollary 3.3 and Theorem 3.4. These computational data of Table 2 illustrate our theoretic results.

Example 5.2. This example is prepared for the monotone convergence of L-matrix. In this test, matrix M is a symmetric tridiagonal matrix with its diagonal elements be 8, whose nondiagonal elements are chosen randomly from interval $[-5\ 0]$. Obviously, M is an L-matrix. The elements of q are random points in the interval $[100, 200]$ in order that $\Psi = \{z \in R^n | z \geqslant 0,\ Mz + q \geqslant 0\}$ is nonempty.

For these tests, we also let in.no and cputime denote, respectively, the number of iteration and the executive time of iterative process. The n parameters ω_i, $i = 1, 2, \ldots, n$, are taken from the n equal-partitioned points of the given interval.

We generate two test problems for different parameters. The initial vector $z^0 \in \Psi$ is always set to $(25, 25, \ldots, 25)$. The termination criterion for the GAOR and GSOR is $\varepsilon(z^k) := \|\min(Mz^k + q, z^k)\|_\infty < 0.001$. The results of test is in Table 3, which confirm Theorems 4.2 and 4.3.

6. Conclusion

In this paper, we have proposed a class of generalized AOR (GAOR) methods for linear complementarity problem, whose special case reduces to GSOR. We also have presented some sufficient conditions for convergence of the GAOR and GSOR methods, when M is an H-matrix, M-matrix and a strictly or irreducible diagonally dominant matrix. Besides, the monotone convergence of the new methods have been discussed when M is an L-matrix. From the numerical results, the proposed methods are effective for large and sparse linear complementarity problems when M is an H-matrix with positive diagonals.

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