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A complete global solution to the pressure gradient equation

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Abstract

We study the domain of existence of a solution to a Riemann problem for the pressure gradient equation in two space dimensions. The Riemann problem is the expansion of a quadrant of gas of constant state into the other three vacuum quadrants. The global existence of a smooth solution was established in Dai and Zhang [Z. Dai, T. Zhang, Existence of a global smooth solution for a degenerate Goursat problem of gas dynamics, Arch. Ration. Mech. Anal. 155 (2000) 277–298] up to the free boundary of vacuum. We prove that the vacuum boundary is the coordinate axes.

MSC: primary 35L65, 35J70, 35R35; secondary 35J65

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1. Introduction

The pressure gradient system

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ E_t + (pu)_x + (pv)_y = 0, \end{cases} \tag{1.1}$$

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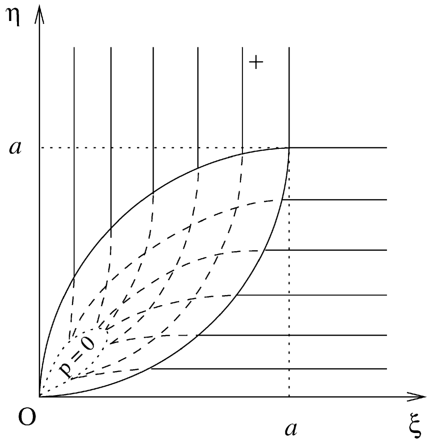

Fig. 1. Solution with a presumed vacuum bubble.

where $E = (u^2 + v^2)/2 + p$, appeared first in the flux-splitting method of Li and Cao [9] and Agarwal and Halt [1] in numerical computation of the Euler system of a compressible gas. Later, an asymptotic derivation was given in Zheng [13,16] from the two-dimensional full Euler system for an ideal fluid

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + pI) = 0, \\ (\rho E)_t + \nabla \cdot (\rho E \mathbf{u} + p\mathbf{u}) = 0, \end{cases}$$

where $\mathbf{u} = (u, v)$, $E = (u^2 + v^2)/2 + p/((\gamma - 1)\rho)$, and $\gamma > 1$ is a gas constant. We refer the reader to the books of Zheng [14] and Li et al. [6] for more background information, and the papers [2,3,7,10,12,13,15,16] for recent studies. After being decoupled from system (1.1), the pressure satisfies the following second order quasi-linear hyperbolic equation:

$$\left(\frac{p_t}{p}\right)_t - \Delta_{(x,y)} p = 0. \tag{1.2}$$

Dai and Zhang [2] studied a Riemann problem for system (1.1), see also Yang and Zhang [11] by the hodograph method. In the self-similar variables $\xi = x/t$, $\eta = y/t$, the value of the pressure variable of the Riemann data is

$$\begin{cases} p(\xi, \eta) = \xi^2 \quad \text{for } 0 < \xi, \eta \leqslant \sqrt{p_1},\ (\xi - \sqrt{p_1})^2 + \eta^2 = p_1, \\ p(\xi, \eta) = \eta^2 \quad \text{for } 0 < \xi, \eta \leqslant \sqrt{p_1},\ \xi^2 + (\eta - \sqrt{p_1})^2 = p_1. \end{cases} \tag{1.3}$$

Here p_1 is any positive number. They showed that the Goursat problem for Eq. (1.2) in the self-similar plane admits a global solution, which is smooth with a possible vacuum near the origin (see Fig. 1, where $a = \sqrt{p_1}$). *We are interested in the size of the vacuum boundary* $\{(\xi, \eta) \mid p(\xi, \eta) = 0\}$ *where the pressure gradient equation* (1.2) *is degenerate*. Somewhat surprisingly, our result shows that the vacuum bubble is trivial and the entire vacuum boundary is the trivial

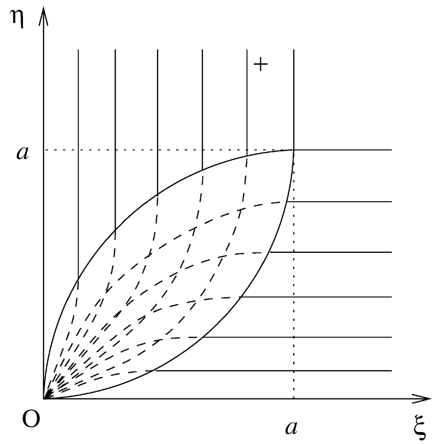

Fig. 2. The global solution.

coordinate axes in the self-similar plane (see Fig. 2, where $a = \sqrt{p_1}$). The result is stated in our main theorem at the end of Section 3.

