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Analytic study on optical solitons in media with Kerr- and parabolic-law nonlinearities

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The nonlinear optical transmission equation in the presence of detuning, inter-modal dispersion, time- and space-modulated external potential and dissipation has been investigated. Optical solitons, triangular periodic solutions and Jacobian elliptic periodic solutions are constructed with the aids of the self-similarity transformation and Jacobian elliptic equation expansion method. The conditions for analytical solutions to exist are obtained. The Kerr-law as well as the parabolic-law nonlinear media is considered.

Keywords: nonlinear Schrödinger equation; Jacobian elliptic equation; self-similarity transformation; soliton-like; Kerr-law, parabolic-law.

1. Introduction

The nonlinear evolution equations appear in many areas of natural sciences. In fluid mechanics, the Korteweg–de Vries (KdV) equation [1-2] describes surface waves in shallow water. In nonlinear optics, the nonlinear Schrödinger equation (NLSE) [3-12] describes the propagation of optical pulse in Kerr-law or (and) non-Kerr-law media. In condensed matter physics, the Gross-Pitaevskii (GP) equation [13-19] describes the dilute atomic gases for Bose-Einstein condensates (BECs) at near-zero-temperature. In magnetism, the sine–Gordon (SG) equation [20-22] describes the propagation of fluxons in Josephson junctions, and so on.

With the rapid development of optical communication systems, information is encoded into optical pulses and transmitted through optical fibers over long distances [9,10,12], optical solitons as a carrier of information, information capacity and efficiency in transmission are irreplaceable position due to the fact that they retain shapes in the procedure of transmission and collision [6,9,12], so the

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optical solitons are often applied to optical communication area. In the presence of detuning, inter-modal dispersion, time- and space-modulated external potential and dissipation, the dynamics of the propagation of spatial optical soliton through an optical fiber is ruled by

$$iu_t + au_{xx} + bF(|u|^2)u - k^2u - i\lambda u_x = -v(x,t)u - i\Gamma(x,t)u\,. \tag{1}$$

In Equation (1), the unknown variable $u(x,t)$ represents the complex wave envelope of the electric field in a comoving frame, x and t represent the spatial and temporal variables. On the left-hand side of Equation (1), the first term, the second term, the third term, the fourth term and the fifth term are respectively, the time evolution, the group velocity dispersion (GVD), the nonlinear term, the detuning and inter-modal dispersion (IMD) respectively. Here a, F, k and λ are respectively, the coefficient of GVD, the real function [7, 9] (it determines the form of nonlinear term), the coefficient of detuning [7], the coefficient of IMD respectively. On the right-hand side of Equation (1), the first and second terms represent external potential and dissipation (or gain), while $v(x,t)$ and $\Gamma(x,t)$ are the time- and space-dependent coefficients of external potential and dissipation ($\Gamma > 0$) or gain ($\Gamma < 0$), for a Gaussian dissipation $\Gamma = e^{-\delta(x)t^2/\sigma^2}/\sqrt{\pi\sigma}$, where σ and $\delta(x)$ define the width of dissipation and the effect of the space.

Very recently, many nonlinear media such as power-, parabolic-, dual-power-, log-, square-root-, saturating-, exponential-, higher-order polynomial- law media and so on, have been studied widely [7,9,10,12]. In this paper, we consider the Kerr- and parabolic-law media. The Kerr-law nonlinearity (cubic nonlinearity) arises in an optical pulse propagating in optical fiber faces nonlinear responses from nonharmonic motion of electrons bound in molecules [10], the expression of F has the form of $F(|u|^2) = |u|^2$, the corresponding refractive index of medium is $n = n_0 + n_2 I$, while n_0 is the linear refractive index of the medium, $n_2 = 3\chi^{(3)}/8n_0$ is the third-order nonlinear refractive index coefficient, $\chi^{(3)}$ is third-order susceptibility. The parabolic law nonlinearity (cubic-quintic nonlinearity) originates from the nonlinear interaction between Langmuir waves and electrons [9,10], the expression of F has the form of $F(|u|^2) = |u|^2 + s|u|^4$, the corresponding refractive index of medium is $n = n_0 + n_2 I + n_4 I^2$, here $n_4 = 5\chi^{(5)}/16n_0$ is the fifth-order nonlinear refractive

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index coefficients, $\chi^{(5)}$ is the fifth-order susceptibility.

