Almost every complement of a tadpole graph is not chromatically unique*

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Abstract

The study of chromatically unique graphs has been drawing much attention and many results are surveyed in [4, 12, 13]. The notion of adjoint polynomials of graphs was first introduced and applied to the study of the chromaticity of the complements of the graphs by Liu [17] (see also [4]). Two invariants for adjoint equivalent graphs that have been employed successfully to determine chromatic unique graphs were introduced by Liu [17] and Dong et al. [4] respectively. In the paper, we shall utilize, among other things, these two invariants to investigate the chromaticity of the complement of the tadpole graphs $C_n(P_m)$, the graph obtained from a path P_m and a cycle C_n by identifying a pendant vertex of the path with a vertex of the cycle. Let $\overline{G}$ stand for the complement of a graph G. We prove the following results:

1. The graph $\overline{C_{n-1}(P_2)}$ is chromatically unique if and only if $n \neq 5, 7$.
2. Almost every $\overline{C_n(P_m)}$ is not chromatically unique, where $n \geq 4$ and $m \geq 2$.

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1 Introduction

All graphs considered here are finite and simple. Undefined notation and terminology will conform to that in [2]. For a graph G, let $\overline{G}$, $V(G)$, $E(G)$,

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$\chi(G)$, $P(G,\lambda)$ and $\sigma(G,x)$ be, respectively, the complement, vertex set, edge set, chromatic number, chromatic polynomial and σ-polynomial of G.

A partition $\{A_1, A_2, \cdots, A_k\}$ of $V(G)$, where k is a positive integer, is called a *k-independent partition* of a graph G if each A_i is a nonempty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions of G. Then

$$P(G,\lambda) = \sum_{k=1}^{p} \alpha(G,k)(\lambda)_k \quad \text{and} \quad \sigma(G,x) = \sum_{k=\chi(G)}^{p} \alpha(G,k)x^{k-\chi(G)},$$

where $|V(G)| = p$, $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$ (see [14, 21]).

Two graphs G and H are said to be *chromatically equivalent* (or simply *χ-equivalent*), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph G is *chromatically unique* or in short *χ-unique* if $H \cong G$ whenever $G \sim H$. The questions on chromatic equivalence and uniqueness are said to be the chromaticity problem of graphs. See [21] and [4, 12, 13, 24] for details on chromatic polynomials and the chromaticity of graphs respectively.

Let G be a graph of order p. An *ideal subgraph* of a graph G is a spanning subgraph of G whose components are all complete graphs. Let $N(G,k)$ denote the number of ideal subgraphs with k components. Note that $N(G,k) = \alpha(\overline{G},k)$, where k is a positive integer. The *adjoint polynomial* of a graph G is defined as follows [4, 17, 22]:

$$h(G,x) = \sum_{k=1}^{p} N(G,k)x^k.$$

Two graphs G and H are said to be *adjointly equivalent*, denoted by $G \overset{h}{\sim} H$, if $h(G,x) = h(H,x)$. A graph G is said to be *adjointly unique* if $H \cong G$ whenever $H \overset{h}{\sim} G$. The following two results follow directly from the definitions of $P(G,\lambda)$, $h(G,x)$, and $\sigma(G,x)$.

Theorem 1.1. **([4, 17])**

(1) *$G \overset{h}{\sim} H$ if and only if $\overline{G} \sim \overline{H}$.*

(2) *G is χ-unique if and only if $\overline{G}$ is adjointly unique.*

Theorem 1.2. **([6, 17])** $h(G,x) = x^{\chi(\overline{G})}\sigma(\overline{G},x)$.

In what follows we will write $h_1(G,x) = \sigma(\overline{G},x)$.

Remark 1.1. E.J. Farrell [9, 10] studied the relations between chromatic, adjoint, clique, matching, σ-polynomials and uniquely colourable graphs. For details we refer the readers to see his papers.

Definition 1.1. The *adjoint roots* (simply adj-roots) of a graph G are the roots of its adjoint polynomial.

Let G be a graph of order $p(G) = p$ and size $q(G) = q$. For convenience, we simply denote $h(G, x)$ by $h(G)$ and $h_1(G, x)$ by $h_1(G)$. By $\beta(G)$ we denote the smallest real adj-root of $h(G)$. For each $v \in G$, let $d_G(v)$, or simply $d(v)$, be the degree of v in G. For two graphs G and H, $G \bigcup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies of H. By K_n and $K_{1,n-1}$ respectively, we denote the complete graph and star with order n. Let $n_G(K_3)$ and $n_G(K_4)$ denote the number of subgraphs in G isomorphic to K_3 and K_4, respectively. Let $g(x)|f(x)$(resp. $g(x) \nmid f(x)$) denote $g(x)$ divides $f(x)$(resp. $g(x)$ does not divide $f(x)$) on the rational field and $\partial(f(x))$ denotes the degree of $f(x)$.

The notion of χ-unique graphs was first introduced by Chao et al. [3]. By (2) of Theorem 2.1, we know that the goal of searching for the χ-unique graphs can be realized by looking for adjointly unique graphs. In order to search for them, it is very helpful to find as many as possible necessary conditions for two graphs to be adjointly equivalent. A quantity $\zeta(G)$ is called an invariant for adjointly equivalent graphs (or adj-invariant in short) if $h(G, x) = h(H, x)$ implies the $\zeta(G) = \zeta(H)$, where G and H are graphs [4].

Some researchers, such as Du [8] and Li and Whitehead [15], have used σ-polynomials to study the chromaticity of some dense graphs, but one disadvantage is that $\sigma(G, x)$ does not determine the order of G. This can be seen from the fact that $\sigma(\overline{G}, x) = \sigma(\overline{G \bigcup mK_1})$ for any integer $m \geq 1$. The adjoint polynomial does not have this fault, and it contains all the information that the σ-polynomial has. Hence in this paper we shall use adjoint polynomials rather than σ-polynomials.

Now we define some classes of graphs which will be used throughout the paper.

