Results. Math. 63 (2013), 451–465

1422-6383/13/010451-15
published online September 30, 2011
DOI 10.1007/s00025-011-0208-8

Results in Mathematics

Using Time Scales to Study Multi-Interval Sturm–Liouville Problems with Interface Conditions

Qingkai Kong and Qi-Ru Wang

Abstract. We consider a Sturm–Liouville problem defined on multiple intervals with interface conditions. The existence of a sequence of eigenvalues is established and the zero counts of associated eigenfunctions are determined. Moreover, we reveal the continuous and discontinuous nature of the eigenvalues on the boundary condition. The approach in this paper is different from those in the literature: We transfer the Sturm–Liouville problem with interface conditions to a Sturm–Liouville problem on a time scale without interface conditions and then apply the Sturm–Liouville theory for equations on time scales. In this way, we are able to investigate the problem in a global view. Consequently, our results cover the cases when the potential function in the equation is not strictly greater than zero and when the domain consists of an infinite number of intervals.

Mathematics Subject Classification (2010). 34B24, 34N05.

Keywords. Multi-interval Sturm–Liouville problems, interface conditions, time scales, existence of eigenvalues, dependence on the problem.

1. Introduction

The Sturm–Liouville theory plays an important role in mathematical physics. In the past decades, there has been growing interest in Sturm–Liouville problems (SLPs) with interface (or transmission) conditions. Such research is motivated by the problems of heat and mass transfer, string vibration with loaded point masses, and the thermal conduction for a strip with piecewise

This paper is supported by the NNSF of China (No. 10971231).

continuous coefficients, etc., see [14,16,18,19]. For the recent development of SLPs with interface conditions, the reader is referred to [3,8,9,14,15,17,20–22] and the references therein.

In this paper, we consider the SLP on multiple intervals consisting of the equation

$$(p(t)x')' + q(t)x = \lambda w(t)x \quad \text{on } I := \bigcup_{i=1}^{j} [a_i, b_i], \tag{1.1}$$

where $j \leq \infty$ and $b_i < a_{i+1}$ for $i = 1, \ldots, j-1$, with the non-self-adjoint interface conditions that for $i = 1, \ldots, j-1$,

$$\begin{aligned} x(a_{i+1}) &= x(b_i) + l_i \quad (px')(b_i), \\ (px')(a_{i+1}) &= h_i\, x(b_i) + k_i \quad (px')(b_i); \end{aligned} \tag{1.2}$$

and boundary conditions (BCs)

$$\begin{aligned} \cos\alpha\, x(a_1) - \sin\alpha\, (px')(a_1) &= 0, \quad 0 \leq \alpha < \pi, \\ \cos\beta\, x(b_j) - \sin\beta\, (px')(b_j) &= 0, \quad 0 < \beta \leq \pi; \end{aligned} \tag{1.3}$$

where l_i, h_i, and k_i are constants, and the quasi-derivative px' at a_i and b_i means the appropriate one-sided quasi-derivative.

This problem and its variations have been discussed by many authors, see [8,9,17,20–22], mainly for the case with $j < \infty$. Under certain assumptions, nice results have been obtained on the structure of the spectrum and properties of the eigenfunctions. The main approach in their work is to construct a direct sum of Hilbert spaces of functions defined on each interval $[a_i, b_i], i = 1, \ldots, j$, or one Hilbert space of functions with the inner product as a sum of inner products defined on each interval $[a_i, b_i], i = 1, \ldots, j$; and then apply the spectral theory for self-adjoint Sturm–Liouville operators. However, there is a major limitation to this method: To form such weighted Hilbert spaces with weight w, there must be a sign requirement for the function w. In fact, all existing results are under the restriction that $w(t)$ is strictly greater than zero a.e. on I, and many of them require $w(t) \equiv 1$. Although some special cases of the results may be potentially extended, using substantially more sophisticated arguments, to some cases where $w(t)$ is allowed to be zero on a subset of I with positive Lebesgue measure; we will never expect that this method will work when $w(t) \equiv 0$ on a whole interval $[a_i, b_i]$ for some $i \in \{1, \ldots, j\}$.

Here, we study the spectrum of SLP (1.1)–(1.3) by a different approach: we first change SLP (1.1)–(1.3) to a SLP on a time scale and then apply the Sturm–Liouville theory for equations on time scales. In this way, the interface condition are absorbed into the equation and hence the resulting SLP on time scale becomes a two-point SLP without interface conditions. Therefore, we are able to investigate the problem in a global view, i.e., using an inner product defined on one time scale interval rather than a combination of inner products defined on each $[a_i, b_i], i = 1, \ldots, j$, separately. Consequently, the assumption for w is much released. Actually, w can be zero on any subset of I as long as its Lebesgue measure is less than $\sum_{i=0}^{j}(b_i - a_i)$. In particular, our results allow that $w(t) \equiv 0$ on all intervals $[a_i, b_i]$ except one where $w(t) \geq 0$ and $w(t) > 0$ on a subset with positive Lebesgue measure.

Our main purpose is to establish the existence of a sequence of real eigenvalues of SLP (1.1)–(1.3) and determine the zero counts of associated eigenfunctions including the case when $j = \infty$. Moreover, we will reveal the continuous and discontinuous nature of the eigenvalues on the BC parameters α and β which is difficult to obtain without using time scales. We also hope that the ideas in this paper will provide a foundation for the further study of second order linear and nonlinear problems with interface conditions or jumping conditions.