The optical solitons are the result of a delicate balance the linear effects (GVD, detuning, external potential and dissipation) with the nonlinearities (Kerr-law nonlinearity, parabolic law nonlinearity, and IMD) in this model. The main objective of this paper is to seek exact solutions (especially the solitons) of Equation (1) with cubic and cubic-quintic nonlinearities. The tasks of constructing analytical solutions to nonlinear evolution equations have attracted much attention, many methods are proposed, for example the Jacobian elliptic equation (JEE) expansion method [6,8,12], the hirota bilinear method [23,24], the self-similar method [25], the solitary wave ansatz method [7,26]. But generally speaking, Equation (1) is non-integrable. In recently years, various reduced and extended equations from Equation (1) have attracted a lot of interest. Masemola et al. [7] studied the NLSE with linear attenuation and detuning, and obtained the bright solitons, Zhu and Ma [11] studied the NLSE with parabolic law nonlinearity, and obtained the bright and dark solitons, Toikka et al. [13] studied the Radial Radial GP equation with an external potential, and obtained the ring soliton-like solutions, and so on [27-29]. In this paper, we use the self-similarity transformation and JEE expansion method to study Equation (1) with Kerr- and parabolic- law nonlinearity. As a consequence, Jacobian elliptic periodic solutions are got, in particular, these Jacobian elliptic periodic solutions degenerate to the solitons (when the modulus $m \to 1$) and the triangular periodic solutions (when the modulus $m \to 0$). In addition, we get the constraint conditions for explicit solutions to exist.

2. Solitons to Equation (1) with Kerr-law nonlinearity

In this section, we construct explicit solutions of Equation (1) with cubic nonlinearity, in this case, Equation (1) reduces to

$$iu_t + au_{xx} + b|u|^2 u - k^2 u - i\lambda u_x = -v(x,t)u - i\Gamma(x,t)u . \tag{2}$$

we first make the following self-similarity transformation

$$u(x,t) = A(x,t)B[\eta(x,t)]\exp[i\varphi(x,t)], \tag{3}$$

and $B(\eta)$ satisfies

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$$\frac{d^2B}{d\eta^2} = h_1 B + h_3 B^3 , \tag{4}$$

where $A(x,t)$, $\eta(x,t)$ and $\varphi(x,t)$ are the real functions of space and time to be determined later, Equation (4) is the famous Jacobian elliptic equation [6,8], the solutions are listed (see Table 1), while m ($0 \le m \le 1$) is the modulus, the cases of $m = 0$ and $m = 1$ are also considered.

Table 1. Solutions to the Jacobian elliptic equation (4) with different values of h_1 and h_3.

	h_1	h_3	$B(\eta,m)$	$B(\eta,0)$	$B(\eta,1)$
1	$-(1+m^2)$	$2m^2$	$sn(\eta,m)$	$\sin(\eta)$	$\tanh(\eta)$
2	$2m^2-1$	$-2m^2$	$cn(\eta,m)$	$\cos(\eta)$	$\sec h(\eta)$
3	$2-m^2$	-2	$dn(\eta,m)$	1	$\sec h(\eta)$
4	$-(1+m^2)$	2	$ns(\eta,m)$	$\csc(\eta)$	$\coth(\eta)$
5	$2m^2-1$	$2(1-m^2)$	$nc(\eta,m)$	$\sec(\eta)$	$ch(\eta)$
6	$2-m^2$	$-2(1-m^2)$	$nd(\eta,m)$	1	$ch(\eta)$
7	$2-m^2$	$2(1-m^2)$	$sc(\eta,m)$	$\tan(\eta)$	$sh(\eta)$
8	$2m^2-1$	$-2m^2(1-m^2)$	$sd(\eta,m)$	$\sin(\eta)$	$sh(\eta)$
9	$2-m^2$	2	$cs(\eta,m)$	$\cot(\eta)$	$\csc h(\eta)$
10	$-(1+m^2)$	$2m^2$	$cd(\eta,m)$	$\cos(\eta)$	1
11	$2m^2-1$	2	$ds(\eta,m)$	$\csc(\eta)$	$\csc h(\eta)$
12	$-(1+m^2)$	2	$dc(\eta,m)$	$\sec(\eta)$	1

Substituting Equations (3) and (4) into Equation (2), and separating the real and imaginary parts respectively, we get the following coupled equations

