Degree and connectivity conditions for IM-extendibility and vertex-deletable IM-extendibility*

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Abstract

A graph is called induced matching extendable, if every induced matching of it is contained in a perfect matching of it. A graph G is called $2k$-vertex deletable induced matching extendable, if $G-S$ is induced matching extendable for every $S \subset V(G)$ with $|S| = 2k$. The following results are proved in this paper. (1) If $\kappa(G) \geq \lceil \frac{\nu(G)}{3} \rceil + 1$ and $max\{d(u),\ d(v)\} \geq \frac{2\nu(G)+1}{3}$ for every two nonadjacent vertices u and v, then G is induced matching extendable. (2) If $\kappa(G) \geq \lceil \frac{\nu(G)+4k}{3} \rceil + 1$ and $max\{d(u),\ d(v)\} \geq \frac{2\nu(G)+2k+1}{3}$ for every two nonadjacent vertices u and v, then G is $2k$-vertex deletable induced matching extendable. (3) If $d(u) + d(v) \geq 2\lceil \frac{2\nu(G)+2k}{3} \rceil - 1$ for every two nonadjacent vertices u and v, then G is $2k$-vertex deletable IM-extendable. Examples are given to show the tightness of all the conditions.

Keywords: induced matching, IM-extendable, $2k$-vertex deletable IM-extendable

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1 Introduction and preliminary results

Graphs considered in this paper are finite and simple. Terminologies and notations which are not defined here can be found in [1] or [4].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $\nu(G)$ the order of $V(G)$. For any vertex subset $S \subseteq V(G)$, set

$$E(S) = \{uv \in E(G) |\ u, v \in S\}.$$

For any edge set $M \subseteq E(G)$, set

$$V(M) = \{u \in V(G) |\ \text{there is a vertex } v \text{ of } G \text{ such that } uv \in M\}.$$

For any vertex $v \in V(G)$, we denote by $N(v)$ the neighbor set of v in G, by $d_G(v)$ or $d(v)$ the degree of v in G. The minimum degree of vertices of G is denoted by $\delta(G)$. The connectivity of G is denoted by $\kappa(G)$. A complete graph on n vertices is denoted by K_n, while a graph with n independent vertices is denoted by $\overline{K_n}$.

A set of edges $M \subseteq E(G)$ is called a *matching* of G if no two of them share a common endvertex. A matching is *perfect* if it covers all vertices in G. We call a connected graph G *k-extendable*, for $1 \leq k \leq \frac{\nu(G)}{2} - 1$, if there is a matching of size k in G, and every such matching is contained in a perfect matching of G. A matching M is *induced* [2], if $E(V(M)) = M$. We say that a graph G is *induced matching extendable* [9], shortly *IM-extendable*, if every induced matching M of G is contained in a perfect matching of G. Researches on IM-extendable graphs can be found in [6, 7, 8, 9]. G is called *$2k$-vertex deletable IM-extendable* if for every $S \subseteq V(G)$ with $|S| = 2k$, $G - S$ is IM-extendable.

In this paper we prove some degree and connectivity conditions for IM-extendibility and $2k$-vertex deletable IM-extendibility, and provide examples to show that the conditions are tight. The following lemmas will be used in our proofs.

Lemma 1.1. *(Fan, [3]) Let G be a 2-connected graph with $\nu(G) \geq 3$. If for each pair vertices $\{u, v\}$ with $d(u, v) = 2$, $max\{d_G(u), d_G(v)\} \geq \frac{\nu(G)}{2}$, then G is hamiltonian.*

Lemma 1.2. *(Dirac Theorem [1]) Let G be a simple graph with $\delta(G) \geq \frac{\nu(G)}{2}$. Then G is hamiltonian.*

Lemma 1.3. *(Plummer, [5]) Let G be a graph with p vertices, where p is even, and let k be an integer with $1 \leq k < \frac{p}{2}$. Suppose that for each pair of nonadjacent vertices u and $v \in V(G)$, $d(u) + d(v) \geq p + 2k - 1$. Then G is k-extendable.*

2 Main results

Theorem 2.1. *Let G be a graph with $2n$ vertices. If $\kappa(G) \geq \lceil \frac{2n}{3} \rceil + 1$ and $max\{d(u), d(v)\} \geq \frac{4n+1}{3}$ for every two nonadjacent vertices u and v in G, then G is IM-extendable.*

