Abundant soliton solutions of general nonlocal nonlinear Schrödinger system with external field

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Periodic and quasi-periodic breather multi-solitons solutions, the dipole-type breather soliton solution, the rogue wave solution, and the fission soliton solution of the general nonlocal Schrödinger equation are derived by using the similarity transformation and manipulating the external potential function. The stability of the exact solitary wave solutions with the white noise perturbation also is investigated numerically.

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1. INTRODUCTION

Properties of solitons supported by media with local nonlinear response are now well established. However, recent interest in the study of nonlocal optical solitons has increased because of experimental observations and theoretical treatments of self-trapping effects and spatial solitary waves in different types of nonlocal nonlinear media. In particular, new effects of nonlocality have been studied in photorefractive crystals [1,2], nematic liquid crystals [3,4], plasmas [5], thermo-optical materials [6], and Bose–Einstein condensates with long-range interparticle interactions [7,8]. The nonlocality of nonlinear response, which has been shown to profoundly affect the properties and interactions of solitons, is a characteristic feature of nonlocal nonlinear media. For example, the nonlocal nonlinear response suppresses the modulation instability of the plane waves in focusing media [9,10]; it can arrest the catastrophic collapse of multidimensional beams [11–14] and stabilize complex soliton structures, such as vortex solitons [15]; it also can make the soliton form bound states [16] and the colliding solitons merge into a standing wave. Stable dipole solitons in a medium with a Gaussian response function also have been predicted recently [17]. Periodic solitons were identified in nonlocal lead-glass displaying a thermal optical nonlinearity [18]. Analytical soliton solutions of the nonlocal focusing Kerr-type medium with exponential response function have been derived by the symmetry reduction method [19].

The propagating properties of solitons were studied in nonlocal media with the spatially growing self-repulsive local nonlinearity [20–24]. The critical behavior of solitary waves was investigated by using the time-dependent variational principle in a nonlinear nonlocal medium with a power-law response function [25]. Many different types of nonlocal solitons are found in the nonlocal media by manipulating the external field. The dipole solitons that are strongly asymmetric and stable are revealed at the interface of optical lattice [26]. A 2D nonlocal gap soliton is formed in the nonlocal medium with liquid-infiltrated photonic crystal fibers [27]. Mixed gap solitons exist in the nonlocal nonlinear Kerr-type medium with two different imprinted semi-infinite periodic lattices [28]. Studying explicit solutions of nonlinear Schrödinger (NLS) equations with inhomogeneous coefficients is interesting because these NLS equations are applied to nonlinear optics, Bose–Einstein condensates, and plasma [29–33]. In this paper, we construct different types of explicit solutions in the nonlocal nonlinear Kerr media with space dependent nonlinearity, space-dependent nonlocality, and the external field.

The paper is organized as follows: in Section 2, we introduce a general nonlocal NLS model and the corresponding order differential system by a similarity transformation. In Section 3, we express various types of soliton waves in the nonlocal media with different external fields, nonlocalities and nonlinearities. These soliton waves include breather bright solitons, double- periodic solitons, quasi-periodic solitons, Saddle-type rogue waves, and fisson soliton waves. Section 4 is the conclusion.

2. GENERAL THEORY

We consider the dimensionless nonlocal nonlinear Schrödinger equation with the external field and the alterable modulation nonlinearity, i.e.,

$$\begin{aligned} &i\psi_z + \frac{1}{2}\psi_{xx} - V(x,z)\psi - g(x,z)n\psi = 0, \\ &n - A(x,z)n_{xx} = B(x,z)|\psi|^2, \end{aligned} \tag{2.1}$$

where $\psi(x,z)$ is the complex amplitude envelope of the laser beam, the nonlinear contribution to the refractive index is given by $n(x,z) = \int_{-\infty}^{\infty} R(x-x')|\psi(x',z)\mathrm{d}x'$ in which $R(x) \sim \exp(-|x|/\sigma)$ is the response function of the nonlocal medium, x and z denote the transverse and longitudinal coordinates scaled to the beam width ω_0 and diffraction length L_d, respectively. $V(x,z)$ is an external potential, $g(x,z)$ and $B(x,z)$ describe the nonlinearity, and $A(x,z)$ describes the nonlocality of the nonlinear response. As far as we know, the relationship between the width of the response function and the width of the intensity profile divides the degree of nonlocality into four types, namely the local, weakly nonlocal, general, and strongly responses. When $A(x,z)\to 0$, Eq. (2.1) describes a local nonlinear response. In contrast, Eq. (2.1) describes a strongly nonlocal response as $A(x,z)\to\infty$.