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$$A_t B + A\eta_t B' + 2aA_x\varphi_x B + 2aA\eta_x\varphi_x B' + aA\varphi_{xx}B - \lambda A_x B - \lambda A\eta_x B' + \Gamma AB = 0\,, \tag{5}$$

$$-A\varphi_t B + aA_{xx}B + 2aA_x\eta_x B' + aA\eta_{xx}B' - aA\varphi_x^2 B + bA^3B^3$$

$$+ aA\eta_x^2(h_1 + h_3B^2)B + (v - k^2)AB + \lambda AB\varphi_x = 0\,. \tag{6}$$

Then using the homogeneous balance principle [6], and setting the coefficients of each power of B to zero, one gets

$$A_t + 2aA_x\varphi_x + aA\varphi_{xx} - \lambda A_x + \Gamma A = 0\,, \tag{7}$$

$$\eta_t + 2a\eta_x\varphi_x - \lambda\eta_x = 0\,, \tag{8}$$

$$-A\varphi_t + aA_{xx} - aA\varphi_x^2 + aA\eta_x^2 h_1 + (v - k^2)A + \lambda A\varphi_x = 0\,, \tag{9}$$

$$2A_x\eta_x + A\eta_{xx} = 0\,. \tag{10}$$

$$bA^2 + a\eta_x^2 h_3 = 0\,. \tag{11}$$

Solving the Equations (7)-(11) above, the unknown functions $A(x,t)$, $\eta(x,t)$ and $\varphi(x,t)$ are obtained

$$A(x,t) = -\left(\frac{ah_3}{b}\right)^{\frac{1}{6}}\left[\frac{1}{6\Gamma(t)}\right]^{\frac{1}{3}}, \tag{12}$$

$$\eta(x,t) = \left[\frac{b}{6ah_3\Gamma(t)}\right]^{\frac{1}{3}}x + f_1\,, \tag{13}$$

$$\varphi(x,t) = \frac{\lambda}{2a}x + \frac{\Gamma(t)}{2a}x^2 + f_2(t)\,, \tag{14}$$

and the external potential $v(x,t)$ is given by

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$$v(x,t)=k^2-\frac{\lambda^2}{4a}+f_2'(t)-h_1a^{\frac{1}{3}}[\frac{b}{6h_3\Gamma(t)}]^{\frac{2}{3}}+\frac{4\Gamma^2(t)}{a}x^2, \tag{15}$$

where f_1 is an arbitrary real constant, and $f_2(t)$ is an arbitrary real function of time.

We find that three conditions are needed for analytical solutions to exist, the first one is that the dissipation is only the time-modulated, and the second one is that the coefficient of dissipation is non-zero, i.e. $\Gamma=\Gamma(t)\neq 0$, the last one is that the external potential has the form of Equation (15).

Inserting Equations (12)-(14) into Equation (3), and utilizing the solutions of Jacobian elliptic equation (4), exact Jacobian elliptic periodic solutions to Equation (1) with Kerr-law nonlinearity are constructed. These solutions and corresponding external potential are listed in Table 2.

Table 2. Jacobian elliptic periodic solutions to the Equation (2) with the corresponding external potential.

$v(x,t)$	$u(x,t)$
$k^2-\frac{\lambda^2}{4a}+f_2'+\frac{(1+m^2)a^{\frac{1}{3}}b^{\frac{2}{3}}}{(12m^2\Gamma)^{\frac{2}{3}}}+\frac{4\Gamma^2x}{a}$	$-(\frac{2am^2}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}sn\{(\frac{b}{12am^2\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$
$k^2-\frac{\lambda^2}{4a}+f_2'-\frac{(2m^2-1)a^{\frac{1}{3}}b^{\frac{2}{3}}}{(12m^2\Gamma)^{\frac{2}{3}}}+\frac{4\Gamma^2}{a}$	$-(-\frac{2am^2}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}cn\{(-\frac{b}{12am^2\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$
$k^2-\frac{\lambda^2}{4a}+f_2'-\frac{(2-m^2)a^{\frac{1}{3}}b^{\frac{2}{3}}}{(12\Gamma)^{\frac{2}{3}}}+\frac{4\Gamma^2x^2}{a}$	$-(-\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}dn\{(-\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$