Further motivation for the study of the current problem is that the study of boundaries such as a sonic curve is important in establishing the global existence of a solution to a general two-dimensional Riemann problem of the pressure gradient system. In addition, the solution of the current problem covers wave interaction problems in which only some fractions of the plane waves are involved. Wave interactions of these kinds are common in two-dimensional Riemann problems. Finally, the study of the pressure gradient system has motivated work on two-dimensional full Euler systems, see Li [5], Zheng [17,18], and Li and Zheng [8]. In particular, the latest work [8] has resolved completely the location of the vacuum boundary for the gas expansion problem for the adiabatic Euler system.

2. Integration along characteristics

In the self-similar variables $\xi = x/t,\ \eta = y/t$, the pressure gradient equation (1.2) takes the form

$$\frac{(\xi\partial_\xi + \eta\partial_\eta)^2 p}{p} - \Delta_{(\xi,\eta)} p + \frac{(\xi\partial_\xi + \eta\partial_\eta) p}{p} - \frac{((\xi\partial_\xi + \eta\partial_\eta) p)^2}{p^2} = 0. \tag{2.1}$$

In the polar coordinates

$$\begin{cases} r = \sqrt{\xi^2 + \eta^2}, \\ \theta = \arctan\frac{\eta}{\xi}, \end{cases}$$

Eq. (2.1) can be decomposed along the characteristics into the following form (see [2,4,7]):

$$\begin{cases} \partial_+\partial_- p = m p_r \partial_- p, \\ \partial_-\partial_+ p = -m p_r \partial_+ p, \end{cases} \tag{2.2}$$

provided that $p < r^2$, where

$$m = \frac{\lambda r^4}{2p^2},$$

and

$$\begin{cases} \partial_\pm = \partial_\theta \pm \frac{1}{\lambda}\partial_r, \\ \lambda = \sqrt{\frac{p}{r^2(r^2-p)}}. \end{cases} \tag{2.3}$$

Note that the equation is invariant under the following scaling transformation:

$$(\xi, \eta, p) \to \left(\frac{\xi}{\sqrt{p_1}}, \frac{\eta}{\sqrt{p_1}}, \frac{p}{p_1}\right) \quad (p_1 > 0).$$

Thus, without loss of generality, the corresponding boundary condition (1.3) in the polar coordinates can be set in the form

$$\begin{cases} p = \xi^2 = r^2\cos^2\theta = 4\cos^4\theta & \text{on } r = 2\cos\theta,\ \pi/4 \leqslant \theta \leqslant \pi/2; \\ p = \eta^2 = r^2\sin^2\theta = 4\sin^4\theta & \text{on } r = 2\sin\theta,\ 0 \leqslant \theta \leqslant \pi/4. \end{cases} \tag{2.4}$$

The solution exists in the interaction zone up to a possible vacuum bubble. See Fig. 3.

The characteristic form (2.2) of the pressure gradient equation enjoys a number of useful properties. For example, the quantities $\partial_\pm p$ keep their positivities/negativities along characteristics of a plus/minus family, and the sign-persevering quantities yield monotonicity of the primary variable p (see [2,7], also [4] where the authors propose to call them *Riemann sign-persevering variables*), and the fact that a state adjacent to a constant state for the pressure gradient system must be a simple wave in which p is constant along the characteristics of a plus/minus family [7].

The characteristic decomposition (2.2) has played an important role and was a powerful tool for building the existence of smooth solutions in the work of Dai and Zhang [2]. We are interested in the size of the vacuum boundary $\{(r,\theta) \mid p(r,\theta) = 0\}$ where the pressure gradient equation is degenerate. For this purpose, we rewrite (2.2) as

$$\begin{cases} \partial_+\partial_- p = q\partial_+ p\partial_- p - q(\partial_- p)^2, \\ \partial_-\partial_+ p = q\partial_+ p\partial_- p - q(\partial_+ p)^2, \end{cases} \tag{2.5}$$

where

$$q = \frac{r^2}{4p(r^2 - p)}. \tag{2.6}$$

Define the characteristic curves $r_-^a(\theta)$ and $r_+^b(\theta)$ by

$$\begin{cases} \frac{dr_-^a(\theta)}{d\theta} = -\frac{1}{\lambda(r_-^a(\theta),\theta)}, \\ r_-^a(\theta_a) = 2\sin\theta_a, \end{cases} \quad \text{and} \quad \begin{cases} \frac{dr_+^b(\theta)}{d\theta} = \frac{1}{\lambda(r_+^b(\theta),\theta)}, \\ r_+^b(\theta_b) = 2\cos\theta_b. \end{cases} \tag{2.7}$$

We point out here that for convenience, we also use the notation $r_-^a(r,\theta)$ ($r_+^b(r,\theta)$, respectively) which represents the characteristic curve passing through the point (r,θ) and intersecting the lower (upper, respectively) boundary at point a (b, respectively). See Fig. 3.

