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A HAMILTON SOLVER FOR FINDING MODAL EIGENVALUES

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A method is proposed for finding the normal mode eigenvalues in shallow water waveguide. We transform the problem determining eigenvalues in complex wave number plane into solving a "true" one dimensional Hamilton system. Simulations are performed, the result agrees very well with that calculated by the normal mode program KrakanC.

Keywords: Normal mode; eigenvalue; Hamilton system.

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1. Introduction

Normal mode methods are well developed for the solution of propagation of underwater acoustics. A number of well tested computer programs are available for applications to a wide variety of practical situations.[1,2] A crucial step in a normal mode solution is to find numerically modal eigenvalues, that is, numerically solve dispersion/eigenvalue equation. A main difficulty in finding modal eigenvalues for most general case where eigen-horizontal wavenumber is complex and may not close to the real axis. As has been argued by Tindle and Chapman,[3] in these cases, finding modal eigenvalues becomes a two-dimensional (**2D**) search in the complex horizontal wavenumber plane. They also proposed an elegant method to find quickly the eigenvalues. We refer interested readers to as their paper, where merits and demerits of several current methods have been discussed clearly.

In this short paper, motivated by the paper of Tindle and Chapman, we develop a method for finding modal eigenvalues. This method is more effective in the sense of that if we consider the method proposed by Tindle and Chapman to be a **1.5D** search in the complex plane, then our method is **1.1D** search problem. We will explain the terms **1.5D** or **1.1D** in the following section. Our idea is to transform the problem finding eigenvalues in the complex plane to solve a **1D** Hamilton equation. We would like to emphasize that the method can also be used in other fields of classical waveguide problems like ultrasonic's, optics.

For self-contained, in Sec. 2, we recall briefly the normal mode solution, relevant concepts: phase function and Wronskian. In Sec. 3, we illustrate our idea; how to transform finding eigenvalue problem to a problem of Hamilton mechanics. Two simple numerical examples are shown in Sec. 4.

2. Normal Mode Theory

The normal mode theory is well-developed in underwater acoustics, the details will not be repeated here, referring readers to standard text books. We here base the works of Bucker.[4]

For a range independent waveguide, sound pressure of frequency f at the range r, depth z, which generated by a harmonic point source at the coordinate origin with depth z_s can be expressed as

$$p(f,r,z) = -\int_{-\infty}^{+\infty} \frac{U(z_<)V(z_>)}{W} H_0^{(1)}(kr)kdk \tag{1}$$

where k is the horizontal wavenumber. The parameters $z_<$ and $z_>$ are the lesser and greater respectively of z and z_s. The functions U and V are two independent solutions which satisfies the lower and upper boundary condition of the considering waveguide and the following equation

$$\frac{d^2U}{dz^2} + \left(\frac{\omega^2}{c(z)^2} - k^2\right) U = 0 \tag{2}$$

The W called Wronskian of the ordinary differential equation for U and V is defined by

$$W \equiv UV' - U'V \tag{3}$$

where the prime denotes the derivative with respect to z. It can be shown in very general situation that U and V are function of the horizontal wavenumber with some isolated branch points, acrossing these points we need clearly point out the branch line e.g. Pekeris branch cut. There exist other singular points in Eq. (1) where the Wronskian are zero and corresponding wave numbers k are eigen-wavenumbers. By performing a contour integral with a suitable branch cut, Eq. (1) can be rewritten to

$$p(f,r,z) = 2\pi i \sum_n \frac{U(z_<,k_n)V(z_>,k_n)}{\partial W|_{kn}} H_0^{(1)}(k_n r)k_n + c.p \tag{4}$$

where c.p denotes the contribution coming from continuous spectrum, which cannot be omitted for near field or the problem of coupled modes,[5,6] while or far field discrete normal modes dominate.

Alternatively, we can use WKB solution for U and V, and then eigenvalues satisfies the following modal equation[3,4]

$$\varphi_{w,m} - \frac{i}{2}\ln(R_1) - \frac{i}{2}\ln(R_1) + \pi = m\pi \tag{5}$$

where φ_m denotes the one-way phase shift in water column, R_1 and R_2 the reflection coefficients of upper and lower boundaries, respectively. Equation (5) is similar to the semi-classical quantization condition in quantum mechanics, and the horizontal wavenumber is quantized. In Ref. 3, the authors coined the left hand side of Eq. (5) as phase function Φ. The phase function is in general a complex function of the complex k. In principle, there are two equivalent methods; one is to find zeros of Wronskian in the complex k plane and the other is proposed by Tindle and Chapman by first finding the curve $\mathrm{Im}\,\Phi = 0$ then identifying the positions $\mathrm{Re}\,\Phi = \mathrm{m}\pi$. Their method is powerful in the sense that the search is constrained to essentially one dimension after one found the curve $\mathrm{Im}\,\Phi = 0$. In this sense, we say it 1.5 dimension; 0.5 coming from finding the curve.