Finally, we comment that although we assume that $b_i < a_{i+1}$ for $i = 1, \ldots, j-1$ at the beginning, all work in this paper can be easily extended to the case with $b_i = a_{i+1}$ for some or all $i \in \{1, \ldots, j-1\}$. In the latter case, SLP (1.1)–(1.3) becomes a SLP on one interval with interior jumping discontinuous points.

This paper is organized as follows: following this introduction, we present our main results in Sect. 2, their proofs are given in Sect. 4 after related results on time scales are derived in Sect. 3. In Sect. 5, we summarize the basic knowledge on time scales used in this paper just for the convenience of the reader.

2. Main Results

We assume throughout this paper that

(A1) p, q, $w \in C(I), p > 0, w \geq 0$, and $w \not\equiv 0$ a.e. on I;

(A2) $l_i > 0$ and $h_i, k_i \in \mathbb{R}$ such that $k_i - l_i h_i > 0$;

(A3) $\sum_{i=1}^{j}[(b_i - a_i) + l_i p(b_i)] < \infty$.

We note that assumption (A3) is automatically satisfied when $j < \infty$.

A solution x of Eq. (1.1) is said to have a zero at $t \in I$ if $x(t) = 0$, and to have a node between b_i and a_{i+1} for some $i \in \{1, \ldots, j-1\}$ if $x(b_i)x(a_{i+1}) < 0$. A generalized zero of x is then defined as a zero or a node.

Now we present the main results of this paper. The first one is on the existence of eigenvalues and the zero counts of associated eigenfunctions.

Theorem 2.1. *Under the assumptions (A1)–(A3), the non-self-adjoint SLP (1.1)–(1.3) has an infinite number of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ which are real, simple, bounded below, and can be ordered to satisfy that*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \ \lambda_n \to \infty \ \textit{as} \ n \to \infty.$$

Moreover, each eigenfunction u_n *associated with* λ_n *has exactly* n *generalized zeros in* (a_1, b_j).

In the next two theorems, we show how the eigenvalues λ_n, $n \in \mathbb{N}_0$, depend on the BC α and β. By a normalized eigenfunction u of SLP (1.1)–(1.3) we mean a real valued eigenfunction which satisfies that

$$\int_a^b [u(t)]^2 w(t)dt = 1.$$

Theorem 2.2 below reveals the continuous and jump-discontinuous nature of λ_n with respect to (α, β) at different points in $[0, \pi) \times (0, \pi]$, and in Theorem 2.3, it is shown that λ_n is continuously differentiable whenever it is continuous and derivative formulas are obtained.

Theorem 2.2. *For* $n \in \mathbb{N}_0, \lambda_n(\alpha, \beta)$ *is continuous on* $[0, \pi) \times (0, \pi]$. *Moreover,*

- (i) *for each* $\beta \in (0, \pi]$,
$\lim_{\alpha \to \pi^-} \lambda_0(\alpha, \beta) = -\infty$ *and* $\lim_{\alpha \to \pi^-} \lambda_n(\alpha, \beta) = \lambda_{n-1}(0, \beta)$ *for* $n = 1, 2, \ldots$;
- (ii) *for each* $\alpha \in [0, \pi)$,
$\lim_{\beta \to 0^+} \lambda_0(\alpha, \beta) = -\infty$ *and* $\lim_{\beta \to 0^+} \lambda_n(\alpha, \beta) = \lambda_{n-1}(\alpha, \pi)$ *for* $n = 1, 2, \ldots$;
- (iii) *in general, when taking limits within the region* $(0, \pi) \times (0, \pi)$,
$\lim_{(\alpha,\beta) \to (\pi^-, 0^+)} \lambda_n(\alpha, \beta) = -\infty$ *for* $n = 0, 1,$ *and*
$\lim_{(\alpha,\beta) \to (\pi^-, 0^+)} \lambda_n(\alpha, \beta) = \lambda_{n-2}(0, \pi)$ *for* $n = 2, 3, \ldots$.

Theorem 2.3. *For* $n \in \mathbb{N}_0$ *and* $(\alpha, \beta) \in [0, \pi) \times (0, \pi]$, *let* $u_n = u_n(\cdot; \alpha, \beta)$ *be the normalized eigenfunction associated with* $\lambda_n = \lambda_n(\alpha, \beta)$. *Then*

- (i) *for a fixed* $\beta \in (0, \pi], \lambda_n$ *is continuously differentiable in* α *on* $[0, \pi)$ *and*
$$\frac{d\lambda_n}{d\alpha} = -u_n^2(a) - (pu_n')^2(a);$$
- (ii) *for a fixed* $\alpha \in [0, \pi), \lambda_n$ *is continuously differentiable in* β *on* $(0, \pi]$ *and*
$$\frac{d\lambda_n}{d\beta} = u_n^2(b) + (pu_n')^2(b).$$

The following result regarding the monotone properties of λ_n is an immediate consequence of Theorem 2.3.

Corollary 2.1. *For* $n \in \mathbb{N}_0$ *and* $(\alpha, \beta) \in [0, \pi) \times (0, \pi]$, *let* $\lambda_n = \lambda_n(\alpha, \beta)$. *Then*

- (i) *for a fixed* $\beta \in (0, \pi], \lambda_n$ *is strictly decreasing in* α *on* $[0, \pi)$;
- (ii) *for a fixed* $\alpha \in [0, \pi), \lambda_n$ *is strictly increasing in* β *on* $(0, \pi]$.