Now let us rewrite system (2.5) in the form

$$\begin{cases} \partial_+\big(\frac{1}{\partial_- p}\exp\int_{\theta_b}^{\theta} q\,\partial_+ p(r_+^b(\phi),\phi)\,d\phi\big) = q\exp\int_{\theta_b}^{\theta} q\,\partial_+ p(r_+^b(\phi),\phi)\,d\phi, \\ \partial_-\big(\frac{1}{\partial_+ p}\exp\int_{\theta_a}^{\theta} q\,\partial_- p(r_-^a(\phi),\phi)\,d\phi\big) = q\exp\int_{\theta_a}^{\theta} q\,\partial_- p(r_-^a(\phi),\phi)\,d\phi. \end{cases}$$

Integrate the above equations along the positive and negative characteristics $r_+^b(\theta)$ and $r_-^a(\theta)$ from θ_b and θ_a to θ, respectively, with respect to θ, one obtain the iterative expressions of $\partial_+ p$ and $\partial_- p$:

$$\begin{cases} \frac{1}{\partial_- p}\exp\int_{\theta_b}^{\theta} q\,\partial_+ p(r_+^b(\phi),\phi)\,d\phi \\ \quad = \frac{1}{\partial_- p}(2\cos\theta_b,\theta_b) + \int_{\theta_b}^{\theta} q(r_+^b(\psi),\psi)\exp\int_{\theta_b}^{\psi} q\,\partial_+ p(r_+^b(\phi),\phi)\,d\phi\,d\psi, \\ \frac{1}{\partial_+ p}\exp\int_{\theta_a}^{\theta} q\,\partial_- p(r_-^a(\phi),\phi)\,d\phi \\ \quad = \frac{1}{\partial_+ p}(2\sin\theta_a,\theta_a) + \int_{\theta_a}^{\theta} q(r_-^a(\psi),\psi)\exp\int_{\theta_a}^{\psi} q\,\partial_- p(r_-^a(\phi),\phi)\,d\phi\,d\psi. \end{cases} \tag{2.8}$$

On the other hand, we use boundary condition (2.4) to find

$$\begin{aligned} &\exp\int_{\theta_b}^{\theta} q\,\partial_+ p\big(r_+^b(\phi),\phi\big)\,d\phi \\ &\quad = \exp\frac{1}{4}\int_{\theta_b}^{\theta}\left(\frac{1}{p}+\frac{1}{r^2-p}\right)\partial_+ p\big(r_+^b(\phi),\phi\big)\,d\phi \\ &\quad = \frac{p^{\frac14}(r_+^b(\theta),\theta)}{\sqrt{2}\cos\theta_b}\exp\left\{\frac14\int_{p(2\cos\theta_b,\theta_b)}^{p(r_+^b(\theta),\theta)}\frac{1}{r^2(r_+^b)-p_1}\,dp_1\right\}. \end{aligned} \tag{2.9}$$

Similarly, one has

$$\begin{aligned} &\exp\int_{\theta_a}^{\theta} q\,\partial_- p\big(r_-^a(\phi),\phi\big)\,d\phi \\ &\quad = \frac{p^{\frac14}\big(r_-^a(\theta),\theta\big)}{\sqrt{2}\sin\theta_a}\exp\left\{\frac14\int_{p(2\sin\theta_a,\theta_a)}^{p(r_-^a(\theta),\theta)}\frac{1}{r^2(r_-^a)-p_1}\,dp_1\right\}. \end{aligned} \tag{2.10}$$

Then, we have

$$\int_{\theta_b}^{\theta} q\big(r_+^b(\psi),\psi\big)\left\{\exp\int_{\theta_b}^{\psi} q\partial_+ p\big(r_+^b(\phi),\phi\big)\,d\phi\right\}d\psi$$
$$= \frac{1}{4\sqrt{2}\cos\theta_b}\int_{\theta_b}^{\theta}\frac{r^2 p^{-\frac{3}{4}}}{r^2-p}\big(r_+^b(\psi),\psi\big)$$
$$\times\exp\left\{\frac{1}{4}\int_{p(2\cos\theta_b,\theta_b)}^{p(r_+^b(\psi),\psi)}\frac{1}{r^2(r_+^b)-p_1}\,dp_1\right\}d\psi, \tag{2.11}$$