(1) C_p (resp. P_p) denotes the cycle (resp. the path) of order p, and we write $\mathcal{C} = \{C_p|p\geq3\}$, $\mathcal{P} = \{P_p|p\geq2\}$ and $\mathcal{U} = \{U_{1,1,t,1,1}|t\geq1\}$.

(2) $D_p(p\geq4)$ denotes the graph obtained from C_3 and P_{p-2} by identifying a vertex C_3 with a pendant vertex of P_{p-2}.

(3) T_{l_1,l_2,l_3} is the tree with a vertex v of degree 3 such that $T_{l_1,l_2,l_3} - v = P_{l_1}\bigcup P_{l_2}\bigcup P_{l_3}$ and $l_3 \geq l_2 \geq l_1$, write $\mathcal{T}_1 = \{T_{1,1,l_3}|l_3\geq1)$ and $\mathcal{T} = \{T_{l_1,l_2,l_3}|(l_1, l_2, l_3)\neq(1, 1, 1)\}$.

(4) $\vartheta = \{C_p, D_p, K_1, T_{l_1,l_2,l_3}|n \geq 4,\ l_3 \geq l_2 \geq l_1 \geq 1\}$.

(5) $\xi = \{C_r(P_s), Q_{r,s}, B_{r,s,t}, F_p, U_{r,s,t,a,b}, K_4^-\}$.

(6) $\psi = \{\psi_p^1, \psi_p^2, \psi_p^3(r, s), \psi_p^4(r, s), \psi_p^5(r, s, t), \psi_5^6\}$.

Some of the graphs with orders n used in the paper are shown in Table 1.

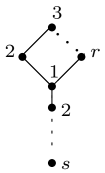
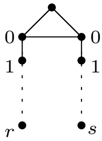
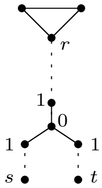
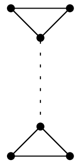
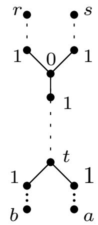
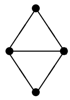
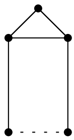
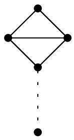
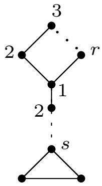
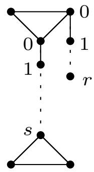
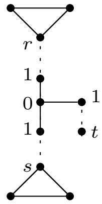
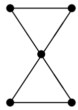
ξ						
	$C_r(P_s)$	$Q_{r,s}$	$B_{r,s,t}$	F_p	$U_{r,s,t,a,b}$	K_4^-
	$r \geq 4, s \geq 2$	$r, s \geq 1$	$r, s, t \geq 1$	$p \geq 6$	$r, s, t, a, b \geq 1$	$p = 4$
ψ						
	ψ_p^1	ψ_p^2	$\psi_p^3(r,s)$	$\psi_p^4(r,s)$	$\psi_p^5(r,s,t)$	ψ_5^6
	$p \geq 5$	$p \geq 5$	$r \geq 4, s \geq 2$	$r, s \geq 1$	$r, s, t \geq 1$	$p = 5$

Table 1

The organization of the paper is the following. In Section 2 we introduce some basic lemmas and results of adjoint polynomials, such as the first character and the second character of a graph, the smallest real adj-roots and the divisibility of adjoint polynomials and so on. In Section 3, by making use of these algebraic properties of adjoint polynomials, we research into the chromaticity of the complement of the tadpole $C_n(P_m)$, the graph obtained from a path P_m and a cycle C_n by identifying a pendant vertex of the path P_m with a vertex of the cycle C_n, where $n \geq 4$ and $m \geq 2$. Finally in Section 4 we give a conjecture for the chromaticity of $\overline{C_n(P_{n-1})}$.

2 Some Algebraic Properties of Adjoint Polynomials

For an edge $e = v_1v_2$ of a graph G, the graph $G{*}e$ is defined as follows: the vertex set of $G{*}e$ is $(V(G) - \{v_1, v_2\})\bigcup\{v\}$, where $v \notin V(G)$, and the edge set of $G{*}e$ is $\{e^{'} | e^{'} \in E(G)$, $e^{'}$ is not incident with v_1 or $v_2\}\bigcup\{uv | u \in N_G(v_1)\bigcap N_G(v_2)\}$, where $N_G(v)$ is the set of vertices of G which are adjacent to v.

Lemma 2.1 ([4, 17]). *Let G be a graph with $e\in E(G)$. Then*

$$h(G,x) = h(G-e,x) + h(G{*}e,x),$$

In particular, if $e = uv \in E(G)$ is not an edge of any triangle of G, then

$$h(G,x) = h(G-e,x) + xh(G-\{u,v\},x),$$

where $G-e$ and $G-\{u,v\}$ are, respectively, the graphs obtained from G by deleting the edge e and deleting the vertices u,v and their incident edges in G.

Lemma 2.2 ([17]).

(1) *For $n\geq 3$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$.*

(2) *For $n\geq 6$, $h(C_n) = x(h(C_{n-1}) + h(C_{n-2}))$.*

Theorem 2.1.

(1) *For $n\geq 4$, $m\geq 1$, $h(C_n(P_m)) = h(P_{n-1})h(P_m) + 2xh(P_{n-2})h(P_{m-1})$.*

(2) *For $n\geq 4$ and $m\geq 3$, $h(C_n(P_m)) = h(C_{m+1}(P_{n-1}))$.*

(3) *For $n\geq 6$ and $m\geq 2$, $h(C_n(P_m)) = x(h(C_{n-1}(P_m)) + h(C_{n-2}(P_m))$.*

(4) *$h(C_4(P_2)) = h(K_4^- \bigcup K_1)$, $h(C_6(P_2)) = h(B(2,1,1))$, $h(C_4(P_3)) = h(Q_{1,2})$.*

Proof.