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$$k^2-\frac{\lambda^2}{4a}+f_2'+\frac{(1+m^2)a^{\frac{1}{3}}b^{\frac{2}{3}}}{(12\Gamma)^{\frac{2}{3}}}+\frac{4\Gamma^2x^2}{a}$$

$$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}ns\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$$

$$k^2-\frac{\lambda^2}{4a}+f_2'-\frac{(2m^2-1)a^{\frac{1}{3}}b^{\frac{2}{3}}}{[12(1-m^2)\Gamma]^{\frac{2}{3}}}+\frac{4\Gamma^2x}{a}$$

$$-[\frac{2a(1-m^2)}{b}]^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}nc\{[\frac{b}{12a(1-m^2)\Gamma}]^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}}$$

$$k^2-\frac{\lambda^2}{4a}+f_2'-\frac{(2-m^2)a^{\frac{1}{3}}b^{\frac{2}{3}}}{[12(1-m^2)\Gamma]^{\frac{2}{3}}}+\frac{4\Gamma^2x}{a}$$

$$-[\frac{2a(m^2-1)}{b}]^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}nd\{[\frac{b}{12a(m^2-1)\Gamma}]^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}}$$

$$k^2-\frac{\lambda^2}{4a}+f_2'-\frac{(2-m^2)a^{\frac{1}{3}}b^{\frac{2}{3}}}{[12(1-m^2)\Gamma]^{\frac{2}{3}}}+\frac{4\Gamma^2x}{a}$$

$$-[\frac{2a(1-m^2)}{b}]^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}sc\{[\frac{b}{12a(1-m^2)\Gamma}]^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}}$$

$$k^2-\frac{\lambda^2}{4a}+f_2'-\frac{(2m^2-1)a^{\frac{1}{3}}b^{\frac{2}{3}}}{[12m^2(1-m^2)\Gamma]^{\frac{2}{3}}}+$$

$$-[\frac{2am^2(m^2-1)}{b}]^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}sd\{[\frac{b}{12am^2(m^2-1)\Gamma}]^{\frac{1}{3}}x+f_1\}e$$

$$k^2-\frac{\lambda^2}{4a}+f_2'-\frac{(2-m^2)a^{\frac{1}{3}}b^{\frac{2}{3}}}{(12\Gamma)^{\frac{2}{3}}}+\frac{4\Gamma^2x}{a}$$

$$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}cs\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$$

$$-(\frac{2am^2}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}cd\{(\frac{b}{12am^2\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$$

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$$k^2-\frac{\lambda^2}{4a}+f_2'+\frac{(1+m^2)a^{\frac{1}{3}}b^{\frac{2}{3}}}{(12m^2\Gamma)^{\frac{2}{3}}}+\frac{4\Gamma^2x^2}{a}$$

$$k^2-\frac{\lambda^2}{4a}+f_2'-\frac{(2m^2-1)a^{\frac{1}{3}}b^{\frac{2}{3}}}{(12\Gamma)^{\frac{2}{3}}}+\frac{4\Gamma^2}{a}$$

$$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}ds\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$$

$$k^2-\frac{\lambda^2}{4a}+f_2'+\frac{(1+m^2)a^{\frac{1}{3}}b^{\frac{2}{3}}}{(12\Gamma)^{\frac{2}{3}}}+\frac{4\Gamma^2x}{a}$$

$$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}dc\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$$

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3. Solitons to Equation (1) with parabolic law nonlinearity

In this section, we start looking for the analytical solutions to Equation (1) with cubic-quintic nonlinearity, we consider a medium with the refractive index in form of $n(x,t)=n_0+n_2I+n_4I^2$, then Equation (1) reduces to

$$iu_t+au_{xx}+b|u|^2u+c|u|^4u-k^2u-i\lambda u_x=-v(x,t)u-i\Gamma(x,t)u\,, \tag{16}$$

Following the same ideas in section 2, a difference is that here we use the B^6 model [30] to replace the Jacobian elliptic equation (4) as the expansion equation, the B^6 model reads

$$\frac{d^2B}{d\eta^2}=h_1B+h_3B^3+h_5B^5\,, \tag{17}$$

As a result, the expressions of $A(x,t)$, $\eta(x,t)$, $\varphi(x,t)$ and $v(x,t)$ are the same as in Equations (12)-(15), but we need to add a new condition for analytical solutions to exist, that is

$$c=-h_5(\frac{1}{a})^{\frac{1}{3}}(\frac{b}{h_3})^{\frac{4}{3}}[6\Gamma(t)]^{\frac{2}{3}}\,. \tag{18}$$