Our goal is to research for the analytical solutions of Eq. (2.1) by the proper similarity transformation, which was proposed in [29,31,34]:

$$\psi(x,z)=\rho(x,z)e^{i\tau(x,z)}\Phi(\xi(x,z)),$$
$$n(x,z)=\rho(x,z)\chi(\xi(x,z)). \tag{2.2}$$

We use the above transformation in Eq. (2.2) and require $\Phi(\xi)$ and $\chi(\xi)$ to satisfy the following stationary nonlocal nonlinear Schrödinger equation:

$$\mu\Phi+\Phi_{\xi\xi}+G\Phi\chi=0,\qquad \chi-D\chi_{\xi\xi}-\Phi^2=0, \tag{2.3}$$

where μ is the eigenvalue of the nonlinear equation (or named the chemical potential), and G and D are constants. And we find a set of constraint equations for the functions ρ, τ, and ξ,

$$2\rho\rho_z+(\rho^2\tau_x)_x=0, \tag{2.4}$$

$$(\rho^2\xi_x)_x=0,\qquad 2g\rho=-G\xi_x^2 \tag{2.5}$$

$$V+\tau_z+(\tau_x^2+\mu\xi_x^2-\rho^{-1}\rho_{xx})/2=0, \tag{2.6}$$

$$\xi_z+\tau_x\xi_x=0, \tag{2.7}$$

$$\rho-A\rho_{xx}-B\rho^2=0, \tag{2.8}$$

$$A\xi_x^2-DB\rho=0. \tag{2.9}$$

If we can get exact solutions of these functions ρ, τ, and ξ from Eqs. (2.4) to (2.9), we finally obtain the solutions of ψ and n by combining the similarity transformation in Eq. (2.2) with the solutions Φ and χ of Eq. (2.3). In order to get the solutions of ρ, τ, and ξ, we take $\xi(x,z)=F(\eta(x,z))$, in which $\eta(x,z)=\gamma(z)x+\delta(z)$. The function $\gamma(z)$ denotes the inverse of the width of the soliton solution, and $-\delta(z)/\gamma(z)$ denotes the central position of its mass. Obviously, Eqs. (2.4)–(2.9) are overdetermined for the functions ρ, τ, and ξ. However, we can handle the problem to confirm that functions $V(x,z)$, $g(x,z)$, $A(x,z)$, and $B(x,z)$ are expressed by the functions ρ, τ, and ξ. Therefore, we obtain

$$\rho(x,z)=\sqrt{\frac{\gamma(z)}{F'(\eta)}}, \tag{2.10}$$

$$\tau(x,z)=-\frac{\gamma_z}{2\gamma}x^2-\frac{\delta_z}{\gamma}x+\alpha(z) \tag{2.11}$$

$$g(x,z)=-\frac{G}{2}\frac{\gamma^4}{\rho^5}, \tag{2.12}$$

$$V(x,z)=\frac{\rho_{xx}}{2\rho}-\tau_z-\frac{\tau_x^2}{2}-\frac{\mu\gamma^4}{2\rho^4} \tag{2.13}$$

$$A(x,z)=\frac{\rho^4 D}{\gamma^4+\rho^3\rho_{xx}D}, \tag{2.14}$$

$$B(x,z)=\frac{\gamma^4}{\rho\gamma^4+\rho^4\rho_{xx}D}, \tag{2.15}$$

where $\alpha(z)$, $\gamma(z)$, and $\delta(z)$ are arbitrary functions with respect to z. We can choose the appropriate forms of the functions $\gamma(z)$ and $\delta(z)$ to construct the external potential function $V(x,z)$, the nonlocality of the nonlinear responses $A(x,z)$ and the nonlinearity $g(x,z)$ [and $B(x,z)$]. The nonlocality function $A(x,z)$ in Eq. (2.14) and the parameter D are required to maintain their positive values for nonlocal nonlinearity media materials in reality. However, the nonlocal Kerr system is equivalent to the quadratic $\chi^{(2)}$ nonlinear system when the nonlocality $A(x,z)<0$ (or the parameter $D<0$) [35, 35]. So that seeking the solutions of this case is useful in physics.

3. NONLOCAL SOLITON SOLUTIONS

We take that the nonlinearity $g(x,z)$ has the exponential function type as

$$g(x,z)=-\frac{G}{2}\gamma^{\frac{3}{2}}e^{\frac{5\eta^2}{2}}. \tag{3.1}$$

And then from Eqs. (3.1) and (2.10), we obtain the external potential function about harmonic and parabolic types and the nonlocality function

$$V(x,z)=\omega^2(z)x^2+f(z)x+h(z)-\frac{\mu}{2}\gamma^2e^{2\eta^2}, \tag{3.2}$$

$$A(x,z)=\frac{\gamma^{-2}e^{-2\eta^2}D}{1+(\eta^2-1)e^{-2\eta^2}D}, \tag{3.3}$$

where

$$\omega^2(z)=\frac{\gamma^6+\gamma\gamma_{zz}-2\gamma_z^2}{2\gamma^2}, \tag{3.4}$$

$$f(z)=\frac{\gamma^5\delta+\gamma\delta_{zz}-2\gamma_z\delta_z}{\gamma^2}, \tag{3.5}$$

$$h(z)=\frac{\gamma^4\delta^2-\gamma^4-\delta_z^2-2\gamma^2\alpha_z}{2\gamma^2}, \tag{3.6}$$

and the functions $\rho(x,z) = \sqrt{\gamma}e^{-(\eta^2/2)}$ and $B(x,z) = \gamma^{-1/2}e^{\eta^2/2}/(1+(\eta^2-1)e^{-2\eta^2}/D)$. In order to investigate the nonlocal soliton solutions of Eq. (2.1) with different external potential functions, we consider three different cases of the chemical potential μ.