(1) Choosing v_1 on the path joining to u_1 such that $d(u_1) = 3$ and $d(v_1) = 2$, we have, by Lemma 2.1, that

$$h(C_n(P_m)) = h(P_{m-1})h(C_n) + xh(P_{m-2})h(P_{n-1}). \tag{2.1}$$

Let $e_2 \in E(C_n)$, it follows, from Lemma 2.1, that

$$h(C_n) = h(P_n) + xh(P_{n-2}). \tag{2.2}$$

Combining (2.1) and (2.2), we obtain, together with (1) of Lemma 2.2, that

$$\begin{aligned}
h(C_n(P_m)) &= h(P_{m-1})h(C_n) + xh(P_{m-2})h(P_{n-1}) \\
&= h(P_{m-1})(h(P_n) + xh(P_{n-2})) + xh(P_{m-2})h(P_{n-1}) \\
&= h(P_{m-1})(xh(P_{n-1}) + xh(P_{n-2})) + xh(P_{m-1})h(P_{n-2}) \\
&\quad +xh(P_{m-2})h(P_{n-1}) \\
&= h(P_{n-1})(xh(P_{m-1}) + xh(P_{m-2})) + 2xh(P_{m-1})h(P_{n-2}) \\
&= h(P_{n-1})h(P_m) + 2xh(P_{n-2})h(P_{m-1}).
\end{aligned}$$

(2) Assertion (2) follows directly from (1).

(3) By using (2.1) and Lemma 2.2, we arrive at

$$\begin{aligned}
h(C_n(P_m)) &= h(P_{m-1})h(C_n) + xh(P_{m-2})h(P_{n-1}) \\
&= h(P_{m-1})(xh(C_{n-1}) + xh(C_{n-2})) + xh(P_{m-2}))(xh(P_{n-2}) \\
&\quad +xh(P_{n-3})) \\
&= x(h(P_{m-1})h(C_{n-1}) + h(P_{m-2})h(P_{n-2})) \\
&\quad +x(h(P_{m-1})h(C_{n-2}) + h(P_{m-2})h(P_{n-3})) \\
&= x(h(C_{n-1}(P_m)) + h(C_{n-2}(P_m)))
\end{aligned}$$

(4) Part (4) can be similarly proved. □

Lemma 2.3 ([4, 17]). *Let G be a graph with p vertices and q edges. Let M denote the set of vertices of the triangles in G and $M(i)$ denote the number of triangles which cover the vertex i in G. If the degree sequence of G is $(d_1, d_2, \cdots, d_n)$, then*

(**1**) $N(G,p) = 1,\ N(G,p-1) = q$;

(**2**) $N(G,p-2) = \binom{q+1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2 + n_G(K_3)$;

(**3**) $N(G,p-3) = \frac{q}{6}(q^2+3q+4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 + \sum_{ij\in E(G)} d_i d_j - \sum_{i\in M} M(i)d_i + (q+2)n_G(K_3) + n_G(K_4)$.

We next define two invariants for adjoint equivalent graphs.

Definition 2.1([5, 18]). Let G be a graph with p vertices and q edges. Let $b_i(G) = N(G, p-i)$ for $i = 1, 2, 3$.

The *first character* of G is defined as

$$R_1(G) = \begin{cases} 0 & \text{if } q = 0; \\ b_2(G) - \binom{b_1(G)-1}{2} + 1 & \text{if } q > 0. \end{cases}$$

The *second character* of G is defined as

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G)-2)\left(b_2(G) - \binom{b_1(G)}{2}\right) - b_1(G),$$

Remark 2.1. It deserves to be pointed out that the parameter $\pi(G) = N(G, n-2) - (m^2 - 3m)/2$, defined by Du [8] independently, is in fact the same as $R_1(G)$. Very good work was done by Du [7] and Mao [19], respectively, who used a recursive method to construct graphs with $R_1(G) = i$ for $i \leq 1$ and $R_2(G) = j$ for $j \geq -2$.

Lemma 2.4 ([5, 18]). *Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then $h(G) = \prod_{i=1}^{k} h(G_i)$ and $R_j(G) = \sum_{i=1}^{k} R_j(G_i)$ for $j = 1, 2$.*

By the definition of adjoint polynomial, Theorem 1.2 and Lemma 2.3, we state some adj-invariants in the following theorem.

Theorem 2.2. *Let G and H be graphs such that $H \overset{h}{\sim} G$. Then*

(1) $N(G,k) = N(H,k)$, *where k is a non-negative integer.*

(2) $|V(G)| = |V(H)|$ *and* $|E(G)| = |E(H)|$.

(3) $R_i(G) = R_i(H)$ *for* $i = 1, 2$

(4) $\beta(G) = \beta(H)$

(5) $\chi(\overline{G}) = \chi(\overline{H})$. $\square$

Remark 2.2. The first four conclusions also hold for $h_1(G) = h_1(H)$ (see Theorem 1.2).

Lemma 2.5([7, 17]). *Let G be a nontrivial connected graph with p vertices and q edges.*

(1) $R_1(G) \le 1$, *and the equality holds iff* $G \cong P_p$ $(p \ge 2)$ *or* $G \cong K_3$.

(2) $R_1(G) = 0$ *iff* $G \in \vartheta$.

(3) $R_1(G) = -1$ *iff* $G \in \xi$. *In particular,* $R_1(G) = -1$ *and* $G \in \{F_p | n \ge 6\} \bigcup \{K_4^-\}$ *iff* $q = p + 1$.

(4) $R_1(G) = -2$ *iff* $G \in \psi$ *for* $q = p + 1$ *and* $G \cong K_4^-$ *for* $q = p + 2$.

Lemma 2.6 ([11]). *For $k \ge 0$, let $G^{(-k)}$ denote the union of the components of G, whose first character is $-k$, and s_k denote the number of components of $G^{(-k)}$.*

(**1**) *If $k \in \{0, 1, 2\}$, then $q(G^{(-k)}) - p(G^{(-k)}) \le ks_k$ with equality holding iff $q(X) - p(X) = k$ for each component X of $G^{(-k)}$.*

(**2**) *If $k = 3$, then $q(G^{(-k)}) - p(G^{(-k)}) \le 2s_3$ with equality holding iff $q(X) - p(X) = 2$ for each component $G^{(-3)}$.*

Some alternative formulas for computing $R_2(G)$ are given by Dong et al. [4], we prefer the one below.