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Now we mainly focus on solving the Equation (17). Using the homogeneous balance principle and the Jacobian elliptic equations (JEEs) expansion method, the starting hypothesis is

$$B(\eta)=\frac{U(\eta)}{\sqrt{fU^2(\eta)+g}}, \tag{19}$$

where f and g are real constants to be determined later, and $U(\eta)$ is the solution (see Table 3) of the JEEs,

$$\frac{d^2U}{d\eta^2}=l_1U+l_3U^3 \text{ and } (\frac{dU}{d\eta})^2=l_0+l_1U^2+\frac{l_3}{2}U^4. \tag{20}$$

Table 3. Solutions to the JEEs (20) with different values of l_0, l_1 and l_3.

	l_0	l_1	l_3	$U(\eta,m)$	$m=0$	$m=1$
1	1	$-(1+m^2)$	$2m^2$	$sn(\eta,m)$	$\sin(\eta)$	$\tanh(\eta)$
2	$1-m^2$	$2m^2-1$	$-2m^2$	$cn(\eta,m)$	$\cos(\eta)$	$\sec h(\eta)$
3	$-(1-m^2)$	$2-m^2$	-2	$dn(\eta,m)$	1	$\sec h(\eta)$
4	m^2	$-(1+m^2)$	2	$ns(\eta,m)$	$\csc(\eta)$	$\coth(\eta)$
5	$-m^2$	$2m^2-1$	$2(1-m^2)$	$nc(\eta,m)$	$\sec(\eta)$	$ch(\eta)$
6	-1	$2-m^2$	$-2(1-m^2)$	$nd(\eta,m)$	1	$ch(\eta)$
7	1	$2-m^2$	$2(1-m^2)$	$sc(\eta,m)$	$\tan(\eta)$	$sh(\eta)$
8	1	$2m^2-1$	$-2m^2(1-m^2)$	$sd(\eta,m)$	$\sin(\eta)$	$sh(\eta)$
9	$1-m^2$	$2-m^2$	2	$cs(\eta,m)$	$\cot(\eta)$	$\csc h(\eta)$
10	1	$-(1+m^2)$	$2m^2$	$cd(\eta,m)$	$\cos(\eta)$	1
11	$-m^2(1-m^2)$	$2m^2-1$	2	$ds(\eta,m)$	$\csc(\eta)$	$\csc h(\eta)$

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12	m^2	$-(1+m^2)$	2	$dc(\eta,m)$	$\sec(\eta)$	1

From Equations (17), (19) and (20), we get

$$f=\frac{h_3(l_1-h_1)}{2(l_1-h_1)^2+3l_0l_3-4l_1(l_1-h_1)}, \tag{21}$$

$$g=\frac{3l_0h_3}{2(l_1-h_1)^2+3l_0l_3-4l_1(l_1-h_1)}, \tag{22}$$

and the constraint condition on the constants l_i (i=0,1,3) and h_j (j=1,3,5) is given by

$$h_3^2[\frac{9l_0l_3}{(l_1-h_1)^2}-6l_1+2]+2h_5[2(l_1-h_1)+\frac{3l_0l_3}{l_1-h_1}-4l_1]^2=0, \tag{23}$$

Finally, Substituting Equations (12)-(14), (19), (21) and (22) into Equation (3), and combining the solutions of JEEs, analytical Jacobian elliptic periodic solutions of Equation (16) are constructed.

4. Results and discussion

In the previous sections, exact Jacobian elliptic periodic solutions to Equation (1) with Kerr-law nonlinearity and parabolic law nonlinearity are got. It's worth mentioning that the triangular periodic solutions will appear when the modulus $m\to 0$, and the bell- and kink-type solitons will appear when the modulus $m\to 1$, Without loss of generality, as a example, let's discuss the case of Equation (1) with Kerr-law nonlinearity, the singular solutions (which are expressed in terms of the singular functions) are listed in Table 4. In addition, we are focusing on the soliton-like solutions, the bright-type soliton and dark-type soliton are listed in Table 5.

Table 4. Singular solutions to the Equation (2) with the corresponding external potential.