Proof. It is easily checked that the theorem holds for $n \leq 3$. Therefore, we assume that $n \geq 4$. Let M be an induced matching of G and $G' = G - V(M)$. We need to find a perfect matching of G'.
We claim that $|M| \leq \frac{n+1}{3}$. For if $|M| = 1$ the claim holds. If $|M| \geq 2$, then, there exist $x, y \in V(M)$, $xy \notin M$. We have $\frac{4n+1}{3} \leq max\{d(x), d(y)\} \leq 2n - 1 - (2|M| - 2) = 2n - 2|M| + 1$, that is, $|M| \leq \frac{n+1}{3}$.
For any two nonadjacent vertices u and v in G',

$$\begin{aligned} max\{d_{G'}(u), d_{G'}(v)\} &\geq max\{d(u), d(v)\} - 2|M| \\ &\geq \frac{4n+1}{3} - \frac{n+1}{3} - |M| \\ &= n - |M| \\ &= \frac{\nu(G')}{2}. \end{aligned} \tag{1}$$

If $|M| \leq \frac{n+1}{3} - 1 = \frac{n-2}{3}$, then $\kappa(G') \geq \kappa(G) - 2|M| \geq \lceil \frac{2n}{3} \rceil + 1 - \frac{2(n-2)}{3} > 2$. By Lemma 1.1, G' is hamiltonian, and hence has a perfect matching.
What left is the case that $|M| = \lfloor \frac{n+1}{3} \rfloor$. Since $\kappa(G') \geq \kappa(G) - 2|M| \geq \lceil \frac{2n}{3} \rceil + 1 - 2\lfloor \frac{n+1}{3} \rfloor \geq 1$, G' is connected. If G' is 2-connected, then again by Lemma 1.1, G' is hamiltonian and has a perfect matching. So, we assume that G' has a cut vertex u_0.

Let G_1, G_2, …, G_l, $l \geq 2$, be the components of $G - u_0$. We prove that $l = 2$. Suppose that $l \geq 3$ and let $u_i \in V(G_i)$, $i = 1, 2, 3$. By (1), at least two of u_1, u_2 and u_3, say u_1 and u_2, are of degree no less than $\frac{\nu(G')}{2}$. But then $\nu(G') \geq \nu(G_1) + \nu(G_2) + 2 \geq d_{G'}(u_1) + d_{G'}(u_2) + 2 \geq \nu(G') + 2$, a contradiction. Hence $l = 2$.

Without lost of generality, we assume that $\nu(G_1) \geq \frac{\nu(G')}{2} > \frac{\nu(G')}{2} - 1 \geq \nu(G_2)$. Then G_2 must be a complete graph, or, for any two nonadjacent vertices u, v in G_2, we have

$$\nu(G_2) - 2 \geq max\{d_{G_2}(u), d_{G_2}(v)\} \geq max\{d_{G'}(u), d_{G'}(v)\} - 1 \geq \frac{\nu(G')}{2} - 1,$$

a contradiction.

For any $v \in V(G_2)$, $d_{G'}(v) \leq \nu(G_2) < \frac{\nu(G')}{2}$. By (1), for any $u \in V(G_1)$, we have $d_{G'}(u) \geq \frac{\nu(G')}{2}$, and $d_{G_1}(u) \geq d_{G'}(u) - 1 \geq \frac{\nu(G')}{2} - 1 \geq \frac{\nu(G_1)}{2}$. So, G_1 is hamiltonian by Lemma 1.2. Then it is easy to find a perfect matching of G'. $\square$

Theorem 2.2. *The bounds for connectivity and degree in Theorem 2.1 are tight.*

Proof. We give two examples, which show that the connectivity condition and the degree condition are tight, respectively.

Example 1. Let $n = 3m$, where m is a nonnegative integer, and G_m be a 1-regular graph with $2m$ vertices, and G is obtained by joining every vertex of G_m to every vertex of a K_1 and a K_{4m-1}. Then, $\nu(G) = 2m+1+4m-1 = 6m = 2n$, $\kappa(G) = 2m = \frac{2n}{3}$ and $max\{d(u), d(v)\} \geq 4m + 1 = \lceil \frac{4n+1}{3} \rceil$ for every two nonadjacent vertices u and v in G. However, $G' = G - V(G_m)$ has no perfect matching, hence G is not IM-extendable. So the bound for connectivity is tight.