Lemma 2.7 ([4]). *Let G be a graph with p vertices and q edges. Then*

$$R_2(G) = \frac{4q}{3} - 2\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 + \sum_{ij \in E(G)} d_i d_j + \sum_{i \in M} M(i) d_i + 4n_G(K_3) + n_G(K_4),$$

where the notation has the same meaning as in Lemma 2.3.

Lemma 2.8 ([19]). *If $G \in \eta \bigcup \{P_2\}$, then $-1 \le R_2(G) \le 2$.*

In particular,

(1) $R_2(G) = -1$ *iff* $G \in \{T_{1,1,1}, P_2\}$.

(2) $R_2(G) = 0$ *iff* $G \in \{C_n, D_4, K_1, T_{1,1,l_3} | n \ge 4, l_3 \ge 2\}$.

(3) $R_2(G) = 1$ *iff* $G \in \{T_{1,l_2,l_3} | l_3 \ge l_2 \ge 2\} \bigcup \{D_n | n \ge 5\}$.

(4) $R_2(G) = 2$ *iff* $G \in \{T_{l_1,l_2,l_3} | l_3 \ge l_2 \ge l_1 \ge 2\}$.

By Lemma 2.4 and straightforward calculation, in the next lemma we classify the graphs in ξ by the second character and give the lower and upper bounds for $R_2(G)$, where $G \in \psi \bigcup \{K_4\}$.

Lemma 2.9. *Let G be a graph such that $G \in \psi \bigcup \{K_4\}$, then $7 \leq R_2(G) \leq 10$; if $G \in \xi - \{U_{r,s,t,a,b}\}$, then $3 \leq R_2(G) \leq 6$. In particular,*

(1) $R_2(G) = 3$ *iff* $G \in \{K_4^-, B_{r,1,1}, Q_{1,1} | r \geq 2\} \bigcup \{C_r(P_2) | r \geq 4\}$.

(2) $R_2(G) = 4$ *iff*
$G \in \{C_r(P_s), F_n | r \geq 4, s \geq 3, n \geq 7\} \bigcup \{B_{1,1,1}, B_{r,1,t}, Q_{1,s} | r, s, t \geq 2\}$.

(3) $R_2(G) = 5$ *iff* $G \in \{B_{1,1,t}, B_{r,s,t}, Q_{r,s}, F_6 | r, s, t \geq 2\}$.

(4) $R_2(G) = 6$ *iff* $G \in \{B_{1,s,t} | s, t \geq 2\}$. $\square$

Now, we discuss the smallest real adj-roots of adjoint polynomials, which play an important role in studying the chromaticity of graphs.

Lemma 2.10 ([23]). *Let $f_1(x), f_2(x)$ and $f_3(x)$ be polynomials with real positive coefficients. If $f_3(x) = f_2(x) + f_1(x)$, $\partial f_3(x) - \partial f_1(x) \equiv 1 (mod\, 2)$ and $\beta_2 < \beta_1$, then $f_3(x)$ has at least one real root such that $\beta_3 < \beta_2$, where β_i denotes the smallest real root of $f_i(x)$ $(i = 1, 2, 3)$.*

Lemma 2.11 ([24]).

(1) *For $n \geq 8$ and $r \geq 6$,*

$$\begin{aligned}&\beta(C_{n-1}(P_2)) < \beta(C_6(P_2)) = \beta(B(2,1,1)) < \beta(B(3,1,1))\\ &< \beta(B(4,1,1)) < \beta(B(5,1,1)) = \beta(C_5(P_2)) < \beta(B(r,1,1)).\end{aligned}$$

(2) *For $n \geq 10, 12 \leq r \leq 16$ and $m \geq 18$,*

$$\begin{aligned}&\beta(F_6) < \beta(F_7) < \beta(F_8) < \beta(C_{n-1}(P_2)) < \beta(C_8(P_2))\\ &= \beta(F_9) < \beta(C_7(P_2)) < \beta(F_{10}) < \beta(F_{11})\\ &= \beta(C_6(P_2)) < \beta(F_r) < \beta(C_5(P_2)) = \beta(F_{17}) < \beta(F_m) < \beta(C_4(P_2)).\end{aligned}$$

(3) *For $r \geq 2, n \geq 6$ and $m \geq 4$,*

$$\begin{aligned}&\beta(F_n) < \beta(F_{n+1}) < \beta(D_m),\\ &\beta(B(r-1,1,1)) < \beta(B(r,1,1)) < \beta(D_m).\end{aligned}$$

Note, the coefficients of an adjoint polynomial of a graph G are positive and the constant term is zero, so all real adj-roots of G are non-positive. The following lemma characterizes the graphs whose smallest real adj-root is in the interval [-4,0].

Lemma 2.12 ([24]). *Let G be a connected graph. Then*

(1) $\beta(G) = -4$ *iff*

$$G \in \{T_{1,2,5}, T_{2,2,2}, T_{1,3,3}, K_{1,4}, C_4(P_2), Q_{1,1}, K_4^-, D_8\} \bigcup \mathcal{U}.$$

(2) $-4 < \beta(G) \leq 0$ *iff*

$$G \in \{K_1, T_{1,2,i}(2 \leq i \leq 4), D_i(4 \leq i \leq 7)\} \bigcup \mathcal{P} \bigcup \mathcal{C} \bigcup \mathcal{T}_1.$$

(3) *If H is a proper subgraph of G, then $\beta(G) < \beta(H)$.*

Theorem 2.3.

(1) *For $n \geq 3$ and $m \geq 2$, $\beta(C_n(P_m)) < \beta(C_n(P_{m-1}))$.*

(2) *For $n \geq 5$ and $m \geq 4$, $\beta(C_{n-1}(P_2) \leq \beta(D_m)$ with equality holding if and only if $n = 5$ and $m = 4$.*

(3) *For $m \geq 2$ and $n \geq 5$, $\beta(C_n(P_m)) < \beta(C_{n-1}(P_m))$.*

(4) *For $n \geq 5$, $\beta(C_{n-1}(P_2)) \leq \beta(Q_{1,1})$ with equality holding if and only if $n = 5$.*

Proof.

(1) It is evident that $C_n(P_{m-1})$ is a proper subgraph of $C_n(P_m)$, which results in $\beta(C_n(P_m)) < \beta(C_n(P_{m-1}))$ by (3) of Lemma 2.11.