A. Case A. $\mu = 0$

In this case, choosing $\delta(z) = 0$ and $\alpha(z) = -\int \gamma^2(z)/2\mathrm{d}z$, we can obtain the simplest form of the external potential

$$V(x,z) = \omega^2(z)x^2. \tag{3.7}$$

Let $\varphi = 1/\gamma$, we transform Eq. (3.4) into the Mathieu equation:

$$\varphi_{zz} + 2\omega^2(z)\varphi = \frac{1}{\varphi^3}. \tag{3.8}$$

Considering the periodic function with respect to z for the external potential, we take two choices for $\omega(z)$ in Eq. (3.8).

The first choice is

$$\omega^2(z) = (1+\cos(\omega_0 z))/2, \tag{3.9}$$

where ω_0 is a real constant. To illustrate the solution of the Mathieu Eq. (3.8), we take $\omega_0 = 6$ and obtain the function γ:

$$\gamma = \sqrt{\frac{3\sqrt{4c_1c_2^2 - c_1^2}\left(C\left(\frac{1}{9},-\frac{1}{18},3z\right)S'\left(\frac{1}{9},-\frac{1}{18},3z\right) - C'\left(\frac{1}{9},-\frac{1}{18},3z\right)S\left(\frac{1}{9},-\frac{1}{18},3z\right)\right)}{2c_1C\left(\frac{1}{9},-\frac{1}{18},3z\right)S\left(\frac{1}{9},-\frac{1}{18},3z\right) + 2c_2S\left(\frac{1}{9},-\frac{1}{18},3z\right)^2 + 2c_1c_2C\left(\frac{1}{9},-\frac{1}{18},3z\right)^2}}, \tag{3.10}$$

where $S((1/9),-(1/18),3z)$ and $C((1/9),-(1/18),3z)$ stand for the odd and even periodic Mathieu functions, respectively, the prime stands for the first derivative of the Mathieu function, and c_1, c_2 are arbitrary real constants. From Eq. (2.10), we find the similarity invariant ξ is

$$\xi(x,z) = \int_0^{\gamma(z)x} e^{\eta^2}\mathrm{d}\eta = -\frac{i}{2}\sqrt{\pi}\,\mathrm{erf}(i\eta). \tag{3.11}$$

On the other hand, taking $\mu = 0$ into Eq. (2.3) and solving it, we have derived

$$\Phi(\xi) = \frac{3m}{(m^2+1)\sqrt{GD}}\mathrm{cn}(k\xi,m)\mathrm{dn}(k\xi,m),$$
$$\chi(\xi) = \frac{1}{2GD} - \frac{3\,m^2}{(1+m^2)GD}\mathrm{sn}(k\xi,m)^2, \tag{3.12}$$

where $k = (1/\sqrt{2(1+m^2)D})$ and $m = \sqrt{3}-\sqrt{2}$. Finally, we can obtain that the profile of the laser pulse and the refractive index are

$$|\psi|^2 = \frac{9\,m^2\gamma e^{-\eta^2}}{(1+m^2)^2GD}\mathrm{cn}(k\xi,m)^2\mathrm{dn}(k\xi,m)^2,$$
$$n = \gamma^{\frac{1}{2}}e^{-\frac{\eta^2}{2}}\left(\frac{1}{2GD} - \frac{3m^3}{(1+m^2)GD}\mathrm{sn}(k\xi,m)^2\right). \tag{3.13}$$

It is evident the solution in Eq. (3.13) of ψ is a multibright soliton, in which amplitude is periodically changed along with the propagating direction. The breather frequency of this soliton solution is not the unitary value. It has a multivalue, which is determined by the frequencies of the function $\mathrm{cn}((k/6m)\xi,m)\mathrm{dn}((k/6m)\xi,m)$. Figure 1 plots the evolutions of the pulse profile $|\psi|^2$ and the refractive index n with $G = 3$, $D = 0.2$, and $c_1 = 2$, $c_2 = 3$ in Eq. (3.10). From Figs. 1(a) and 1(b), we understand that the evolutions of profile $|\psi|^2$ and the refractive index n are the proceeding propagation of multisolitons, in which amplitudes are periodically modulated. These amplitudes of the bright soliotns ($x = 0$) of $|\psi|^2$ and n are modulated by two periods. Not only the amplitudes but also the routes of bright solitons distributed in both sides of $x = 0$ are periodically changed. Figure 1(c) shows the form of the external potential $V(x,z) = (1+\cos(6z))x^2$, respectively.