3. Right-Semi-Definite SLPs on Time Scales

Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$ such that $a < b$ and consider the SLP consisting of the equation

$$-(p(t)y^{\Delta})^{\Delta} + q(t)y^{\sigma} = \lambda w(t)y^{\sigma} \quad \text{on } [a,b] \cap \mathbb{T} \tag{3.1}$$

and the separated BC

$$\begin{cases} R_a(y) := \cos\alpha\, y(\rho(a)) - \sin\alpha\, (py^{\Delta})(\rho(a)) = 0, & \alpha \in [\hat{0}, \hat{\pi}) \\ R_b(y) := \cos\beta\, y(b) - \sin\beta\, (py^{\Delta})(b) = 0, & \beta \in (0, \pi], \end{cases} \tag{3.2}$$

where $\hat{0} = \begin{cases} 0, & \text{if } \rho(a) = a \\ -\tan^{-1}\dfrac{\mu(\rho(a))}{p(\rho(a))}, & \text{if } \rho(a) < a \end{cases}$ and $\hat{\pi} = \pi + \hat{0}$.

In this section we assume that

(I) $p, q, w : [\rho(a), \sigma(b)] \cap \mathbb{T} \to \mathbb{R}$ are rd-continuous, $p > 0, w \geq 0$, and $w \not\equiv 0$ a.e.;

(II) a is right-scattered implies that a is left-scattered, and b is left-scattered implies that b is right-scattered.

The following remark was given in [11]. We include it in this paper for the convenience of the reader.

Remark 3.1. (i) When $\alpha = \hat{0}$ or $\hat{\pi}, R_a(y) = 0$ is equivalent to $y(a) = 0$. In fact, this is obviously true when $\rho(a) = a$. Now we assume $\rho(a) < a$. Then when $\alpha = \hat{0}, R_a(y) = 0$ means that

$$\begin{aligned} 0 &= y(\rho(a)) - \tan\hat{0}\, p(\rho(a))[y(a) - y(\rho(a))]/\mu(\rho(a)) \\ &= y(\rho(a)) + [y(a) - y(\rho(a))] = y(a). \end{aligned}$$

Similarly for $\alpha = \hat{\pi}$.

(ii) Assumption (II) is not essential. In fact, when a is right-scattered but left-dense, $R_a(y) = 0$ becomes

$$R_{\sigma(a)}(y) := \cos\alpha\, y(a) - \sin\alpha\, (py^{\Delta})(a) = 0, \tag{3.3}$$

which is a BC at $\sigma(a)$ satisfying the first condition in (II) with a replaced by $\sigma(a)$. Assume b is left-scattered but right-dense. If $\sin\beta = 0$, then the BVP is defined only on $[a, b] \cap \mathbb{T}$ and hence is not affected by the nature of $\mathbb{T}$ to the right of b; and if $\sin\beta > 0$, then

$$y^{\Delta}(\rho(b)) = \frac{y(b) - y(\rho(b))}{\mu(\rho(b))} \quad \text{and} \quad y^{\Delta}(b) = y'(b).$$

Hence by Eq. (3.1)

$$\begin{aligned} (py^{\Delta})^{\Delta}(\rho(b)) &= [(py')(b) - p(\rho(b))\,(y(b) - y(\rho(b)))\,/\mu(\rho(b))]\,/\mu(\rho(b)) \\ &= [q(\rho(b)) - \lambda w(\rho(b))]y(b). \end{aligned}$$

This together with the relation $(py')(b) = \cot\beta\, y(b)$ provides a new BC at $\rho(b)$ with a new angle γ determined by the values of p, q, w at $\rho(b)$ and β

$$R_{\rho(b)}(y) = \cos\gamma\, y(\rho(b)) - \sin\gamma\,(py^{\Delta})(\rho(b)) = 0,\ \gamma \in (0, \pi],$$

which satisfies the second condition of (II) with b replaced by $\rho(b)$.

For any $x \in \mathbb{R}$ we define the deficiency of x by $\operatorname{def}(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$; and for any subset E of $\mathbb{R}$ we define by $|E|$ the number of points in E.

We first state a result in [11] on the existence of eigenvalues and the number of the generalized zeros of corresponding eigenfunctions of SLP (3.1), (3.2), which is a generalization of Theorems 1 and 8 in Agarwal et al. [2] for the special case where $p \equiv 1$ and $w \equiv 1$.

Proposition 3.1. *In addition to Assumptions (I) and (II), assume $w(t) > 0$ on $[\rho(a), \sigma(b)] \cap \mathbb{T}$. Then the number of eigenvalues of SLP (3.1), (3.2) is given by*

$$k := |[a, b] \cap \mathbb{T}| - \operatorname{def}\left(\tan\alpha + \frac{\mu(\rho(a))}{p(\rho(a))}\right) - \operatorname{def}(\sin\beta). \tag{3.4}$$

Moreover,

(*a*) *all eigenvalues of SLP (3.1), (3.2) are real, simple, bounded below, and can be ordered to satisfy that*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,\ n \in \mathbb{N}_0^k,$$

and

$$\lambda_n \to \infty \quad \text{as } n \to \infty \quad \text{when } k = \infty;$$

where $\mathbb{N}_0^k := \begin{cases} \{0, 1, 2, \ldots\}, & k = \infty \\ \{0, 1, 2, \ldots k-1\}, & k < \infty \end{cases}$, *and $\lambda_n \to \infty$ as $n \to \infty$ when $k = \infty$;*

(*b*) *each eigenfunction u_n associated with λ_n has exactly n generalized zeros in (a, b) for $n \in \mathbb{N}_0^k$.*

We now extend Proposition 3.1 without assuming $w(t) > 0$ on $[\rho(a), \sigma(b)] \cap \mathbb{T}$.