Example 2. Let $n = 3m + 1$, where $m \geq 1$ is an integer. Let G_{m+1} be a 1-regular graph with $2(m + 1)$ vertices, and G is obtained by joining every vertex of G_{m+1} to every vertex of a K_1 and a K_{4m-1}. Then $\nu(G) = 6m + 2 = 2n$, $\kappa(G) = 2m + 2 = \lceil \frac{2n}{3} \rceil + 1$, and $max\{d(u), d(v)\} \geq 4m + 1 = \lceil \frac{4n+1}{3} \rceil - 1$ for every two nonadjacent vertices u and v in G. But $G' =$

$G - V(G_{m+1})$ has no perfect matching and so G is not IM-extendable. Hence, the degree condition is tight. $\square$

Theorem 2.1 can be used to obtain the following result for $2k$-vertex deletable IM-extednable graphs.

Theorem 2.3. *Let G be a graph with $2n$ vertices. If $\kappa(G) \geq \lceil \frac{2n+4k}{3} \rceil + 1$ and $max\{d(u), d(v)\} \geq \frac{4n+2k+1}{3}$ for every two nonadjacent vertices u and v, then G is $2k$-vertex deletable IM-extendable.*

Proof. Let G be a graph with $\nu(G) = 2n$, $\kappa(G) \geq \lceil \frac{2n+4k}{3} \rceil + 1$ and $max\{d(u), d(v)\} \geq \frac{4n+2k+1}{3}$ for every two nonadjacent vertices u and v. Let $S \subseteq V(G)$ with $|S| = 2k$ and $H = G - S$. Then, $\nu(H) = 2n - 2k$ and

$$\kappa(H) \geq \lceil \frac{2n+4k}{3} \rceil + 1 - 2k = \lceil \frac{2(n-k)}{3} \rceil + 1 = \lceil \frac{\nu(H)}{3} \rceil + 1.$$

For every two nonadjacent vertices u and v,

$$max\{d_H(u), d_H(v)\} \geq \frac{4n+2k+1}{3} - 2k = \frac{4(n-k)+1}{3} = \frac{2\nu(H)+1}{3}.$$

By Theorem 2.1, H is IM-extendable. Therefor G is a $2k$-vertex deletable IM-extendable graph. $\square$

We give examples to show that the bounds in Theorem 2.3 are tight .

Theorem 2.4. *The bounds for connectivity and degree in Theorem 2.3 are tight.*

Proof. We show the tightness by two examples.
Example 1. Let G_m be a 1-regular graph with $2m$ vertices, where $m \geq 2$, H be obtained by joining every vertices of G_m to every vertices of a K_1 and a K_{4m-1}, and $G = H \vee K_{2k}$. Let $\nu(G) = 6m + 2k = 2n$. Then, $\kappa(G) = 2k + 2m = \lceil \frac{2n+4k}{3} \rceil$, and $max\{d(u), d(v)\} \geq 4m+2k+1 = \frac{4n+2k}{3} + 1 > \frac{4n+2k+1}{3}$ for every two nonadjacent vertices u and v in G.
Removing the $2k$ vertices in K_{2k} from G, we get H, which is not IM-extendable. Therefore, G is not $2k$-vertex deletable IM-extendable and the connectivity condition is tight.

Example 2. Let G_{m+1} be a 1-regular graph with $\nu(G_{m+1}) = 2m + 2$, H be obtained by joining every vertices of G_m to a K_1 and a K_{4m-1}, and $G = H \vee K_{2k}$. Let $\nu(G) = 6m + 2k + 2 = 2n$.
It is easy to check that $\kappa(G) = 2k+2m+2 = \lceil \frac{2n+4k}{3} \rceil + 1$, $max\{d(u), d(v)\} \geq 2k + 4m + 1 = \frac{4n+2k+1}{3} - \frac{2}{3}$ for every two nonadjacent vertices u and v in G.
Removing the $2k$ vertices in K_{2k} from G, we get H, which is not IM-extendable. Therefore, G is not $2k$-vertex deletable IM-extendable. So, the degree condition is tight. □

Now, we consider a degree sum condition for $2k$-vertex deletable IM-extendable graphs.