3. A Hamilton Solver

It has been seen that the crucial step in the application of the method proposed in Ref. 3 is to find isovalue curve $\mathrm{Im}\,\Phi = 0$. In this section, we show that it is not needed to find the whole curve, only one or two initial points are needed. A method traces automatically the curve $\mathrm{Im}\,\Phi = 0$ is proposed.

At first we recall some basic ingredients of Hamilton mechanics, see e.g. Ref. 7. In mathematical terms, a Hamilton mechanics is defined over an even $2n$-dimensional symplectic space $\boldsymbol{M}$. A symplectic space (or manifold) is equipped by a canonical structure called symplectic structure given by a rank two exterior differential-form

$$\Omega = \sum_{m,n=1}^{2n} \omega_{mn} dx^n \wedge dx^m. \tag{6}$$

where ω_{mn} are antisymmetric with respect to the subscripts m and n. A standard canonical symplectic structure is given by e.g. $n = 1$,

$$\Omega = dx^1 \wedge dx^2. \tag{7}$$

Given a symplectic structure, one can define a Poisson bracket as follows

$$\{F, G\} \equiv \sum_{m,n=1} \omega^{mn} \frac{\partial F}{\partial x^m} \frac{\partial G}{\partial x^n}, \tag{8}$$

where functions F and G are differentiable. The Poisson bracket has some elegant properties:

$$\begin{gathered} \{F, G\} = -\{G, F\}, \\ \{F, GH\} = \{F, G\}H + H\{F, G\}, \\ \{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \end{gathered} \tag{9}$$

the last one is the Jacobi identity. For the symplectic structure Eq. (7), we have

$$\{F, G\} \equiv \frac{\partial F}{\partial x^1}\frac{\partial G}{\partial x^2} - \frac{\partial F}{\partial x^2}\frac{\partial G}{\partial x^1}, \tag{10}$$

Equation (9) can be verified directly by using Eq. (10). Having a symplectic structure or Poisson bracket, given a function called to Hamiltonian (being not explicitly dependent on time variable t) we can define Hamilton equation

$$\frac{dx^m(t)}{dt} = \{x^m, H(x(t))\}, \tag{11}$$

e.g. given (10), we have

$$\begin{aligned} \frac{dx^1(t)}{dt} &= \frac{\partial H(x^1, x^2)}{\partial x^2}, \\ \frac{dx^2(t)}{dt} &= -\frac{\partial H(x^1, x^2)}{\partial x^1}. \end{aligned} \tag{12}$$

The following relation can be shown by using the first property in Eq. (9),

$$\frac{d}{dt}H(x^1, x^2) = \frac{dx^1(t)}{dt}\frac{\partial H}{\partial x^1} - \frac{dx^2}{dt}\frac{\partial H}{\partial x^2} \equiv 0. \tag{13}$$

Now we have all tools to treat the problem at hand by using Hamilton mechanics. The complex plane in k has a natural symplectic structure

$$\Omega = dk_R \wedge dk_I, \tag{14}$$

where k_R and k_I denote the real and imaginary part of complex k, respectively. Now identifying $\operatorname{Im}\Phi$ as a Hamiltonian, we have

$$\begin{aligned} \frac{dk_R(t)}{dt} &= \frac{\partial \operatorname{Im}\Phi(k_R, k_I)}{\partial k_I}, \\ \frac{dk_I(t)}{dt} &= -\frac{\partial \operatorname{Im}\Phi(k_R, k_I)}{\partial k_R} \end{aligned} \tag{15}$$

where "time variable" is now just a parameter describing curve. Furthermore, Eq. (13) implies along the curve determined by Hamilton equation,

$$\operatorname{Im}\Phi = const. \tag{16}$$

We give a geometric interpretation. The isovalue curve $\operatorname{Im}\Phi = 0$ is expressed by

$$\operatorname{Im}\Phi(k_R, k_I) = 0, \tag{17}$$

its normal vector is given by

$$\nabla[\operatorname{Im}\Phi] = \left(\frac{\partial \operatorname{Im}\Phi}{\partial k_R}, \frac{\partial \operatorname{Im}\Phi}{\partial k_I}\right). \tag{18}$$

Equations (13) and (18) mean that the trajectory or curve determined by the Hamilton Eq. (15) is tangent automatically to the normal vector. Given an initial point on the curve $\operatorname{Im}\Phi = 0$, then Eq. (13) assures the trajectory on the curve for all the time.