The second choice of the function ω is $\omega^2(z) = 1$. From Eq. (3.8), we obtain

$$\varphi = \frac{1}{\gamma} = \sqrt{\frac{1+\cos\left(\sqrt{2}z\right)^2}{2}}. \tag{3.14}$$

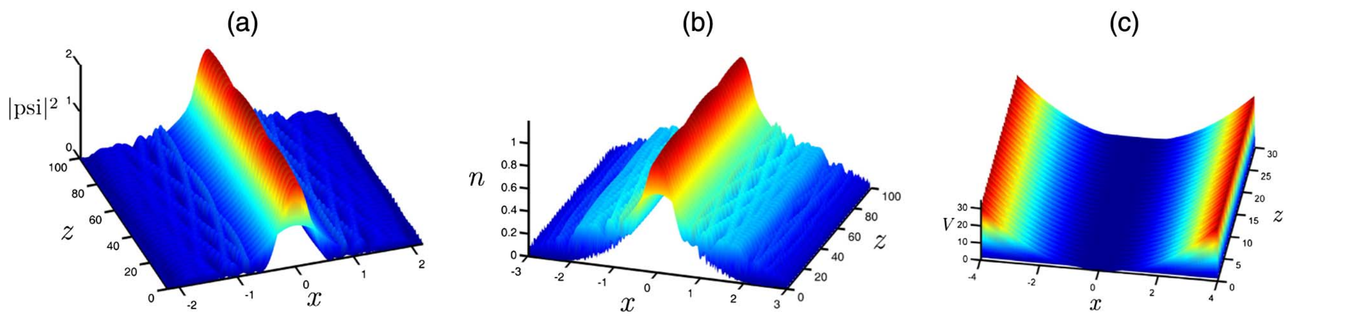

Fig. 1. Propagating behavior of Eq. (3.13) $|\psi|^2$ and n with $G = 3$, $D = 2$, $c_1 = 2$, and $c_2 = 3$: (a) $|\psi|^2$, (b) n, (c) $V(x,z)$.

Taking the same solution in Eq. (3.12) of the system in Eq. (2.3) and substituting Eq. (3.14) into Eq. (3.13), we find these solutions of $|\psi|^2$ and n of Eq. (2.1) are multiperiodically breather solitons. The width and the amplitude of breather bright soliton $|\psi|^2$ at center ($x = 0$) are variable with the same period along the propagating direction z. The solitons solutions are symmetric on both sides of the center ($x = 0$) and move on the curve $\cos(\sqrt{2}z)$. The intensity region of pulse is distributed in the domain $x \in (-1.8, 1.8)$. The solution of the refractive index n is a multidark solitons solution. The distribution of the refractive index n also has the symmetric structure on both sides ($x = 0$) in the domain $x \in (-3.0, 3.0)$ [see Fig. 2(d)]. These amplitudes of the solitons are periodically changed along the propagating orientation z. The propagating trajectories are the curve of $\cos(\sqrt{2}z)$. The corresponding external potential is just a parabolic surface $V(x, z) = x^2$. The nonlocality function $A(x, z)$ is a periodic bright soliton in which amplitude is periodically varied. Figure 2 shows the evolutions of the breather solitons $|\psi|^2$ and n.

B. Case B. $\mu = (1/2D)$

As we know, the ODE system in Eq. (2.3) has different solutions for the different values of μ. Here we also take $\delta(z) = 0$ and $\alpha(z) = -\int \gamma^2(z)/2\mathrm{d}z$ for the simplicity. For this case, we can get a dipole soliton solution of Φ and a bright soliton solution of χ in Eq. (2.3):

$$\Phi(\xi) = \frac{3}{\sqrt{-GD}}\,\mathrm{sech}\left(\frac{\xi}{\sqrt{-2D}}\right)\tanh\left(\frac{\xi}{\sqrt{-2D}}\right), \tag{3.15}$$

$$\chi(\xi) = -\frac{3}{GD}\,\mathrm{sech}\left(\frac{\xi}{\sqrt{-2D}}\right)^2. \tag{3.16}$$

Meanwhile, we can decide the function $\gamma(z)$ by the external potential function $V(x, z)$ and finally obtain the solutions of ψ and n in Eq. (2.1). We give two choices of the external potential function.