(2) In view of (2) and (3) of Lemma 2.11, together with (1) of Lemma 2.12, the result obviously holds.

(3) We prove $\beta(C_n(P_m)) < \beta(C_{n-1}(P_m))$ by induction on $n + m$. From the condition of (3), it follows that $min\{n + m\} = 7$, which leads to $m = 2$ and $n = 5$. By software *Mathematica* we obtain that $\beta(C_5(P_2)) = -4.11494 < \beta(C_4(P_2)) = -4$. Suppose that the result holds for k when $k < n + m$. Let $k = n + m$, by (3) of Theorem 2.1, we have that

$$h(C_n(P_m)) = x(h(C_{n-1}(P_m)) + h(C_{n-2}(P_m))).$$

By the induction hypothesis, we get $\beta(C_{n-1}(P_m)) < \beta(C_{n-2}(P_m))$. From Lemma 2.10, it follows that $\beta(C_n(P_m)) < \beta(C_{n-1}(P_m))$.

(4) From (3) of the theorem and (1) of Lemma 2.12, we have that $\beta(C_{n-1}(P_2)) < \beta(C_4(P_2)) = \beta(Q_{1,1}) = -4$ for $n \geq 6$, which illustrates that the result holds. $\square$

We conclude this section by establishing some results concerning the divisibility of adjoint polynomials, which are helpful in the study of the chromaticity of graphs.

Lemma 2.13 ([24]).

(1) *Let $\{g_i(x)\}$ be a polynomial sequence with integer coefficients and $g_n(x) = x(g_n(x) + g_{n-1}(x))$. Then*

$$g_n(x) = h(P_k)g_{n-k}(x) + xh(P_{k-1})g_{n-k-1}(x).$$

(2) *For $m \geq 2$ and $n \geq 6$, $h(P_m)|h(C_{n-1}(P_2))$ if and only if $m = 2$ and $n = 3k + 2$, where $k \geq 1$.*

Theorem 2.4. *For $n \geq 5$, $h^2(P_2) \nmid h(C_{n-1}(P_2))$.*

Proof. For $n \geq 7$, according to (3) of Theorem 2.1, we arrive at

$$h(C_{n-1}(P_2)) = x(h(C_{n-2}(P_2) + h(C_{n-3}(P_2))). \tag{2.3}$$

Let $g_n(x) = h(C_{n-1}(P_2))$ which implies, from (2.3), that

$$g_n(x) = x(g_{n-1}(x) + g_{n-2}(x)). \tag{2.4}$$

Noting (2) of Lemma 2.13, we obtain that $h(P_m)|g_n(x)$ if and only if $m = 2$ and $n = 3k + 2$, where $k \geq 1$. Suppose that $h^2(P_2)|h(C_{n-1}(P_2))$, that is, $h^2(P_2)|g_n(x)$. It follows, from (2.4) and (1) of Lemma 2.13, that

$$\begin{aligned}
g_n(x) &= h(P_2)g_{n-2}(x) + x^2 g_{n-3}(x) \\
&= h^2(P_2)g_{n-4}(x) + 2x^2 h(P_2)g_{n-5}(x) + x^4 g_{n-6}(x) \\
&= h^2(P_2)(g_{n-4}(x) + 2x^2 g_{n-7}(x)) + 3x^4 h(P_2)g_{n-8}(x) + x^6 g_{n-9}(x) \\
&= h^2(P_2)(g_{n-4}(x) + 2x^2 g_{n-7}(x) + 3x^4 g_{n-10}(x)) \\
&\quad +4x^6 h(P_2)g_{n-11}(x) + x^8 g_{n-12}(x) \\
&\cdots \\
&= h^2(P_2)\sum_{s=1}^{k-2} g_{n-3s-1}(x) + (k-1)x^{2k-4}h(P_2)g_{n+1-3(k-1)}(x) \\
&\quad +x^{2k-2}g_{n-3(k-1)}(x),
\end{aligned}$$

which, together with the assumption and $n = 3k + 2$, results in

$$h^2(P_2)|((k-1)x^{2k-4}h(P_2)g_6(x) + x^{2k-2}g_5(x)). \tag{2.5}$$

By Lemma 2.1 and calculation, we arrive at

$$\begin{aligned}
g_5(x) &= h(C_4(P_2)) = x^5 + 5x^4 + 4x^3, \\
g_6(x) &= h(C_5(P_2)) = x^6 + 6x^5 + 8x^4 + x^3.
\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we have, from $h(P_2) = x^2 + x$, that

$$(x^2 + x)|((k-1)x^{2k+2} + (6k-5)x^{2k+1} + (8k-4)x^{2k} + (k-1)x^{2k-1})$$

By direct calculation, it follows that $k = -\frac{1}{2}$ which is in contradiction with $k \geq 1$. $\square$

3 The chromaticity of $\overline{C_n(P_m)}$

By means of the algebraic properties of adjoint polynomials in Section 2, we now investigate the chromaticity of the complement of $C_n(P_m)$.

Theorem 3.1. *Let $G = \bigcup_{i=1} G_i$ be a graph such that $G \overset{h}{\sim} C_{n-1}(P_2)$, where G_i are the components of G. Then G contains at most one component isomorphic to P_2.*

Proof. By $h(G) = h(C_{n-1}(P_2))$ and (2) of Lemma 2.13, we obtain that $h(P_m)|h(G)$ iff $m = 2$. So the other paths P_k ($k \neq 2$) are not components of G. Or else, $h(P_k)|h(G)$, that is, $h(P_k)|h(C_{n-1}(P_2))$ which contradicts to (2) of Lemma 2.13, where $k \neq 2$. It follows, from $h(P_4)\nmid h(G)$ and $h(K_3 \bigcup K_1) = h(P_4)$, that G also contains no K_3 as its component. In view of Theorem 2.4, we arrive at $h^2(P_2)\nmid h(G)$, which implies, from (1) of Lemma 2.5, that G contains at most one component P_2. $\square$