Theorem 3.1. *Let k and $\mathbb{N}_0^k$ be defined as in Proposition 3.1. Under the Assumptions (I) and (II), SLP (3.1), (3.2) has k eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}_0^k}$. Moreover,*

(*a*) *all eigenvalues of SLP (3.1), (3.2) are real, simple, bounded below, and can be ordered to satisfy that*

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,\ n \in \mathbb{N}_0^k$$

and

$$\lambda_n \to \infty \quad \text{as } n \to \infty \quad \text{when } k = \infty;$$

(b) *each eigenfunction* u_n *associated with* λ_n *has exactly* n *generalized zeros in* (a, b) *for* $n \in \mathbb{N}_0^k$.

Proof. It is easy to see that all eigenvalues of SLP (3.1), (3.2) are real.

To prove the existence of eigenvalues of SLP (3.1), (3.2), we employ the eigencurve approach for two-parameter SLPs used in [4,5,12]. Consider the two-parameter problem consisting of the equation

$$(-p(t)y^{\Delta})^{\Delta} + (q(t) - \lambda w(t))y^{\sigma} = \xi y^{\sigma} \quad \text{on } [a, b] \cap \mathbb{T} \tag{3.5}$$

and BC (3.2). By Proposition 3.1, for any $\lambda \in \mathbb{R}$, SLP (3.5), (3.2) has k eigenvalues $\{\xi_n(\lambda)\}_{n\in\mathbb{N}_0^k}$ which are simple, bounded below, and can be ordered to satisfy that

$$-\infty < \xi_0(\lambda) < \xi_1(\lambda) < \cdots < \xi_n(\lambda) < \cdots, \ n \in \mathbb{N}_0^k,$$

and

$$\xi_n(\lambda) \to \infty \ \text{ as } n \to \infty \ \text{ when } k = \infty;$$

and each eigenfunction u_n associated with $\xi(\lambda)$ has exactly n generalized zeros in (a, b) for $n \in \mathbb{N}_0^k$. It is easy to see that λ is an eigenvalue of SLP (3.1), (3.2) if and only if $\xi_n(\lambda) = 0$ for some $n \in \mathbb{N}_0^k$. We call $C_n = \{(\lambda, \xi_n(\lambda)) : \lambda \in \mathbb{R}\}$ the n-th eigencurve of SLP (3.5), (3.2). Similar to the proof of [13, Theorem 2.1] we can show that $\xi_n(\lambda)$ is a continuous function on $\mathbb{R}$ and hence C_n is a continuous curve on $\mathbb{R}$.

Define a Hilbert space

$$\mathcal{H} = \{y \in [\rho(a), b] \cap \mathbb{T} : y^2 \text{ is integrable on } [\rho(a), b] \cap \mathbb{T}\}$$

with the inner product

$$< y_1, y_2 > = \int_{\rho(a)}^{b} y_1(t)y_2(t)\Delta t \quad \text{for } y_1, y_2 \in \mathcal{H}$$

and induced norm $\|y\| = < y, y >^{1/2}$. Define an operator T by

$$Ty = -(py^{\Delta})^{\Delta} + qy^{\sigma}$$

on the domain

$$D = \{y \in \mathcal{H} : y, py^{\Delta} \text{ are } \Delta\text{-integrable}, \ Ty \in \mathcal{H}, \text{ and } y \text{ satisfies BC (3.2)}\}.$$

Let U be the unit sphere in $\mathcal{H}$ and F any subspace of $D(T)$. The by Theorem 1 of [2] and the self-adjoint spectral theory we see that for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}_0^k$

$$\xi_n(\lambda) = \min_F\{\max_y\{< Ty, y^{\sigma} > - \lambda < wy^{\sigma}, y^{\sigma} >: y \in F \cap U\} : \dim F = n + 1\}. \tag{3.6}$$

Let $\sup w$ be the supremum of w on $[\rho(a), \sigma(b)] \cap \mathbb{T}$. By assumption (I) we have that $\sup w > 0$. It is easy to see that for $\epsilon \in (0, \sup w)$, there exists an n-dimensional subspace F of $D(T)$ such that

$$< wy^\sigma, y^\sigma > \geq \sup w - \epsilon > 0 \quad \text{for all } y \in F \cap U. \tag{3.7}$$

Thus for $\lambda > 0$ and $y \in F \cap U$ we have

$$\frac{\xi_n(\lambda)}{\lambda} \leq \frac{< Ty, y^\sigma >}{\lambda} - < wy^\sigma, y^\sigma > \leq \frac{< Ty, y^\sigma >}{\lambda} - \sup w + \epsilon.$$

This, together with the fact that $< Ty, y^\sigma >$ is bounded above on $F \cap U$, implies that $\xi_n(\lambda) < 0$ for λ sufficiently large.

If $\xi_0(0) \geq 0$. then for $n \in \mathbb{N}_0^k, \xi_n(0) \geq 0$. By the continuity of $\xi_n(\lambda)$, there exists $\lambda_n \geq 0$ such that $\xi_n(\lambda_n) = 0$. This shows that λ_n is the n-th eigenvalue of SLP (3.1), (3.2).