and

$$\int_{\theta_a}^{\theta} q\big(r_-^a(\psi),\psi\big)\exp\int_{\theta_a}^{\psi} q\partial_- p\big(r_-^a(\phi),\phi\big)\,d\phi\,d\psi$$
$$= \frac{1}{4\sqrt{2}\sin\theta_a}\int_{\theta_a}^{\theta}\frac{r^2 p^{-\frac{3}{4}}}{r^2-p}\big(r_-^a(\psi),\psi\big)$$
$$\times\exp\left\{\frac{1}{4}\int_{p(2\sin\theta_a,\theta_a)}^{p(r_-^a(\psi),\psi)}\frac{1}{r^2(r_-^a)-p_1}\,dp_1\right\}d\psi. \tag{2.12}$$

Finally, by substituting (2.10) and (2.12) into the second equality of (2.8), we obtain a new iterative expression for $\partial_+ p$:

$$\partial_+ p(r,\theta) = \frac{\exp\int_{\theta_a}^{\theta} q\partial_- p(r_-^a(\phi),\phi)\,d\phi}{\frac{1}{\partial_+ p}(2\sin\theta_a,\theta_a)+\int_{\theta_a}^{\theta} q(r_-^a(\psi),\psi)\exp\int_{\theta_a}^{\psi} q\partial_- p(r_-^a(\phi),\phi)\,d\phi\,d\psi}$$
$$= \frac{4p^{\frac{1}{4}}(r,\theta)\exp\left\{\frac{1}{4}\int_{p(2\sin\theta_a,\theta_a)}^{p(r_-^a(\theta),\theta)}\frac{1}{r^2(r_-^a)-p_1}\,dp_1\right\}}{\frac{1}{2\sqrt{2}\sin^2\theta_a\cos\theta_a}+\int_{\theta_a}^{\theta}\frac{r^2p^{-3/4}}{r^2-p}(r_-^a(\psi),\psi)\exp\left\{\frac{1}{4}\int_{p(2\sin\theta_a,\theta_a)}^{p(r_-^a(\psi),\psi)}\frac{1}{r^2(r_-^a)-p_1}\,dp_1\right\}d\psi}. \tag{2.13}$$

Similarly, by substituting (2.9) and (2.11) into the first equality of (2.8), one obtains the new iterative formula for $\partial_- p$:

$$-\partial_- p(r,\theta) = \frac{-\exp\int_{\theta_b}^{\theta} q\partial_+ p(r_+^b(\phi),\phi)\,d\phi}{\frac{1}{\partial_- p}(2\cos\theta_b,\theta_b)+\int_{\theta_b}^{\theta} q(r_+^b(\psi),\psi)\exp\int_{\theta_b}^{\psi} q\partial_+ p(r_+^b(\phi),\phi)\,d\phi\,d\psi}$$
$$= \frac{4p^{\frac{1}{4}}(r,\theta)\exp\left\{\frac{1}{4}\int_{p(2\cos\theta_b,\theta_b)}^{p(r_+^b(\theta),\theta)}\frac{1}{r^2(r_+^b)-p_1}\,dp_1\right\}}{\frac{1}{2\sqrt{2}\cos^2\theta_b\sin\theta_b}-\int_{\theta_b}^{\theta}\frac{r^2p^{-3/4}}{r^2-p}(r_+^b(\psi),\psi)\exp\left\{\frac{1}{4}\int_{p(2\cos\theta_b,\theta_b)}^{p(r_+^b(\psi),\psi)}\frac{1}{r^2(r_+^b)-p_1}\,dp_1\right\}d\psi}. \tag{2.14}$$

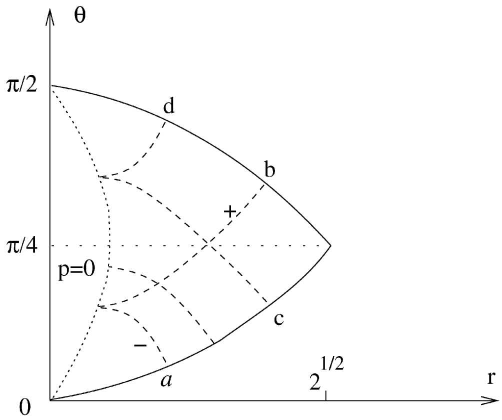

Fig. 3. Presumed vacuum in the polar coordinates.