1. $V(x, z) = (1 + 0.5\cos(2z))x^2 - \gamma^2 e^{2\eta^2}/(4D)$

In this case, we can get the function $\gamma(z)$ from Eq. (3.8):

$$\gamma(z) = \sqrt{\frac{\sqrt{4c_1c_2^2 - c_1^2}\left(C\left(2, -\frac{1}{2}, z\right)S'\left(2, -\frac{1}{2}, z\right) - C'\left(2, -\frac{1}{2}, z\right)S\left(2, -\frac{1}{2}, z\right)\right)}{2c_1C\left(2, -\frac{1}{2}, z\right)S\left(2, -\frac{1}{2}, z\right) + 2c_2S\left(2, -\frac{1}{2}, z\right)^2 + 2c_1c_2C\left(2, -\frac{1}{2}, z\right)^2}}. \tag{3.17}$$

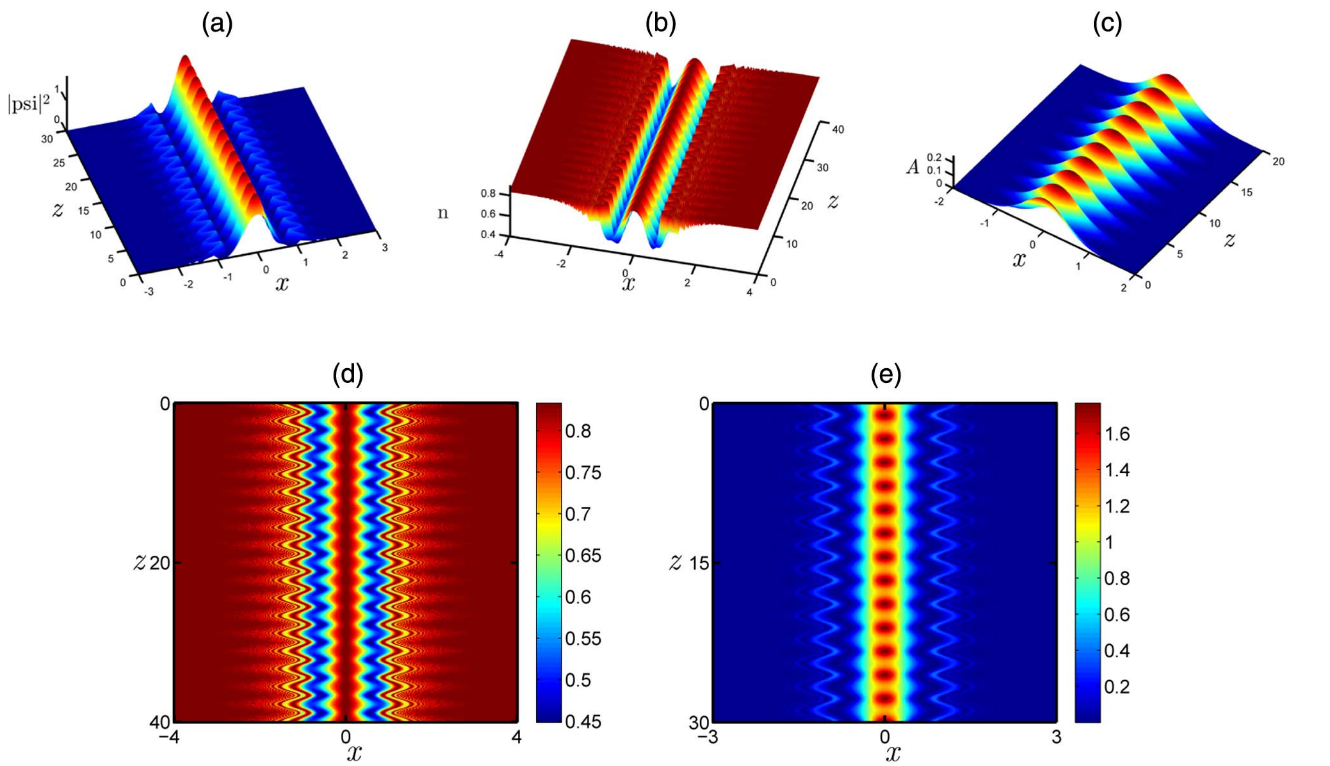

Fig. 2. Propagating behavior of Eq. (3.13) $|\psi|^2$ and n with $G = 3$, $D = 0.2$, $c_1 = 2$, and $c_2 = 3$: (a) $|\psi|^2$, (b) n, (c) $A(x, z)$, and the density distributions of (d) n and (e) $|\psi|^2$.

Taking Eqs. (3.17) and (3.15) into the expressions in Eq. (2.2) of ψ and n, we finally achieve the quasi-periodic soliton solutions of ψ and n. We plot the evolution of intensities of $|\psi|^2$ and n when selecting these parameters $D = -2$, $G = 3$, $c_1 = 2$, $c_2 = 3$ and the external potential function $V(x, z)$ in Fig. 3. From Fig. 3, we know the profile $|\psi|^2$ is a quasi-periodic dipole soliton solution and the refractive index n is a quasi-periodic single bright soliton solution. Their amplitudes and width (Δx_0) are changed periodically with some quasi-periods along the propagating direction z [see Fig. 3(d)].