If $\xi_0(0) \leq 0$, let $\lambda_* < 0$ satisfying

$$< Ty, y^\sigma > -\lambda_* \int_{\rho(a)}^{b} w(t)(y^\sigma(t))^2 \Delta t \geq 0 \tag{3.8}$$

for all $y \in F$ defined by (3.7). Denote $\bar{q} = q - \lambda_* w, \bar{\lambda} = \lambda - \lambda_*$, and $\bar{\xi} = \xi$. Then we may change Eq. (3.5) to

$$(-p(t)y^\Delta)^\Delta + (\bar{q}(t) - \bar{\lambda} w(t))y^\sigma = \bar{\xi} y^\sigma \quad \text{on } [a, b] \cap \mathbb{T}. \tag{3.9}$$

By (3.6), where q, λ, and ξ are replaced by $\bar{q}, \bar{\lambda}$, and $\bar{\xi}$, we have that $\bar{\xi}(\bar{\lambda})$ satisfies that

$$\bar{\xi}_n(0) = \min_F \{\max_y \{< \bar{T}y, y^\sigma >: y \in F \cap U\} : \dim F = n + 1\},$$

where

$$\bar{T}y = -(py^\Delta)^\Delta + \bar{q}y^\sigma = Ty - \lambda_* wy.$$

Hence by (3.8) we have that for $y \in F$,

$$< \bar{T}y, y^\sigma > = < Ty, y^\sigma > -\lambda_* \int_{\rho(a)}^{b} w(t)(y^\sigma(t))^2 \Delta t \geq 0. \tag{3.10}$$

This shows that $\bar{\xi}_n(0) \geq 0$ for $n \in \mathbb{N}_0^k$. Similar to the above we have that $\bar{\xi}_n(\lambda) < 0$ for λ sufficiently large. By the continuity of $\xi_n(\lambda)$, there exists $\bar{\lambda}_n \geq 0$ such that $\bar{\xi}_n(\tilde{\lambda}_n) = 0$. This implies that $\bar{\lambda}_n$ is the n-th eigenvalue of the SLP consisting of the equation

$$-(py^\Delta)^\Delta + \tilde{q}y^\sigma = \bar{\lambda} wy^\sigma \quad \text{on } [a, b] \cap \mathbb{T}$$

and BC (3.2) and hence $\lambda_n = \bar{\lambda}_n - \lambda_*$ is the n-th eigenvalue of the SLP (3.1), (3.2).

Finally, we show that the set $\{\lambda_n\}_{n \in \mathbb{N}_0^k}$ does not have cluster points which follows that $\lambda_n \to \infty$ as $n \to \infty$ when $k = \infty$. Assume the contrary and let

λ^* be such a cluster point. Then the set $\{\xi_n\}_{n\in\mathbb{N}_0^k}$ must have a cluster point $\xi^* \leq 0$, which contradicts that $\xi_n \to \infty$ as $n \to \infty$ when $k = \infty$. This completes the proof. $\square$

4. Proof of the Main Results

We define a time scale as a union of closed intervals $\mathbb{T} = \bigcup_{i=1}^{j} [c_i, d_i]$ in the following way:

$$\begin{aligned} c_1 = a_1, \quad d_i - c_i &= b_i - a_i,\ i = 1, \ldots, j; \quad \text{and} \\ c_{i+1} - d_i &= l_i p(b_i),\ i = 1, \ldots, j-1. \end{aligned} \tag{4.1}$$

It is easy to see that for $i = 1, \ldots, j, [c_i, d_i]$ is a shift of $[a_i, b_i]$ with the same length. Moreover, under assumption (A3), $\mathbb{T}$ is a bounded time scale. Define an operator $\Gamma : I \to \mathbb{T}$ by

$$s = \Gamma(t) = t - a_i + c_i \quad \text{for } t \in [a_i, b_i],\ t = 1, \ldots, j. \tag{4.2}$$

Obviously, Γ is strictly increasing on I and hence its inverse $t = \Gamma^{-1}(s)$ is defined on $\mathbb{T}$. For $s \in \mathbb{T}$, denote $y(s) = x(\Gamma^{-1}(s))$ and let

$$\tilde{p}(s) = p(\Gamma^{-1}(s)),\ \tilde{q}(s) = q(\Gamma^{-1}(s)),\ \text{and } \tilde{w}(s) = w(\Gamma^{-1}(s)).$$

Without confusion we still use y' for the derivative of y with respect to s when $s \in (c_i, d_i)$ and the appropriate one-sided derivative when $s = c_i$ or $d_i, i = 1, \ldots, j$. Let y^Δ be the Δ-derivative of y on the time scale $\mathbb{T}$ except at d_j and $y^\Delta(d_j) := \lim_{s\to d_j-} y^\Delta(s)$.