Theorem 3.2. *For $n \geq 5$, $C_{n-1}(P_2)$ is adjointly unique if and only if $n \neq 5, 7$.*

Proof. The necessity of the theorem is proved by (4) of Theorem 2.1. Now we show the sufficiency of the theorem. Let H be any graph such that $h(H) = h(C_{n-1}(P_2))$ and $H = \bigcup_{i=1}^{l} H_i$, where H_i are the components of H. From $R_1(K_3) = 1$ and Theorem 3.1 we know that H contains no K_3 as its component. Let s_i denote the number of the component H_i with $R(H_i) = -i$. By Theorem 3.1 we obtain that

$$0 \leq s_{-1} \leq 1, \tag{3.1}$$

which leads to $-2 \leq R_1(H_i) \leq 1$ for $1 \leq i \leq l$. In terms of (3) of Lemma 2.5 and (1) of Lemma 2.9, we have that $R_1(C_{n-1}(P_2)) = -1$ and $R_2(C_{n-1}(P_2)) = 3$, which, together with Lemma 2.4 and Theorem 2.2, leads to $R_1(H) = \sum_{i=-1}^{2} s_i = -1$, $R_2(H) = 2$ and $p(H) = q(H)$ implying that

$$\begin{aligned} s_{-1} &= s_1 + 2s_2 - 1 \\ \sum_{-2 \leq R_1(H_i) \leq 0} (q(H_i) - p(H_i)) &= s_{-1}. \end{aligned} \tag{3.2}$$

According to Lemma 2.6, we have the following inequalities:

$$\begin{aligned} \sum_{R_1(H_i)=-1} (q(H_i) - p(H_i)) &\leq s_1 \\ \sum_{R_1(H_i)=-2} (q(H_i) - p(H_i)) &\leq s_2. \end{aligned} \tag{3.3}$$

In view of (3.2) and (3.3), it is not difficult to obtain that

$$\begin{aligned} s_1 - 1 \leq \sum_{R_1(H_i)=-1} (q(H_i) - p(H_i)) &\leq s_1 \\ 2s_2 - 1 \leq \sum_{R_1(H_i)=-2} (q(H_i) - p(H_i)) &\leq 2s_2 \end{aligned} \tag{3.4}$$

We distinguish the following two cases by (3.1):

Case 1. $s_{-1} = 0$.

It follows, from (3.1) and (3.2), that $s_1 = 1, s_2 = 0$ and $0 \leq q(H_1) - p(H_1) \leq 1$ with $R_1(H_1) = -1$.

Subcase 1.1. $q(H_1) = p(H_1) + 1$.

By (3) of Lemma 2.5, we arrive at

$$H_1 \in \{F_t, K_4^- | t \geq 6\}. \tag{3.5}$$

Without loss of generality, let

$$H = H_1 \bigcup rK_1 \bigcup (\bigcup_{i\in A_1} C_i) \bigcup kD_4 \bigcup (\bigcup_{j\in B_1} D_j) \bigcup mT_{1,1,1} \bigcup (\bigcup_{T\in\mathcal{T}} T_{l_1,l_2,l_3}), \tag{3.6}$$

where $A_1 = \{i | i \geq 4\}, B_1 = \{j | j \geq 5\}$ and $r, f, m \geq 0$. From (3.6) it follows that $q(H) = p(H) - r - m - |\mathcal{T}| + 1$, which, together with $q(H) = p(H)$, leads to

$$r + m + |\mathcal{T}| = 1 \text{ and } 0 \leq m \leq 1. \tag{3.7}$$

It follows, from (3.6), Lemma 2.4 and Lemma 2.8, that

$$R_2(H) = 3 \geq R_2(H_1) + |B_1| - m. \tag{3.8}$$

In view of (3.5), the following subcases are discussed.

Subcase 1.1.1. $H_1 \cong F_t$.

If $t = 6$, we have, by (3) of Lemma 2.9, that $R_2(H_1) = R_2(F_6) = 5$. It follows from (3.7) and (3.8) that $|B_1| \leq -1$, which is a contradiction.

If $t \geq 7$, then $R_2(H_1) = R_2(F_t) = 4$. By (3.8) we get that $R_2(H) = 3$ iff $|B_1| = 0$ and $m = 1$, which results in $r = |\mathcal{T}| = 0$. Thus $H = F_t \bigcup (\bigcup_{i\in A_1} C_i) \bigcup kD_4 \bigcup T(1,1,1)$. In view of Lemma 2.12 and (4) of Theorem 2.2, we arrive at $\beta(F_t) = \beta(H) = \beta(C_{n-1}(P_2))$ which, together with (3) of Lemma 2.11, leads to $(t, n) \in \{(9,9), (11,7), (6,17)\}$ that contradicts to $|V(H)| = |V(C_{n-1}(P_2))|$.

Subcase 1.1.2. $H_1 \cong K_4^-$.

From (3.8) and $R_2(K_4^-) = 3$, we have $R_2(H) = 3$ if and only if $|B_1| = m$. We distinguish the following subcases by (3.7):

Subcase 1.1.2.1. $m = |B_1| = 1$.

By (3.7) again we obtain that $r = |\mathcal{T}| = 0$.

So $H = K_4^- \bigcup (\bigcup_{i\in A_1} C_i) \bigcup D_j \bigcup kD_4$.

If $4 < j \leq 8$, we have, from Lemma 2.12, that $\beta(H) = \beta(K_4^-) = -4 > \beta(C_{n-1}(P_2))$, which contradicts to $\beta(H) = \beta(C_{n-1}(P_2))$.

If $j \geq 9$, we obtain, by Lemma 2.12, that $\beta(H) = \beta(D_j)$. From (2) of Theorem 2.3, it follows that $\beta(D_j) > \beta(C_{n-1}(P_2))$ which also contradicts to $\beta(H) = \beta(C_{n-1}(P_2))$.

Subcase 1.1.2.2. $m = |B_1| = 0$.

According to (3.7), we arrive at $r = 1, |\mathcal{T}| = 0$ or $r = 0, |\mathcal{T}| = 1$.