2. $V(x, z) = (\gamma^6 + \gamma\gamma_{zz} - 2\gamma_z^2)x^2/(2\gamma^2) - \gamma^2 e^{2\eta^2}/(4D)$

We take $\gamma = (1 + 0.15\sin(z))^2$ and get the dipole-type breather soliton of $|\psi|$ and the bright-type breather soliton of n. Their amplitudes are changed with a single period along the propagating direction z. Furthermore, the width of the laser beam and the one of the refractive index n take on a periodic variation along the propagating direction z. Figure 4 shows the evolutions of the dipole-type breather soliton solution of ψ and the breather bright-type soliton solution of n with $G = 3$ and $D = -2$. In order to investigate the stability of the exact solitary wave solutions, we take the exact solution with white noise as the initial perturbed solution of Eq. (2.1):

$$\psi = \frac{\sqrt{6}}{2} e^{(-0.5+0.15i)x^2} \operatorname{sech}\left(\frac{\xi}{2}\right) \tanh\left(\frac{\xi}{2}\right)[1 + 0.1\,\mathrm{random}(x)]. \tag{3.18}$$

Then we calculate the evolution of the solution ψ by the numerical simulation technique. Finally, we obtain the profile and the propagating path of the laser beam ψ are all almost not change. So the exact analytical solitary solution is stable. Figure 4(c) shows the numerical evolution of the solution $|\psi|^2$ in Eq. (2.2) with the white noise perturbation.

C. Case C. $\mu = -\frac{1}{D}$

In this case, we can derive the bright-type soliton solutions for Φ and χ in Eq. (2.3):

$$\Phi(\xi) = \frac{3}{2\sqrt{GD}}\left(\tanh\left(\frac{\xi}{2\sqrt{D}}\right) - 1\right),$$
$$\chi(\xi) = \frac{3}{2GD}\left(1 - \tanh\left(\frac{\xi}{2\sqrt{D}}\right)^2\right). \tag{3.19}$$