3. The boundary of the vacuum bubble

In this section, we use the iterative formulas (2.13) and (2.14) to prove that the vacuum bubble $\{(r,\theta) \mid p(r,\theta)=0, \theta \in (0,\pi/2)\}$ is in fact the trivial origin $\{(0,0)\}$ in the self-similar plane. We use the method of contradiction. Assume to the contrary that there is a bubble with boundary $r_0(\theta) \geqslant 0$ for all $\theta \in (0, \frac{\pi}{2})$, and $r_0(\theta) > 0$ for some $\theta \in (0, \frac{\pi}{2})$. That means $p(r_0(\theta), \theta) = 0$ for $\theta \in (0, \frac{\pi}{2})$, and the solution is smooth in the domain bounded by the bubble boundary $r_0(\theta)$ and the upper and lower characteristic boundaries. We intend to deduce contradictions.

Before we start the above procedure, we point out two observations which roughly imply the nonexistence of the vacuum bubble. For presenting the observations, we assume further that $r_0(\theta) > 0$ for all $\theta \in (0, \frac{\pi}{2})$.

First, we can compute easily the characteristic slope

$$d\theta/dr = -\lambda = -\frac{1}{2\sin\theta} \sim -\frac{1}{2}$$

along the upper characteristic boundary and in the limit as $\theta \to \pi/2$. Similar result holds on the lower boundary. Now let us assume that there is a nontrivial bubble with boundary $r_0(\theta) > 0$ for all $\theta \in (0, \frac{\pi}{2})$. We see from the definition of λ in (2.3) that

$$d\theta/dr = \lambda = \sqrt{\frac{p}{r^2(r^2-p)}} \to 0$$

as $p \to 0$ (at the vacuum boundary) except for $(0,0)$ and $(0, \frac{\pi}{2})$. The above computation reveals that there is some kind of inconsistency for the slopes of the characteristics at $(0,0)$ and $(0, \frac{\pi}{2})$ in the polar coordinate plane (i.e. the slope jumps from 0 to $-1/2$).

Next, let us calculate the decay rate of p along the middle line $\theta = \pi/4$ (see Fig. 3). By using the symmetry of system (2.1), we claim that

$$\partial_\pm p \sim \pm M_0 p^{1/2}$$

asymptotically for some $M_0 \neq 0$. To show the details, we propose

$$\partial_\pm p \sim \pm M_0 p^{\frac{1}{4}+\delta}$$

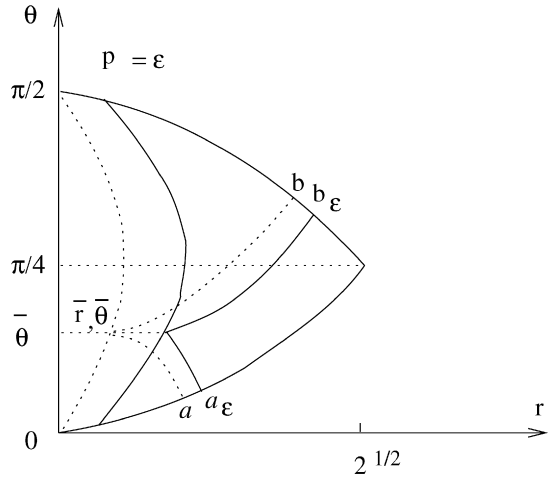

Fig. 4. Domain and notation.

as (r,θ) tends to a point of the bubble boundary. Then, by (2.13), we have

$$\begin{aligned}
\partial_+ p(r,\theta) = 4p^{\frac{1}{4}}(r,\theta)\exp\left\{\frac{1}{4}\int_{p(2\sin\theta_a,\theta_a)}^{p(r_-^a(\theta),\theta)}\frac{1}{r^2(r_-^a)-p_1}\,dp_1\right\}\Big/\left\{\frac{1}{2\sqrt{2}\sin^2\theta_a\cos\theta_a}\right. \\
\left. + \int_{\theta_a}^{\theta}\frac{r^2p^{-\frac{3}{4}}}{(r^2-p)\partial_- p}\big(r_-^a(\psi),\psi\big)\exp\left\{\frac{1}{4}\int_{p(2\sin\theta_a,\theta_a)}^{p(r_-^a(\psi),\psi)}\frac{1}{r^2(r_-^a)-p_1}\,dp_1\right\}dp\right\} \\
= 4\delta M_0 p^{\frac{1}{4}+\delta} + \text{high order terms},
\end{aligned}$$

which forces $4\delta = 1$, i.e., δ should be $\frac{1}{4}$.