If $r = 1, |\mathcal{T}| = 0$, then $H = K_4^- \bigcup (\bigcup_{i\in A_1} C_i) \bigcup kD_4 \bigcup K_1$. So we have, from Lemma 2.12, that $\beta(H) = \beta(K_4^-) > \beta(C_{n-1}(P_2))$ which contradicts to $\beta(H) = \beta(C_{n-1}(P_2))$.

If $r=0, |\mathcal{T}|=1$, then $H=K_4^-\bigcup(\bigcup_{i\in A_1}C_i)\bigcup kD_4\bigcup T_{l_1,l_2,l_3}$. Recalling that $R_2(H)=3=R_2(K_4^-)+\sum_{i\in A_1}R_2(C_i)+fR_2(D_4)+R_2(T_{l_1,l_2,l_3})$, we obtain, from (1) of Lemmas 2.8 and 2.9, that $R_2(T_{l_1,l_2,l_3})=0$ which implies that $l_1=l_2=1$ and $l_3\geq 2$. Note that from Lemma 2.12, it follows $\beta(H)=\beta(K_4^-)=-4>\beta(C_{n-1}(P_2))$ that contradicts to $\beta(H)=\beta(C_{n-1}(P_2))$.

Subcase 1.2. $q(H_1)=p(H_1)$.

By Lemma 2.5, we arrive at $H_1\in\{B_{m_1,m_2,m_3}, C_{m_1}(P_{m_2}), Q_{m_1,m_2}\}$. In terms of (3.6) and $p(H)=q(H)$, it is not difficult to show that $r+m+|\mathcal{T}|=0$, that is, $r=m=|\mathcal{T}|=0$, which results in

$$\begin{array}{l} H=H_1\bigcup(\bigcup_{i\in A_1}C_i)\bigcup(\bigcup_{j\in B_1}D_j)\bigcup kD_4 \\ R_2(H)=3=R_2(H_1)+|B_1| \end{array} \tag{3.9}$$

Subcase 1.2.1. $H_1\cong C_{m_1}(P_{m_2})$.

According to Lemma 2.8 and 2.9, we have that $R_2(H)=3$ if and only if $R_2(C_{m_1}(P_{m_2}))=3$, which leads to $m_1\geq 4$, $m_2=2$ and $|B_1|=0$. Hence $H=C_{m_1}(P_2)\bigcup(\bigcup_{i\in A_1}C_i)\bigcup kD_4$, which results in $\beta(H)=\beta(C_{m_1}(P_2))$ by Lemma 2.12. Note that $\beta(C_{m_1}(P_2))=\beta(H)=\beta(C_{n-1}(P_2))$ and (3) of Theorem 2.3, it follows that $m_1=n-1$. By $p(H)=q(H)$ we arrive at $|A_1|=k=0$. Thus $H\cong C_{n-1}(P_2)$.

Subcase 1.2.2. $H_1\cong B_{m_1,m_2,m_3}$.

In the light of (3) of Lemma 2.8 and (1) of Lemma 2.9, it follows, from (3.9), that $R_2(H)=3$ if and only if $R_2(B_{m_1,m_2,m_3})=3$, which leads to $m_1\geq 2, m_2=m_3=1$ and $|B_1|=0$. Hence $H=B_{m_1,1,1}\bigcup(\bigcup_{i\in A_1}C_i)\bigcup kD_4$ that implies, from Lemma 2.12, that $\beta(H)=\beta(B_{m_1,1,1})$. Note that $\beta(B_{m_1,1,1})=\beta(H)=\beta(C_{n-1}(P_2))$ and by (1) of Lemma 2.11, it follows that $(m_1,n)\in\{(2,7),(5,6)\}$. According to the condition of the theorem that $n\neq 7$, we arrive at $m_1=5$ and $n=6$ which contradicts to $|V(H)|=|V(C_{n-1}(P_2))|$.

Subcase 1.2.3. $H_1\cong Q_{m_1,m_2}$.

In view of (1) of Lemma 2.9 and (3.9), we have that $R_2(H)=3$ iff $R_2(Q_{m_1,m_2})=3$, which results in $m_1=m_2=1$ and $|B_1|=0$. So $H=Q_{1,1}\bigcup(\bigcup_{i\in A_1}C_i)\bigcup kD_4$ which implies, from Lemma 2.12, that $\beta(H)=\beta(Q_{1,1})$. Recalling that $\beta(Q_{1,1})=\beta(H)=\beta(C_{n-1}(P_2))$, we obtain, by (4) of Theorem 2.3, that $n=5$ which contradicts to the assumption of the theorem that $n\neq 5$.

Case 2. $s_{-1}=1$.

We obtain, by (3.2), that $s_1=0, s_2=1$ or $s_1=2, s_2=0$. From Theorem 3.1 and $s_{-1}=1$, we know that H only contains one path P_2 as its component.

Subcase 2.1. $s_1=0$ and $s_2=1$.

It follows, from (3.4), that $1 \leq q(H_1) - p(H_1) \leq 2$ with $R_1(H_1) = -2$. Without loss of generality, let

$H =$
$P_2 \bigcup H_1 \bigcup rK_1 \bigcup (\bigcup_{i\in A_1} C_i) \bigcup kD_4 \bigcup (\bigcup_{j\in B_1} D_j) \bigcup mT_{1,1,1} \bigcup (\bigcup_{T\in \mathcal{T}} T_{l_1,l_2,l_3})$.

It follows, from Lemma 2.8, that

$$R_2(H) \geq R_2(P_2) + R_2(H_1) + |B_1| - m = R_2(H_1) + |B_1| - m - 1. \qquad (3.10)$$

We distinguish the following subcases:

Subcase 2.1.1. $q(H_1) = p(H_1) + 2$.

In this subcase, we have, from (4) of Lemma 2.5, that $H_1 \cong K_4$. Similarly, we have $r + m + |\mathcal{T}| = 1$ and $0 \leq m \leq 1$. From (3.10), Lemmas 2.8 and 2.9, it follows that $|B_1| \leq -2$ which is a contradiction.

Subcase 2.1.2. $q(H_1) = p(H_1) + 1$.