Theorem 2.5. *Let G be a graph with $2n$ vertices. If $d(u) + d(v) \geq 2\lceil \frac{4n+2k}{3} \rceil - 1$ for every two nonadjacent vertices u and v, then G is $2k$-vertex deletable IM-extendable.*

Proof. Let G be a graph with $2n$ vertices, and $d(u) + d(v) \geq 2\lceil \frac{4n+2k}{3} \rceil - 1$ for every two nonadjacent vertices u and v in G. Let S be a subset of $V(G)$ with $|S| = 2k$ and $H = G - S$. We prove that H is IM-extendable.
Let u and v be two nonadjacent vertices in H, then $uv \notin E(G)$. Therefore

$$\begin{aligned} d_H(u) + d_H(v) &\geq d_G(u) + d_G(v) - 4k \\ &\geq 2\lceil \frac{4n + 2k}{3} \rceil - 1 - 4k \\ &= 2\lceil \frac{4n - 4k}{3} \rceil - 1 \\ &= 2\lceil \frac{2\nu(H)}{3} \rceil - 1. \end{aligned} \tag{2}$$

If $\nu(H) = 2$ or 4, then H must be complete and hence IM-extendable. We assume that $\nu(H) \geq 6$. By (2) and Lemma 1.3, H is $\lceil \frac{\nu(H)}{6} \rceil$-extendable.
Let M be any induced matching of H. Suppose that $|M| \geq \lceil \frac{\nu(H)}{6} \rceil + 1$. Then, there must be two nonadjacent vertices u, $v \in V(M)$. By (2), $d_H(u) + d_H(v) \geq 2\lceil \frac{2\nu(H)}{3} \rceil - 1 \geq 4\lfloor \frac{\nu(H)}{3} \rfloor - 1$. Since M is induced, we have $d_H(u) + d_H(v) \leq 2(\nu(H) - 1 - 2(|M| - 1)) \leq 2\nu(H) - 4(\lceil \frac{\nu(H)}{6} \rceil + 1) + 2 = 4\lfloor \frac{\nu(H)}{3} \rfloor - 2$, a contradiction. Hence, $|M| \leq \lceil \frac{\nu(H)}{6} \rceil$.

Since H is $\lceil \frac{\nu(H)}{6} \rceil$-extendable, M is contained in a perfect matching of H. Therefore, H is IM-extendable, and G is $2k$-vertex deletable IM-extendable. $\square$

Similarly, we prove the tightness of the degree condition.

Theorem 2.6. *The bound for degree sum in Theorem 2.5 is tight.*

Proof. To prove the tightness, we construct a graph G, where $\nu(G) = 2n$, $d(u) + d(v) \geq 2\lceil \frac{4n+2k}{3} \rceil - 2$ for every two nonadjacent vertices u and v in G, and there exist $u_0,\ v_0 \in V(G)$ such that $d(u_0) + d(v_0) = 2\lceil \frac{4n+2k}{3} \rceil - 2$, but G is not $2k$-vertex deletable IM-extendable.

Let $G = H \vee K_{2k}$, where H is constructed depending on n and k, as follows.

Case 1. $n - k = 3m$.

Let $H = H_1 \vee H_2 \vee H_3$, where H_1 is an 1-regular graph on $2m$ vertices, $H_2 = \overline{K_{2m-1}}$, $H_3 = \overline{K_{2m+1}}$.

It is easy to check that $\delta(G) = 4m+2k-1$, so $d(u)+d(v) \geq 2(4m+2k-1) = 2\lceil \frac{4n+2k}{3} \rceil - 2$ for every two nonadjacent vertices $u,\ v \in V(G)$, where equality holds if $u,\ v \in V(H_3)$.

Case 2. $n - k = 3m + 1$.

Let $H = H_1 \vee H_2 \vee H_3$, where H_1 is an 1-regular graph on $2m+2$ vertices, $H_2 = \overline{K_{2m-1}}$, $H_3 = \overline{K_{2m+1}}$.

It can be checked that $\delta(G) = 4m+2k+1$, so $d(u)+d(v) \geq 2(4m+2k+1) = 2\lceil \frac{4n+2k}{3} \rceil - 2$ for every two nonadjacent vertices $u,\ v \in V(G)$, where equality holds if $u,\ v \in V(H_3)$.

Case 3. $n - k = 3m + 2$.

Let $H = H_1 \vee H_2 \vee H_3$, where H_1 is an 1-regular graph on $2m+2$ vertices, $H_2 = \overline{K_{2m}}$, $H_3 = \overline{K_{2m+2}}$.

It can be checked that $\delta(G) = 4m+2k+2$, so $d(u)+d(v) \geq 2(4m+2k+2) = 2\lceil \frac{4n+2k}{3} \rceil - 2$ for every two nonadjacent vertices $u,\ v \in V(G)$, where equality holds if $u,\ v \in V(H_3)$.

In all cases above, $H - H_1$ does not have a perfect matching. Therefore H is not IM-extendable and G is not $2k$-vertex deletable IM-extendable. $\square$

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