Thus, by using (2.3), we have

$$\partial_r p \sim \frac{M_0}{r^2}p$$

asymptotically as (r,θ) tends to a point of the bubble boundary. Therefore,

$$p \sim c\exp\left(-\frac{M_0}{r}\right) \tag{3.1}$$

asymptotically as (r,θ) tends to a point of the bubble boundary on the line $\theta = \frac{\pi}{4}$, which implies that there is no interior vacuum at least on $\theta = \frac{\pi}{4}$.

We point out incidentally that Zheng's previous numerical computation of the bubble, referred to in Dai and Zhang [2], is probably caused by the fast exponential decay (3.1).

In what follows, we concentrate on establishing the above formal intuition rigorously. With the symmetry of system (2.5), let us restrict our argument θ to be in $(0,\frac{\pi}{4}]$. Fix a point $(\bar r,\bar\theta)$ on the bubble boundary. Let us use $\mathbb{D}(\bar r,\bar\theta)$ to represent the bounded domain surrounded by the positive and negative characteristic curves starting from $(\bar{r},\bar{\theta})$ and $(\sqrt{2},\frac{\pi}{4})$. See Fig. 4. For $0<\epsilon<1$, define a curve $r_\epsilon(\theta)$, $\theta\in(0,\frac{\pi}{2})$, by

$$p\big(r_\epsilon(\theta),\theta\big)=\epsilon. \tag{3.2}$$

From (2.13) and (2.14), it is easy to see that

$$\begin{cases}\partial_+ p(r,\theta)>0,\\ \partial_- p(r,\theta)<0,\end{cases}\quad \text{for } 0<\theta\leqslant\frac{\pi}{4}. \tag{3.3}$$

Thus, by (2.3) and (3.3), we have

$$p_r(r,\theta)>0, \tag{3.4}$$

which implies that the curve $r_\epsilon(\theta)$, $\theta\in(0,\frac{\pi}{2})$, defined in (3.2) is smooth if $\epsilon\in(0,1)$. Here we point out that we still denote by $r_-^a(\theta)$ and $r_+^b(\theta)$ the characteristics passing through $(r_\epsilon(\theta),\theta)$ although in fact both $a=a_\epsilon$ and b are dependent on ϵ and θ. Further, we still denote by $r_-^a(r,\theta)$ and $r_+^b(r,\theta)$ the characteristics passing through (r,θ), where a,b depend on (r,θ).

Let us now fix an $\epsilon_0\in(0,1)$. Thus the curve $r_{\epsilon_0}(\theta)$ exists. Define

$$M_1=\max_{\text{over } S}\big\{p^{-\frac{1}{2}}\partial_+ p,\,-p^{-\frac{1}{2}}\partial_- p\big\}, \tag{3.5}$$

where

$$S:=\big\{(r,\theta)\,\big|\,\theta\in(0,\pi/2),\ r_{\epsilon_0}(\theta)\leqslant r\leqslant\sqrt{2}\big\}.$$

Then we have

$$\begin{cases}\partial_+ p\leqslant M_1 p^{\frac{1}{2}},\\ -\partial_- p\leqslant M_1 p^{\frac{1}{2}},\end{cases} \tag{3.6}$$

for all (r,θ) with $r\geqslant r_{\epsilon_0}(\theta)$. Note that M_1 depends only on ϵ_0 and does not depend on $(\bar{r},\bar{\theta})$.

Next, for $(r,\theta)\in\mathbb{D}(\bar{r},\bar{\theta})$, we first let a be the intersection of the minus characteristic curve through (r,θ) with the boundary, and b the corresponding counterpart. We then let

$$\begin{cases}A(r,\theta)=\dfrac{\exp[-\frac{1}{4}\int_{p(2\sin\theta_a,\theta_a)}^{p(r_-^a(\theta),\theta)}\frac{1}{r^2(r_-^a)-p_1}\,dp_1]}{2\sqrt{2}\sin^2\theta_a\cos\theta_a},\\[2ex] B(r,\theta)=\dfrac{\exp[-\frac{1}{4}\int_{p(2\cos\theta_b,\theta_b)}^{p(r_+^b(\theta),\theta)}\frac{1}{r^2(r_+^b)-p_1}\,dp_1]}{2\sqrt{2}\cos^2\theta_b\sin\theta_b}.\end{cases} \tag{3.7}$$