We have, by (4) of Lemma 2.5, that $H_1 \in \psi$. Similarly, we obtain $r + m + |\mathcal{T}| = 0$, that is, $r = m = |\mathcal{T}| = 0$ which leads to $H = P_2 \bigcup H_1 \bigcup (\bigcup_{i\in A_1} C_i) \bigcup kD_4 \bigcup (\bigcup_{j\in B_1} D_j)$. From Lemma 2.9 and (3.10), it leads to $R_2(H_1) \geq 7$ and $|B_1| \leq -2$ which is impossible.

Subcase 2.2. $s_1 = 2$ and $s_2 = 0$.

In view of (3.4), we arrive at $1 \leq \sum_{i=1}^{2}[q(H_i) - p(H_i)] \leq 2$ with $R_1(H_i) = -1$. Without loss of generality, let

$$H = P_2 \bigcup (\bigcup_{i=1}^{2} H_i) \bigcup rK_1 \bigcup (\bigcup_{i\in A_1} C_i) \bigcup kD_4 \bigcup (\bigcup_{j\in B_1} D_j) \bigcup mT_{1,1,1} \bigcup (\bigcup_{T\in \mathcal{T}} T_{l_1,l_2,l_3}),$$

From Lemmas 2.8 and 2.9, we have that

$$R_2(H) \geq R_2(P_2) + \sum_{i=1}^{2} R_2(H_i) + |B_1| - m = \sum_{i=1}^{2} R_2(H_i) + |B_1| - m - 1. \qquad (3.11)$$

We distinguish the following two subcases:

Subcase 2.2.1. $\sum_{i=1}^{2}[q(H_i) - p(H_i)] \leq 2$.

By (3) of Lemma 2.5 we obtain that $H_i \in \{F_m, K_4^- | m \geq 6\}$ for $i = 1, 2$. From Lemma 2.9 and the expression of H, we know that $R_2(H_1) + R_2(H_2) \geq 6$, $r + m + |\mathcal{T}| = 1$ and $0 \leq m \leq 1$, which results in $|B_1| \leq -1$ by (3.11).

Subcase 2.2.2. $\sum_{i=1}^{2}[q(H_i) - p(H_i)] \leq 1$.

Without loss of generality, we have, by Lemma 2.5, that $H_1 \in \{F_m, K_4^- | m \geq 6\}$ and $H_2 \in \{C_r(P_s), B_{r,s,t}, Q_{r,s}\}$. In view of Lemma 2.9 and the expression of H, we arrive at $R_2(H_1) + R_2(H_2) \geq 6$ and $r = m = |\mathcal{T}| = 0$. From (3.11) we get $|B_1| \leq -2$, which is a contradiction. $\square$

From (2) of Theorem 1.1, the following corollary is obtained.

Corollary 3.1. For $n \geq 5$, $\overline{C_{n-1}(P_2)}$ is χ-unique if and only if $n \neq 5, 7$.

Theorem 3.3. *Let* $\mathcal{G} = \{\overline{C_n(P_m)} | n \geq 4, m \geq 3, m \neq n-1\} \bigcup \{\overline{C_4(P_3)}\}$, *then any graph in* $\mathcal{G}$ *is not* χ*-unique.*

Proof. By (2) and (4) of Theorem 2.1, we know that any graph in $\mathcal{G}$ is not adjointly unique which implies, from (2) of Theorem 1.1, that it is also not χ-unique. $\square$

Theorem 3.4. For $n \geq 4$ and $m \geq 2$, almost every $C_n(P_m)$ is not adjointly unique.

Proof. We, first of all, calculate the number of $C_n(P_m)$ for the fixed order $p = |V(C_n(P_m)| = n + m - 1$. By the condition of the theorem, it is not difficult to obtain that $n \in [4, p-1]$ and $m \in [2, p-3]$. Thus we have $(p-4)^2$ possibilities to choose the pair (n, m) in the above ranges, that is, the number of graphs $C_n(P_m)$ whose orders do not exceed p is $f(p) = (p-4)^2$. From Theorem 3.3, we know that there are only two graphs $C_4(P_2)$ and $C_6(P_2)$ being not adjointly unique. The number of graphs whose orders do not exceed p in $\mathcal{G}$ of Theorem 3.3 is

$$|\mathcal{G}| = p^2 - \frac{21}{2}p + c,$$

where the constant $c = 27$ if p is even and $c = \frac{53}{2}$ if p is odd. Obviously, among all graphs $C_n(P_m)$, the total number of graphs that are not adjointly unique is at least

$$g(p) = p^2 - \frac{21}{2}p + c + 2.$$

Thus, the proportion of them at least equals

$$\lim_{p\to\infty} \frac{g(p)}{f(p)} = \lim_{p\to\infty} \frac{p^2 - \frac{21}{2}p + c + 2}{(p-4)^2} = \lim_{p\to\infty} \frac{1 - \frac{21}{2p} + \frac{c+2}{p^2}}{1 - \frac{8}{p} + \frac{16}{p^2}} = 1$$

implying that almost every $C_n(P_m)$ is not adjointly unique. $\square$

Now, Our main result in this paper follows from the above theorem.

Theorem 3.5. *For* $n \geq 4$ *and* $m \geq 2$, *almost every* $\overline{C_n(P_m)}$ *is not* χ*-unique.*

B. Bollobás and O. Riordan [1] (or see [20]) conjectured that for the family of all graphs with fixed vertices, almost every graph in the family is χ-unique, that is, almost every complement of a graph in the family is adjointly unique by Theorem 1.1. Another interesting result in contrast with Theorem 3.5 is that almost every K_4-homemorph is chromatically unique proved by Li [16]. On the basis of these two results, we know that to prove their conjecture may be even more difficult.

4 Further Discussion

From the above results, we see that if we are searching for the chromatic uniqueness of $\overline{C_n(P_m)}$ we should consider the graph $\overline{C_n(P_{n-1})}$, where $n \geq 4$. However, we will need to investigate more complex algebraic properties of this family of graphs. The following conjecture is put forward.

Conjecture. *For $n \geq 4$, $\overline{C_n(P_{n-1})}$ is chromatically unique iff $n \neq 4$.*

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