Due to Assumption (A2), we we may define a dynamic equation on $\mathbb{T}$ given as follows:

$$(\tilde{p}y^\Delta)(s))^\Delta + f(s)(\tilde{p}y^\Delta)(\sigma(s)) + g(s)y(\sigma(s)) = \lambda r(s)y(\sigma(s)), \tag{4.3}$$

where for $i = 1, \ldots, j-1$,

$$f(s) = \begin{cases} 0,\ s \in [c_i, d_i) \\ -\dfrac{k_i - 1 - h_i l_i}{l_i \tilde{p}(d_i)(k_i - h_i l_i)},\ s = d_i, \end{cases}$$

$$g(s) = \begin{cases} \tilde{q}(s)(s)),\ s \in [c_i, d_i) \\ -\dfrac{h_i}{l_i \tilde{p}(d_i)(k_i - h_i l_i)},\ s = d_i, \end{cases}$$

and

$$r(s) = \begin{cases} \tilde{w}(s)\ s \in [c_i, d_i) \\ 0,\ s = d_i. \end{cases}$$

We consider the two-point BC

$$\begin{aligned} \cos\alpha\, y(c_1) - \sin\alpha\, (\tilde{p}y^\Delta)(c_1) = 0, \quad 0 \leq \alpha < \pi, \\ \cos\beta\, y(d_j) - \sin\beta\, (\tilde{p}y^\Delta)(d_j) = 0, \quad 0 < \beta \leq \pi. \end{aligned} \tag{4.4}$$

Now we present a result on the relation between the SLP (1.1)–(1.3) on multi-interval I with non-self-adjoint interface conditions and the SLP (4.3), (4.4) on the time scale $\mathbb{T}$ without interface conditions.

Theorem 4.1. *Let assumptions (A1)–(A3) hold. Then λ is an eigenvalue of SLP (1.1)–(1.3) if and only if it is an eigenvalue of SLP (4.3), (4.4). Moreover, the eigenfunctions of SLP (1.1)–(1.3) and SLP (4.3), (4.4) associated with this λ have exactly the same number of generalized zeros.*

Proof. It is easy to see that λ is an eigenvalue of SLP (1.1)–(1.3) with $x(t)$ as an associated eigenfunction if and only if λ is an eigenvalue with $y(s) := x(\Gamma^{-1}(s))$ as an eigenfunction of the SLP consisting of the equation

$$(\tilde{p}(t)y')' + \tilde{q}(t)y = \lambda\tilde{w}(t)y \quad \text{for } s \in \bigcup_{i=1}^{j} [c_i, d_i] \tag{4.5}$$

with the interface condition that for $i = 1, \dots, j-1$,

$$\begin{aligned} y(c_{i+1}) &= y(d_i) + l_i\,(\tilde{p}y')(d_i), \\ (\tilde{p}y')(c_{i+1}) &= h_i\,y(d_i) + k_i\,(\tilde{p}y')(d_i); \end{aligned} \tag{4.6}$$

and boundary conditions (BCs)

$$\begin{aligned} &\cos\alpha\, y(c_1) - \sin\alpha\,(\tilde{p}y')(c_1) = 0, \quad 0 \le \alpha < \pi, \\ &\cos\beta\, y(d_j) - \sin\beta\,(\tilde{p}y')(d_j) = 0, \quad 0 < \beta \le \pi. \end{aligned} \tag{4.7}$$

By (4.1) and the first condition in (4.6) we have

$$(\tilde{p}y^{\Delta})(d_i) = (\tilde{p}y')(d_i) = \lim_{s\to d_i-}(\tilde{p}y')(s) = \lim_{s\to d_i-}(\tilde{p}y^{\Delta})(s).$$

Thus, $\tilde{p}y^{\Delta}$ is continuous at d_i.

It follows from the second condition in (4.6) that

$$(\tilde{p}y^{\Delta})(c_{i+1}) - (\tilde{p}y^{\Delta})(d_i) = (\tilde{p}y')(c_{i+1}) - (\tilde{p}y')(d_i) = h_i y(d_i) + (k_i - 1)(\tilde{p}y^{\Delta})(d_i).$$

This together with (4.1) shows that

$$\begin{aligned} l_i\tilde{p}(d_i)(\tilde{p}y^{\Delta})^{\Delta}(d_i) &= (c_{i+1} - d_i)(\tilde{p}y^{\Delta})^{\Delta}(d_i) \\ &= (\tilde{p}y^{\Delta})(c_{i+1}) - (\tilde{p}y^{\Delta})(d_i) = h_i y(d_i) + (k_i - 1)(\tilde{p}y^{\Delta})(d_i) \\ &= h_i(y(c_{i+1}) - l_i(\tilde{p}y^{\Delta})(d_i)) + (k_i - 1)(\tilde{p}y^{\Delta})(d_i) \\ &= h_i y(c_{i+1}) + (k_i - 1 - h_i l_i)(\tilde{p}y^{\Delta})(d_i). \end{aligned} \tag{4.8}$$

Note that

$$(\tilde{p}y^{\Delta})(d_i) = (\tilde{p}y^{\Delta})(c_{i+1}) - l_i\tilde{p}(d_i)(\tilde{p}y^{\Delta})^{\Delta}(d_i).$$

Then from (4.8) we get

$$\begin{aligned} l_i\tilde{p}(d_i)(\tilde{p}y^{\Delta})^{\Delta}(d_i) = {} & h_i y(c_{i+1}) + (k_i - 1 - h_i l_i)(\tilde{p}y^{\Delta})(c_{i+1}) \\ & -(k_i - 1 - h_i l_i) l_i\tilde{p}(d_i)(\tilde{p}y^{\Delta})^{\Delta}(d_i). \end{aligned}$$

This follows that

$$(\tilde{p}y^{\Delta})^{\Delta}(d_i)-\frac{k_i-1-h_il_i}{l_i\tilde{p}(d_i)(k_i-h_il_i)}(\tilde{p}y^{\Delta})(c_{i+1})-\frac{h_i}{l_i\tilde{p}(d_i)(k_i-h_il_i)}y(c_{i+1})=0. \tag{4.9}$$