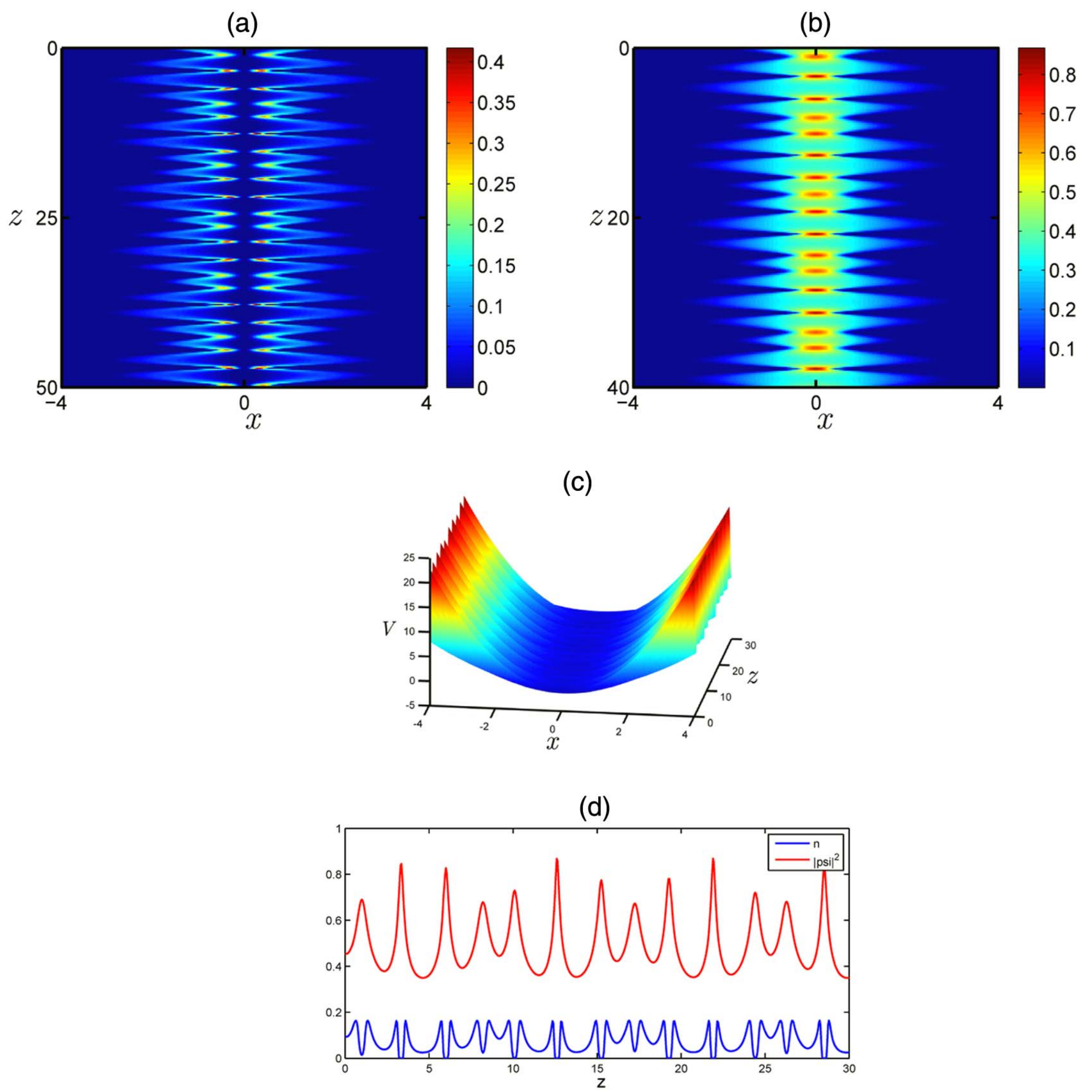

Fig. 3. Intensity $|\psi|^2$ and n with $G = 3$, $D = -2$, $c_1 = 2$, $c_2 = 3$: (a) $|\psi|^2$, (b) n, (c) $V(x, z)$, (d). The evolutions of amplitudes of $|\psi|^2$ at $x = 0.8$ and n at $x = 0$ along the propagating direction.

We investigate the propagating property of the laser beam and the distribution of the refractive index in two different external potentials. First, if we take an external potential function,

$$V(x,z) = ((1+0.2z^2)^4 + 0.4(1-0.6z^2)(1+0.2z^2)^{-2})x^2 + 0.5(1+0.2z^2)^2 e^{2(1+0.2z^2)^2 x^2}), \tag{3.20}$$

we find a fantastic propagating property of the laser beam in the nonlocal media. The laser beam becomes the rogue wave, which has two humps. It means the laser beam can be trapped in the finite domain by manipulating the external potential. It is as if the light was stopped by electromagnetically induced transparency in a doped solid [37]. We hope this theoretical result will be tested by experiment. The distribution of the refractive index is also like the Saddle-type structure in the plane domain ($x \in (-1.4, 1.4)$, $z \in (-20, 20)$) [see Fig. 5(b)]. Figure 5 shows the dynamics of the laser beam, the refractive index, and the external potential function $V(x,z)$.

Second, if we take the external potential function as

$$V(x,z) = 0.5(\mathrm{sech}(0.1z)^4 - 0.01)x^2 + \mathrm{sech}(0.1z)^2 e^{2\mathrm{sech}(0.1z)^2 x^2}), \tag{3.21}$$

we can observe the fissions of the bright solitons of the laser beam and the nonlocality of the refractive index n by manipulating the external potential. We plot the evolutions of the laser beam and the refractive index with the external potential function in Fig. 6.

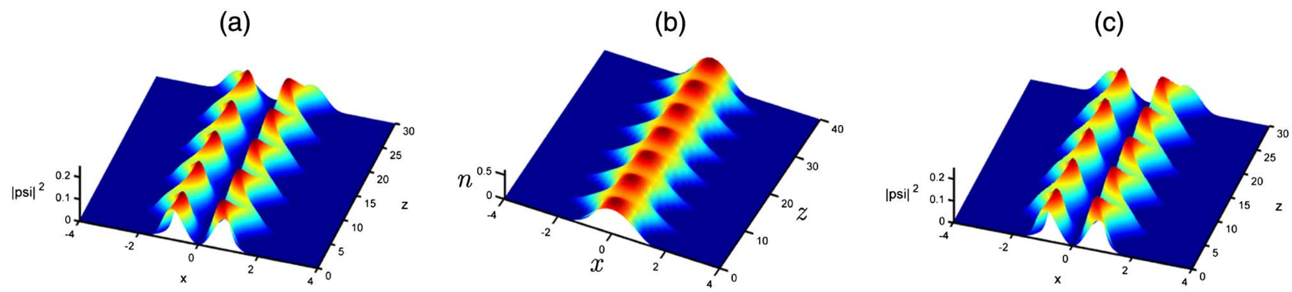

Fig. 4. Propagating behaviors of solutions $|\psi|^2$ and n with Eq. (3.15) and $G = 3$, $D = -2$: (a) $|\psi|^2$, (b) n and the numerical evolution of the exact solution (c) $|\psi|^2$ with the white noise.