Then, let

$$M_2=\max\bigg\{\max_{(r,\theta)\in\mathbb{D}(\bar{r},\bar{\theta})}\frac{4p^{-\frac{1}{4}}(2\sin\theta_a,\theta_a)}{A(r,\theta)},\ \max_{(r,\theta)\in\mathbb{D}(\bar{r},\bar{\theta})}\frac{4p^{-\frac{1}{4}}(2\cos\theta_b,\theta_b)}{B(r,\theta)}\bigg\}, \tag{3.8}$$

and

$$M_3 = \max\{M_1, M_2 + 1\}. \tag{3.9}$$

Just as what we had pointed out before, the intersections a and b in (3.7) and (3.8) vary with (r, θ), which are not expressed explicitly for notational brevity. We intend to prove that inequalities (3.6) are still valid for all points $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$ with M_1 being replaced by M_3. Namely, there hold

$$\begin{cases} \partial_+ p \leqslant M_3 p^{\frac{1}{2}}, \\ -\partial_- p \leqslant M_3 p^{\frac{1}{2}}, \end{cases} \quad (r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta}). \tag{3.10}$$

Note that the positive constant M_3 ($\geqslant M_1$) is independent of $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$, but depends on the fixed $(\bar{r}, \bar{\theta})$.

We start to prove (3.10). Suppose that (3.10) is correct up to a line segment $r_\epsilon(\theta) \subset \mathbb{D}(\bar{r}, \bar{\theta})$, then we improve (3.10) to strict inequalities on the line segment $r_\epsilon(\theta)$ in the same domain. In fact, for $(r, \theta) \in \mathbb{D}(\bar{r}, \bar{\theta})$ with $r \leqslant r_{\epsilon_0}(\theta)$, let us compute:

$$\begin{aligned}
\partial_+ p(r, \theta) &= 4p^{\frac{1}{4}}(r, \theta) \Big/ \Bigg\{ A(r, \theta) + \int_{\theta_a}^{\theta} \frac{r^2 p^{-\frac{3}{4}}}{r^2 - p} \big(r_-^a(\psi), \psi\big) \\
&\quad \times \exp\Bigg[\frac{1}{4} \int_{p(r_-^a(\theta), \theta)}^{p(r_-^a(\psi), \psi)} \frac{1}{r^2(r_-^a) - p_1} \, dp_1 \Bigg] d\psi \Bigg\} \\
&= 4p^{\frac{1}{4}}(r, \theta) \Big/ \Bigg\{ A(r, \theta) + \int_{\theta_a}^{\theta} \frac{-\partial_- p r^2 p^{-\frac{3}{4}}}{(-\partial_- p)(r^2 - p)} \big(r_-^a(\psi), \psi\big) \\
&\quad \times \exp\Bigg[\frac{1}{4} \int_{p(r_-^a(\theta), \theta)}^{p(r_-^a(\psi), \psi)} \frac{1}{r^2(r_-^a) - p_1} \, dp_1 \Bigg] d\psi \Bigg\} \\
&\leqslant M_3 p^{\frac{1}{4}}(r, \theta) \Big/ \Bigg\{ \frac{M_3 A(r, \theta)}{4} + \int_{p(2\sin\theta_a, \theta_a)}^{p(r_-^a(\theta), \theta)} \frac{-\frac{1}{4} p^{-\frac{5}{4}} r^2}{r^2 - p} \big(r_-^a(\psi), \psi\big) \\
&\quad \times \exp\Bigg[\frac{1}{4} \int_{p(r_-^a(\theta), \theta)}^{p(r_-^a(\psi), \psi)} \frac{1}{r^2(r_-^a) - p_1} \, dp_1 \Bigg] dp \Bigg\} \\
&= \frac{M_3 p^{\frac{1}{4}}(r, \theta)}{\frac{M_3 A(r, \theta)}{4} + f(\theta^-) \int_{p(2\sin\theta_a, \theta_a)}^{p(r_-^a(\theta), \theta)} -\frac{1}{4} p^{-\frac{5}{4}} \, dp} \\
&= \frac{M_3 p^{\frac{1}{4}}(r, \theta)}{\frac{M_3 A(r, \theta)}{4} + f(\theta^-)[p^{-\frac{1}{4}}(r, \theta) - p^{-\frac{1}{4}}(2\sin\theta_a, \theta_a)]},
\end{aligned} \tag{3.11}$$

where

$$
\begin{aligned}
f(\theta^-) &= \frac{r^2}{r^2-p}\big(r_-^a(\theta^-),\theta^-\big)\exp\left[\frac{1}{4}\int\limits_{p(r_-^a(\theta),\theta)}^{p(r_-^a(\theta^-),\theta^-)} \frac{1}{r^2(r_-^a)-p_1}\,dp_1\right] \\
&> \exp\left[\frac{1}{4}\int\limits_{p(r_-^a(\theta),\theta)}^{p(r_-^a(\theta^-),\theta^-)} \frac{1}{\sqrt{2}^2}\,dp_1\right] \\
&= \exp\left[\frac{p(r_-^a(\theta^-),\theta^-)-p(r_-^a(\theta),\theta)}{8}\right] > 1 \qquad (3.12)
\end{aligned}
$$