Combining (4.5) and (4.9) we see that $y(s)$ satisfies Eq. (4.3). In view of the facts that $y(s)$ satisfies (4.7), $(\tilde{p}y^{\Delta})(c_1)=(py')(c_1)$, and $(\tilde{p}y^{\Delta})(d_j)=(py')(d_j)$, we have that $y(s)$ satisfies (4.4). □

Proof of Theorem 2.1. We note that f defined in (4.3) is regressive and let $e_f(\cdot,0)$ be the exponential function of f at 0 defined on $\mathbb{T}$. Then we have

$$(e_f(s,0)(\tilde{p}y^{\Delta})(s))^{\Delta}=e_f(s,0)[(\tilde{p}y^{\Delta})^{\Delta}(s)+f(s)(\tilde{p}y^{\Delta})(\sigma(s))].$$

Let

$$P(s)=e_f(s,0)\tilde{p}(s),\ \ Q(s)=e_f(s,0)g(s),\ \text{and}\ W(t)=e_f(s,0)r(s).$$

Then SLP (4.3), (4.4) changes to the problem consisting of the equation

$$(P(s)y^{\Delta}(s))^{\Delta}+Q(s)y(\sigma(s))=\lambda W(s)y(\sigma(s)) \tag{4.10}$$

and the BC

$$\begin{aligned}\cos\tilde{\alpha}\,y(c_1)-\sin\tilde{\alpha}\,(Py^{\Delta})(c_1)=0,\\ \cos\tilde{\beta}\,y(d_j)-\sin\tilde{\beta}\,(Py^{\Delta})(d_j)=0;\end{aligned} \tag{4.11}$$

for some $\tilde{\alpha}\in[0,\pi)$ and $\tilde{\beta}\in(0,\pi]$. This means that SLP (4.3), (4.4) and SLP (4.10), (4.11) have exactly the same eigenvalues and eigenfunctions.

Since

$$1+\mu(d_i)f(d_i)=1-\frac{k_i-1-h_il_i}{k_i-h_il_i}=\frac{1}{k_i-h_il_i}>0,$$

we see that $e_f(s,0)>0$ for all $s\in\mathbb{T}$. Thus Assumptions (I) and (II) in Section 2 are satisfied with p,q,w replaced by P,Q,W and $[a,b]$ replaced by $[c_1,d_j]$; $\hat{0}=0$ and $\hat{\pi}=\pi$ for the time scale $\mathbb{T}$, and $k=\infty$ for the number k defined in (3.4). Thus, Theorem 3.1 applies to SLP (4.10), (4.11). As a result, SLP (4.10), (4.11) has an infinite number of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ which satisfy conclusions (i) and (ii) of Theorem 3.1.

Finally, Theorem 2.1 follows from the equivalent relation between SLP (1.1)–(1.3) and SLP (4.3), (4.4) (hence SLP (4.10), (4.11)) given by Theorem 4.1. □

Proof of Theorem 2.2 *and* 2.3. Note that SLP (1.1)–(1.3) is equivalent to SLP (4.10), (4.11). Then the conclusion follows from Theorems 2.2 and 2.3 in [11] directly. Although these theorems were proved under the assumption that $w(t)>0$, the same proofs work when w satisfy assumption (A1). □

5. Appendix: Preliminaries on Time Scales

In this section, we recall the basic concepts related to time scales used in this paper for the convenience of the reader. For further knowledge on time scales, the reader is referred to [1,6,7,10] and the references therein.

Definition 5.1. A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$ with the inherited Euclidean topology. For $t \in \mathbb{T}$ we define the forward-jump operator σ and the backward-jump operator ρ on $\mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. If $\sigma(t) > t, t$ is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t, t$ is said to be left-scattered; otherwise, it is left-dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is then defined by $\mu(t) := \sigma(t) - t$.

We use the notation $f^\sigma(t) := f(\sigma(t))$ for any function f defined on a time scale $\mathbb{T}$.

Definition 5.2. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (if $t = \sup \mathbb{T}$, assume t is not left-scattered), define the Δ-derivative $f^\Delta(t)$ of $f(t)$ to be the number, provided it exists, with the property that, for any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|$$

for all $s \in U$. For $n > 1$, the n-th Δ-derivative of $f(t)$ is defined by $f^{\Delta^n}(t) := (f^{\Delta^{n-1}})^\Delta(t)$.

It is easily seen that if $f : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$, the set of integers, then

$$f^\Delta(t) = \Delta f(t) = f(t+1) - f(t).$$

If $t \in \mathbb{T}$ is right-dense and $f : \mathbb{T} \to \mathbb{R}$ is differentiable at t, then

$$f^\Delta(t) = f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

The following formula involving the graininess function is valid for all points at which $f^\Delta(t)$ exists:

$$f^\sigma(t) = f(t) + f^\Delta(t)\mu(t).$$

Definition 5.3. If $F^{\Delta}(t)=f(t)$, then we define the integral of f on $[a,b]\cap\mathbb{T}$ by

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a).$$

In this case, we say that f is integrable on $[a,b]\cap\mathbb{T}$.

It has been shown that f is integrable on $[a,b]\cap\mathbb{T}$ if f is rd-continuous on $[a,b]\cap\mathbb{T}$, i.e., f is continuous at each right-dense point in $[a,b]\cap\mathbb{T}$ and $\lim_{s\to t^-} f(s)$ exists as a finite number for all left-dense points in $[a,b]\cap\mathbb{T}$.