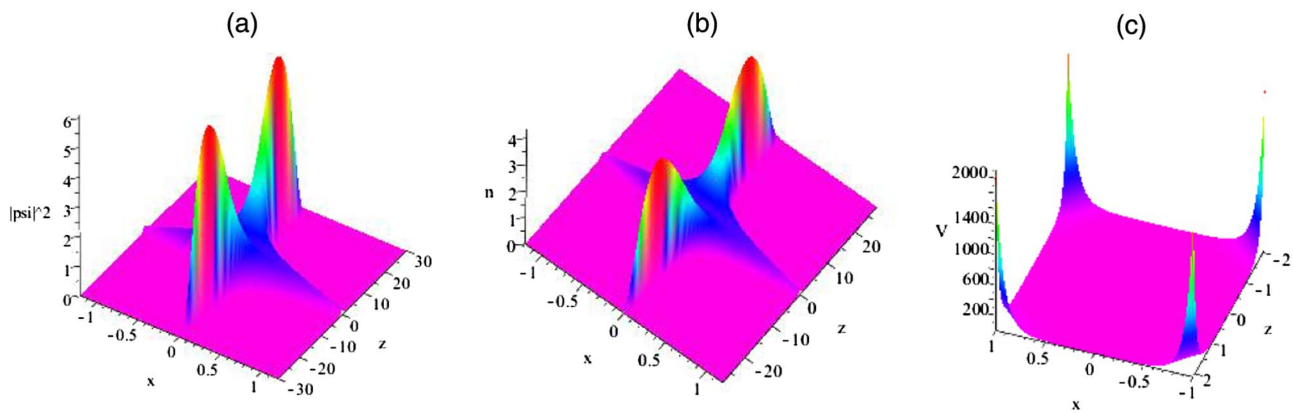

Fig. 5. Dynamics of solution $|\psi|^2$ and n with Eq. (3.19) and $G = 2$, $D = 1$ and the external potential $V(x,z)$: (a) $|\psi|^2$, (b) n, (c) $V(x,z)$.

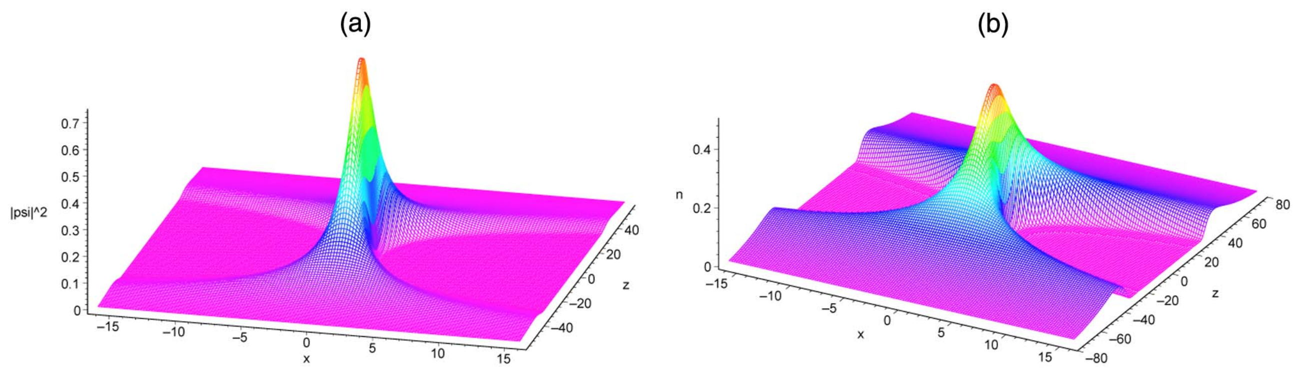

Fig. 6. Propagating behavior of solution $|\psi|$ and n with Eq. (3.19) and $G = 13$, $D = 0.1$: (a) $|\psi|^2$, (b) n.

4. CONCLUSIONS

We first construct various types of exact solutions of the dimensionless nonlocal nonlinear Schrödinger equation with the external potential and the alterable modulation nonlinearity. We use the similarity transformation to reduce Eq. (2.1) to the ordinary differential in Eq. (2.3). For the chemical potential $\mu = 0$, we can obtain periodic and quasi-periodic breather solitons solutions by the periodic-parabolic and parabolic external potential functions, respectively. For $\mu = 1/(2D)$ ($D < 0$), we consider the self-focusing media and must take $D < 0$. We have calculated the quasi-periodic and periodic dipole soliton solutions of the profile of the laser beam and quasi-periodic and periodic single soliton solutions for the refractive index with the external potentials [in Case B]. We hope to use this theoretical result to investigate experimentally the propagating property of the laser beam in SHG by controlling the external potential function. For $\mu = -1/D$ ($D > 0$), we derive the rogue waves and the fission of the soliton wave in Eq. (2.1) by controlling the external potentials. Otherwise, we investigate the stability of the exact solitary wave solution [Case B(2)] with the white noise perturbation numerically. The result reveals the exact analytical solution possesses the stability of propagation. For other cases, we also numerically analyze their evolution stabilities of solitary wave solutions and obtain the same result as Case B(2). In this paper, we only investigate the propagating properties of exact solutions for Eq. (2.1), and we will research for their critical behavior of solutions with a Gaussian-type response function in the future.

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