$V(x,t)$	$u(x,t)$
$k^2-\frac{\lambda^2}{4a}+f_2'+a^{\frac{1}{3}}(\frac{b}{12\Gamma})^{\frac{2}{3}}+\frac{4\Gamma^2x^2}{a}$	$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}\csc\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$

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$k^2-\frac{\lambda^2}{4a}+f_2'-2a^{\frac{1}{3}}(\frac{b}{12\Gamma})^{\frac{2}{3}}+\frac{4\Gamma^2x^2}{a}$	$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}\sec\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$
	$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}\tan\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$
	$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}\cot\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$
$k^2-\frac{\lambda^2}{4a}+f_2'+2a^{\frac{1}{3}}(\frac{b}{12\Gamma})^{\frac{2}{3}}+\frac{4\Gamma^2x^2}{a}$	$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}\coth\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$
$k^2-\frac{\lambda^2}{4a}+f_2'-a^{\frac{1}{3}}(\frac{b}{12\Gamma})^{\frac{2}{3}}+\frac{4\Gamma^2x^2}{a}$	$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}\operatorname{csch}\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$

Table 5. Soliton-like solutions to the Equation (2) with the corresponding external potential.

$V(x,t)$	$u(x,t)$
$k^2-\frac{\lambda^2}{4a}+f_2'-a^{\frac{1}{3}}(\frac{b}{12\Gamma})^{\frac{2}{3}}+\frac{4\Gamma^2x^2}{a}$	$-(-\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}\operatorname{sech}\{(-\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$
$k^2-\frac{\lambda^2}{4a}+f_2'+2a^{\frac{1}{3}}(\frac{b}{12\Gamma})^{\frac{2}{3}}+\frac{4\Gamma^2x^2}{a}$	$-(\frac{2a}{b})^{\frac{1}{6}}(\frac{1}{6\Gamma})^{\frac{1}{3}}\tanh\{(\frac{b}{12a\Gamma})^{\frac{1}{3}}x+f_1\}e^{i(\frac{\lambda}{2a}x+\frac{\Gamma}{2a}x^2+f_2)}$

Finally, in this section, we are focusing on the soliton-like solutions to the system with cubic-quintic nonlinearity, the dark- and bright-type solitons are given as follows:

$$u(x,t)=-\frac{(\frac{ah_3}{b})^{\frac{1}{6}}[\frac{1}{6\Gamma(t)}]^{\frac{1}{3}}\tanh\{[\frac{b}{6ah_3\Gamma(t)}]^{\frac{1}{3}}x+f_1\}}{\sqrt{f\tanh^2\{[\frac{b}{6ah_3\Gamma(t)}]^{\frac{1}{3}}x+f_1\}+g}}\exp\{i[\frac{\lambda}{2a}x+\frac{\Gamma(t)}{2a}x^2+f_2(t)]\},\tag{24}$$

$$u(x,t)=-\frac{(\frac{ah_3}{b})^{\frac{1}{6}}[\frac{1}{6\Gamma(t)}]^{\frac{1}{3}}\operatorname{sech}\{[\frac{b}{6ah_3\Gamma(t)}]^{\frac{1}{3}}x+f_1\}}{\sqrt{f\operatorname{sech}^2\{[\frac{b}{6ah_3\Gamma(t)}]^{\frac{1}{3}}x+f_1,m\}+g}}\exp\{i[\frac{\lambda}{2a}x+\frac{\Gamma(t)}{2a}x^2+f_2(t)]\},\tag{25}$$

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5. Conclusion

In this paper, the NLSE with detuning, inter-modal dispersion, external potential, dissipation is studied analytically. In Kerr-law nonlinear media, the self-similarity transformation and JEE expansion method are used, and in parabolic-law nonlinear media, the self-similarity transformation and B^6 equation expansion method are applied, the solutions of that B^6 equation are got by the JEEs expansion method and homogeneous balance principle. Exact soliton-like solutions are constructed. In addition, there constraint conditions for the existence of analytical solutions are obtained, the dissipation is only the function of time, the dissipation is non-zero, the external potential and quintic nonlinearity satisfy Equations (15) and (18).

The self-similarity transformation and JEE expansion method are very powerful methods to solve nonlinear equations and are worthy of studying further. In future studies, we will consider other types of nonlinearities in the system (1), such as the power law, dual-power law as well as the log law.

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