For $h>0$,we define the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h}\mathrm{Log}\,(1+zh),$$

where Log is the principal logarithm function, and

$$\mathbb{C}_h = \left\{z\in\mathbb{C} : z \neq -\frac{1}{h}\right\}, \quad \mathbb{Z}_h = \left\{z\in\mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h}\right\}.$$

For $h=0$, we define $\xi_0(z)=z$ for all $z\in\mathbb{C}$.

Definition 5.4. A function $p:\mathbb{T}\to\mathbb{R}$ is said to be regressive on $\mathbb{T}$ if $1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}$.

Let p be regressive and $t_0\in\mathbb{T}$. Then we define the exponential function of p at t_0 by

$$e_p(t,t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(t)}(p(\tau))\Delta\tau\right), \quad t\in\mathbb{T}.$$

It is easy to see that $e_p(t,t_0)>0$ on $\mathbb{T}$ for any $t_0\in\mathbb{T}$ if $1+\mu(t)p(t)>0$ on $\mathbb{T}$. Moreover, for any Δ-differentiable function f and any $t_0\in\mathbb{T}$, we have that

$$[e_p(t,t_0)f(t)]^{\Delta} = e_p(t,t_0)[f^{\Delta}(t) + p(t)f^{\sigma}(t)].$$

References

[1] Agarwal, R.P., Bohner, M.: Basic calculus on time scales and some of its applications. Results Math. **35**, 3–22 (1999)

[2] Agarwal, R.P., Bohner, M., Wong, P.J.Y.: Sturm–Liouville eigenvalue problems on time scales. Appl. Math. Comput. **99**, 153–166 (1999)

[3] Altinisik, N., Kadakal, M., Mukhtarov, O.Sh.: Eigenvalues and eigenfunctions of discontinuous Sturm–Liouville problems with eigenparameter-dependent boundary conditions. Acta Math. Hung. **102**, 159–175 (2004)

[4] Binding, P., Browne, P.: Asymptotics of eigencurves for second order ordinary differential equations, I. J. Differ. Equ. **88**, 30–45 (1990)

[5] Binding, P., Volkmer, H.: Eigenvalues for two-parameter Sturm–Liouville equations. SIAM Rev. **38**, 27–48 (1996)

[6] Bohner, M., Peterson, A.: Dynamic Equations on Time Scales, an Introduction with Applications. Birkhanser, Boston (2001)

[7] Erbe, L.H., Hilger, S.: Sturmian theory on measure chains. Differ. Equ. Dyn. Syst. **1**, 223–246 (1993)

[8] Everitt, W.N., Zettl, A.: Sturm–Liouville differential operators in direct sum spaces. Rockey Mt. J. Math. **16**, 497–516 (1986)

[9] Hao, X., Sun, J.: Regular Sturm–Liouville operators with transmission conditions at finite interior discontinuous points. J. Math. Sci. Adv. Appl. **4**(2), 265–277 (2010)

[10] Hilger, S.: Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math. **18**, 18–56 (1990)

[11] Kong, Q.: Sturm–Liouville problems on time scales with separated boundary conditions. Results Math. **52**, 111–121 (2008)

[12] Kong, Q., Wu, H., Zettl, A.: Left-definite Sturm–Liouville problems. J. Differ. Equ. **177**, 1–26 (2001)

[13] Kong, Q., Wu, H., Zettl, A.: Dependence of the n-th Sturm–Liouville eigenvalue on the problem. J. Differ. Equ. **156**, 328–356 (1999)

[14] Lykov, A.V., Mikhailov, Yu.A.: The theory of heat and mass transfer, translated from Russian by I. Shechtman, Israel Program for Scientific Translations, Jerusalem (1965)

[15] Mukhtarov, O.S., Kadakal, M.: Spectral properties of of one Sturm–Liouville type problem with discontinuous weight, (Russian) Sibirsk. Math. Zh., **46**(4), 860–785 (2005); translation in Siberian Math. J., **46**(4), 681–694 (2005)

[16] Mukhtarov, O.S., Yakubov, S.: Problems for differential equations with transmission conditions. Appl. Anal. **81**, 1033–1064 (2002)

[17] Sun, J., Wang, A., Zettl, A.: Two-interval Sturm–Liouville operators in direct sum spaces with inner product multiples. Result. Math. **50**, 155–168 (2007)

[18] Tikhonov, A.N., Samarskii, A.A.: Equations of Mathematical Physics. Dover Publications, New York (1990)

[19] Titeux, I., Yakubov, Y.: Completeness of root functions for thermal conduction in a strip with piecewise continuous coefficients. Math. Models Methods Appl. Sci. **7**, 1035–1050 (1997)

[20] Wang, A., Sun, J., Hao, X., Yao, S.: Completeness of eigenfunctions of Sturm–Liouville problems with transmission conditions. Methods Appl. Anal. **16**, 299–312 (2009)

[21] Wang, A., Sun, J., Zettl, A.: Two-interval Sturm–Liouville operators in Modified Hilbert spaces. J. Math. Anal. Appl. **328**, 390–399 (2007)

[22] Zettl, A.: Adjoint and self-adjoint boundary value problems with interface conditions. SIAM J. Appl. Math. **16**, 851–859 (1968)

Qingkai Kong
Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115, USA
e-mail: kong@math.niu.edu

Qi-Ru Wang
Department of Mathematics
Sun Yat-sen University
Guangzhou 510275, People's Republic of China
e-mail: mcswqr@mail.sysu.edu.cn

Received: June 29, 2011.
Accepted: September 17, 2011.