We have thus shown that the problem finding the curve $\mathrm{Im}\,\Phi = 0$ is equivalent to solve a 1D Hamilton equation, moreover Eq. (18) tells us that if one start with an initial position with $\mathrm{Im}\,\Phi = 0$, then this relation preserves automatically in the calculation. By Hamilton equation, the real part of the phase function, $\mathrm{Re}\,\Phi$ can be calculated at the same time

$$\frac{d\,\mathrm{Re}\,\Phi}{dt} = \{\mathrm{Re}\,\Phi, \mathrm{Im}\,\Phi\}, \tag{19}$$

and the eigenvalues can be identified directly by $\mathrm{Re}\,\Phi = \mathrm{m}\pi$. In other words, we only need an initial point; the remaining is a 1D problem. This is why we refer the present method to as $1.0 + 0.1$D. To our knowledge, no such similar method has been reported. The method presented above find the curve $\mathrm{Im}\,\Phi = 0$ automatically, this improve the method proposed by Ref. 3.

4. Numerical Simulation

In order to show the usefulness of the present method above, in this section, we consider two examples. Both examples are Pekeris waveguide with different bottom: one for fluid and the other is an elastic bottom. An isovelocity layer of water lies over the bottoms. The calculation is programmed by using the software **MatLab**.

Table 1 shows parameters used in the calculation for the fluid bottom. The reflection coefficient of the sea surface is set to -1, and the reflection coefficient of the fluid bottom can be found in a textbook. We here concentrate to trapped modes with phase speeds less than the sound speed of bottom; these modes are most interested in practical applications for long distance propagation. Figure 1(a) gives the calculated line $\mathrm{Im}(\Phi) = 0$ in the complex k plane. Figure 1(b) gives the corresponding $\mathrm{Re}(\Phi)$ and cycles locate at the position of eigenvalues when $\mathrm{Re}(\Phi) = m\pi$. The mode 1 is the lower right corner and mode 9 is the most left one. The curve $\mathrm{Im}(\Phi) = 0$ is calculated by using a Runge-Kutta difference code (**Matlab**) of Eq. (17) starting with the initial value $k_b = \omega/\mathrm{c}_b$, in a step of 10^{-5}. A comparison with the results of KrakanC is given in Figs. 1(c) and 1(d). In the WKB approach, sound field is determined completely by the normal mode wave numbers. The lower panels show a comparison of the transmission Loss (TL) (from $r = 1.0\,\mathrm{km}$ to $r = 100\,\mathrm{km}$) (taking $z_s = z_r = 50\,\mathrm{m}$) of the fluid bottom calculated in the WKB approach and that calculated based on KrakanC with good agreement.

Similarly, Table 2 shows parameters for the elastic bottom. Both longitudinal and shear waves with attenuation. Figure 2 gives the corresponding calculated results.

Table 1. Parameters for the Pekeris waveguide of fluid bottom ($f = 100\,\mathrm{Hz}$).

Layer	Density kg/m^3	Thickness m	Wave Type	Wave speed m/s	Attenuation dB/λ
Water	1000	100	Pressure	1500	0.0
Sediment	1800	Half space	Compressional	2000	0.2