with some $\theta_a < \theta^- < \theta$ and $A(r,\theta)$ is defined in (3.7). Thus, by (3.9), (3.11) and (3.12), we have

$$
\begin{aligned}
\partial_+ p(r,\theta) &< \frac{M_3 p^{\frac14}(r,\theta)}{\frac{M_3 A(r,\theta)}{4} + [p^{-\frac14}(r,\theta) - p^{-\frac14}(2\sin\theta_a,\theta_a)]} \\
&\leqslant M_3 p^{\frac12}(r,\theta). \qquad (3.13)
\end{aligned}
$$

Similarly, by (2.8), (2.9) and (2.11), we have

$$
\begin{aligned}
-\partial_- p(r,\theta) &= 4p^{\frac14}(r,\theta)\Big/\left\{B(r,\theta) - \int\limits_{\theta_b}^{\theta} \frac{r^2 p^{-\frac34}}{r^2-p}\big(r_+^b(\psi),\psi\big)\right. \\
&\qquad \left.\times \exp\left[\frac14 \int\limits_{p(r_+^b(\theta),\theta)}^{p(r_+^b(\psi),\psi)} \frac{1}{r^2(r_+^b)-p_1}\,dp_1\right] d\psi\right\} \\
&\leqslant \frac{M_3 p^{\frac14}(r,\theta)}{\frac{M_3 B(r,\theta)}{4} + g(\theta^+)[p^{-\frac14}(r,\theta) - p^{-\frac14}(2\cos\theta_b,\theta_b)]}, \qquad (3.14)
\end{aligned}
$$

where

$$
\begin{aligned}
g\big(\theta^+\big) &= \frac{r^2}{r^2-p}\big(r_+^b\big(\theta^+\big),\theta^+\big)\exp\left[\frac{1}{4}\int\limits_{p(r_+^b(\theta),\theta)}^{p(r_+^b(\theta^+),\theta^+)} \frac{1}{r^2(r_+^b)-p_1}\,dp_1\right] \\
&> \exp\left[\frac{1}{4}\int\limits_{p(r_+^b(\theta),\theta)}^{p(r_+^b(\theta^+),\theta^+)} \frac{1}{\sqrt{2}^2}\,dp_1\right] \\
&= \exp\left[\frac{p(r_+^b(\theta^+),\theta^+)-p(r_+^b(\theta),\theta)}{8}\right] > 1
\end{aligned}
$$

with some $\theta < \theta^+ < \theta_b$ and $B(r,\theta)$ is given in (3.7). Thus, the similar estimate as (3.13) holds:

$$-\partial_- p(r,\theta) < M_3 p^{\frac{1}{2}}(r,\theta). \tag{3.15}$$

Since M_3 depends only on $(\bar{r},\bar{\theta})$, and in particularly, it is independent of $(r,\theta) \in \mathbb{D}(\bar{r},\bar{\theta})$, the proof of (3.10) follows by (3.13) and (3.15).

Now with the aid of (2.3), we add up (3.13) and (3.15) to yield

$$p_r \leqslant \frac{M_3}{\sqrt{r^2(r^2-p)}} p \leqslant \frac{2M_3}{\bar{r}^2} p$$

for all (r,θ) in a small neighborhood of $(\bar{r},\bar{\theta})$ in $\mathbb{D}(\bar{r},\bar{\theta})$. Thus, a simple integration of the above inequality with respect to r from $\bar{r}$ to r yields

$$p(\bar{r},\bar{\theta}) \geqslant p(r,\bar{\theta}) \exp\left\{-2\frac{M_3}{\bar{r}^2}(r-\bar{r})\right\} \quad \text{for } \bar{r} < r. \tag{3.16}$$

On the other hand, by (3.4) and the fact that $p(r_0(\theta),\theta) = 0$, we have

$$p(r,\theta) > 0$$

for all (r,θ) with $r > r_0(\theta)$, which together with (3.16) result in that $p(\bar{r},\bar{\theta}) > 0$. Since $(\bar{r},\bar{\theta})$ is a point on the bubble, we arrive at a contradiction.

Summing up, we have in fact proven the following theorem.

Theorem 1. *The Riemann problem* (1.2), (2.4) *for the pressure gradient equation admits a unique smooth solution. The pressure of the solution is strictly positive in $\xi > 0$, $\eta > 0$.*

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