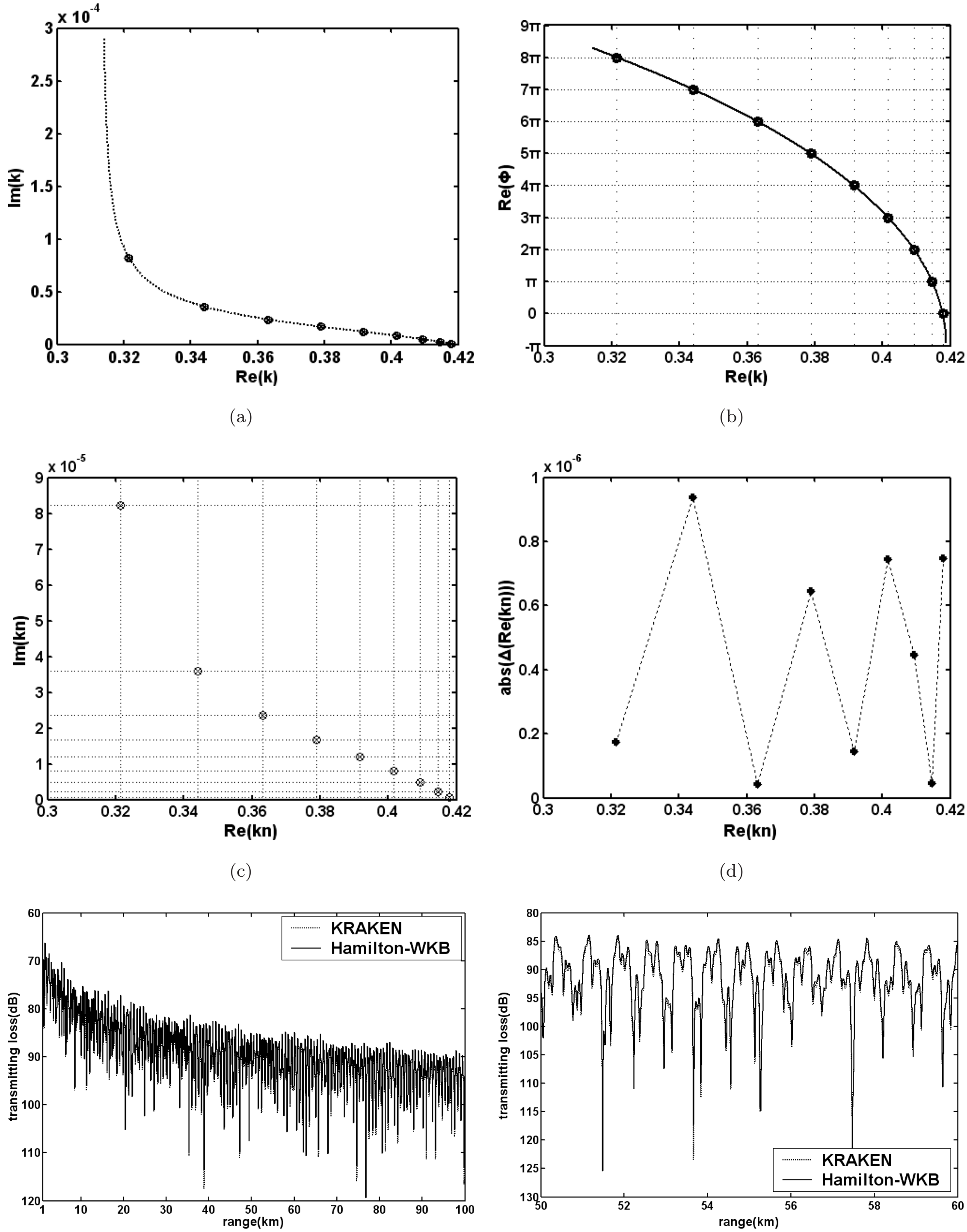

Fig. 1. Calculated results for fluid bottom: (a) curve $\mathrm{Im}\,(\Phi) = 0$ in the complex k-plane, (b) the corresponding $\mathrm{Re}\,(\Phi)$ value. $\circ$ = eigenvalues. (c) eigenvalues k_n: $\otimes$ = Hamiltonian, $*$ = KrakenC (d) $\mathbf{abs}[\Delta\mathbf{Rek}]$: difference of eigen-horizontal wave numbers between the Hamilton Solver and KrakenC (e). Right TL from 1 km–100 km, (f) TL from 50 km–60 km.

Table 2. Parameters for the Pekeris waveguide of elastic bottom ($f = 100\,\text{Hz}$).

Layer	Density kg/m^3	Thickness m	Wave Type	Wave speed m/s	Attenuation dB/λ
Water	1000	100	Pressure	1500	0.0
Sediment	1800	Half space	Longitudinal	2000	0.2
			Shear	800	0.2

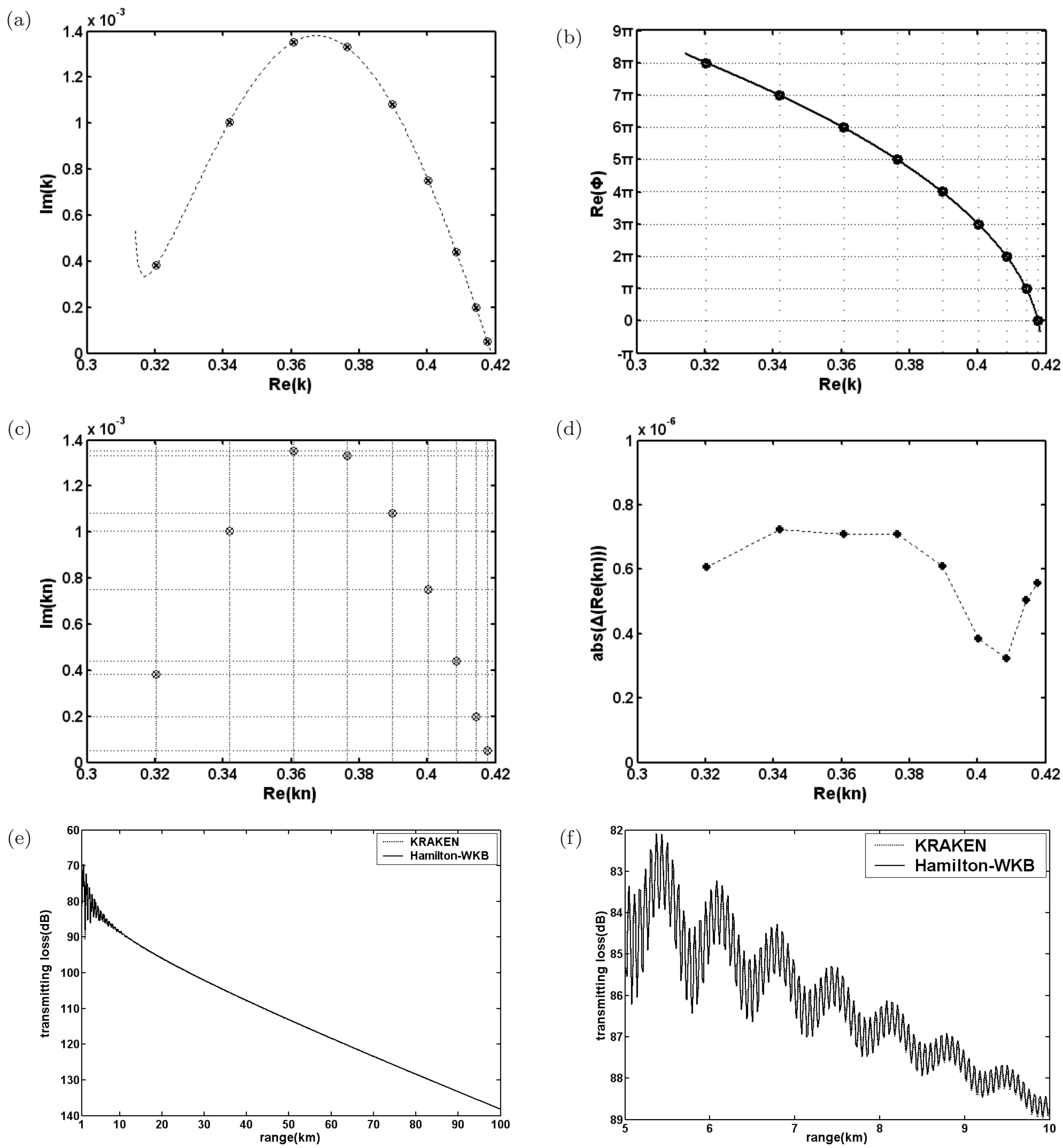

Fig. 2. Calculated results for fluid bottom: (a) curve Im $(\Phi) = 0$ in the complex k-plane, (b) the corresponding Re(Φ) value. $\circ$ = eigenvalues. (c) eigenvalues k_n: $\otimes$ = Hamiltonian, $*$ = KrakenC (d) **abs[ΔRek]**: difference of eigen-horizontal wave numbers between the Hamilton Solver and KrakanC. (e) TL from 1 km to 100 km. (f) TL from 5 km to 10 km.

5. Conclusion

In this short note, we have presented a new method for finding normal mode eigenvalues for underwater acoustic propagation problem. Beside of some initial points must be specified, this method is a true 1D method for finding eigenvalues. It is very interesting to extend the present method to find eigenvalues for frequency broadband and range-dependent environments. It become to constrained Hamilton systems, somewhat complicated than that considered in this note, we shall consider these problems in near future.

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