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Nonholonomic versus vakonomic dynamics on a Riemann–Cartan manifold

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For the Chaplygin's nonholonomic constrained systems, the constraint manifold can be endowed with Riemann–Cartan geometric structure by nonholonomic mapping into a Riemann manifold. The two kinds of existing dynamics, nonholonomic dynamics and vakonomic dynamics, are compared in the framework of Riemann–Cartan geometry. It is proved that the equations of motion for nonholonomic and vakonomic dynamics are described by the equations of autoparallel and geodesic trajectories on the Riemann–Cartan constraint manifold, respectively. If the metricity condition of Riemann–Cartan connection is satisfied, the torsion (contorsion) of the Riemann–Cartan manifold characterizes the difference between the autoparallel and geodesic trajectories as well as the distinction between the nonholonomic and vakonomic equations. © *2005 American Institute of Physics.*
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I. INTRODUCTION

Constrained systems are common dynamical systems in modern physics, mechanics and engineering,[1–13] which can be classified into holonomic and nonholonomic ones according to the Frobenius integrability condition of constraints the systems are subject to. Unlike a holonomic system, a nonholonomic system cannot be reduced to a free system with lower degrees of freedom in general. Furthermore, there exist two inequivalent dynamical theories on nonholonomic constrained systems. One is based on Hamilton's principle of lease action. Similar to treating holonomic constrained systems, the constraints are directly incorporated into Lagrangian $L \to L + \lambda^\alpha f_\alpha$ with λ^α being Lagrange multipliers treated as independent dynamical variables. The dynamical equations derived from the theory can be canonicalized since such a way to incorporate constraints into a dynamical description does not influence on the symplectic structure of phase space of the systems. This dynamics is usually referred to as vakonomic dynamics (variational axiomatic kind).[14,15] The other is based on d'Alembert–Lagrange principle (or Hölder's principle, Gauss's principle) satisfying the condition of ideal constraints. Chetaev's condition on variation of coordinates induced from the nonholonomic constraints is utilized to realize the ideal constraints. Such a theory is not canonically Lagrangian or Hamiltonian, and is called nonholonomic dynamics.[1,11] The above two dynamics are equivalent for holonomic systems.

The paradox that different dynamics can be derived from the same nonholonomic constrained system just because of beginning with different acknowledged principles makes nonholonomic

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constrained systems and their relating theories become a focus of research and disputation.[16–19] Therefore, it has been an important work to compare the two dynamics of nonholonomic systems.

In this paper, nonholonomic dynamics of Chaplygin's linear constrained systems[20] is compared with the corresponding vakonomic dynamics in the framework of Riemann–Cartan geometry.[21,22] In Sec. II, we briefly review the method to construct a Riemann–Cartan manifold by using a nonholonomic mapping[23–28] from a Riemann manifold. In Sec. III, the calculus of nonholonomic variations is discussed on the manifold. The nonholonomic variations are classified into three kinds. In Sec. IV, the two kinds of equations of motion, nonholonomic and vakonomic, for linear nonholonomic constrained systems are proved to describe the autoparallel and geodesic trajectories on the manifold, respectively. Some simple examples are illustrated in Sec. V ended with a concluding remark. The Einstein's summation convention is used throughout this paper and $i, j=1,2,\dots,n$; $\mu, \nu, \sigma, \rho, \lambda, \tau=1,2,\dots,m$; $\alpha, \beta=m+1, m+2,\dots,n$.

II. RIEMANN–CARTAN CONSTRAINT MANIFOLD

A general system is usually subject to two kinds of constraints, the holonomic and the nonholonomic. Suppose that the configuration space of the system is n-dimensional Riemann manifold Q with local coordinates $\{q^i\}$ after the holonomic constraints are reduced. The metric tensor field on manifold Q is defined by the Hessian of nondegenerate Lagrangian L. The configuration space can be further deformed into Riemann–Cartan manifold with both curvature and torsion by the nonholonomic constraints imposed.

Suppose the system is subject to $(n-m)$ linear nonholonomic constraints:

$$\dot{q}^\alpha = \varepsilon^\alpha_\mu(q^\nu)\dot{q}^\mu, \tag{1}$$

where $\{\dot{q}^\mu,\dot{q}^\alpha\}$ are generalized velocities of the system. These constraints are suitable to characterize most linear constrained systems. The systems subjected to such constraints are called Chaplygin's nonholonomic constrained systems.

A m-dimensional constraint manifold M with local coordinates $\{q^\mu\}$ can be constructed by the constraint equations (1). As a subspace of Riemann manifold Q, however, the constraint manifold M is not its invariant embedded submanifold since the vector fields on the constraint manifold M are not involutive due to anholonomy of the constraints (1). Nevertheless, the tangent space of the constraint manifold M can be embedded into the tangent space of Riemann manifold Q by a nonholonomic mapping $i_T: TM \to TQ$:

$$v^i = \varepsilon^i_\mu(q^\nu)v^\mu, \quad v^\mu \in T_qM, \tag{2}$$

induced by the constraints (1), where $\varepsilon^i_\mu=\varepsilon^\alpha_\mu$ if i takes $\alpha=m+1,m+2,\dots,n$; $\varepsilon^i_\mu=\delta^\nu_\mu$ if i takes $\nu=1,2,\dots,m$. This mapping can induce a nonholonomic mapping i_q:

$$q^i(c_q) = \int_{c_q} \varepsilon^i_\mu(q)\mathrm{d}q^\mu \tag{3}$$

from the equivalence class $\langle q^\mu(t)\rangle$ of all paths on manifold M to that on manifold Q, where c_q denotes any path $q^\mu(t)$ on manifold M. The integrals can be classified according to the end points of paths $c_{q(t)}$ if the same initial point of the integrals is fixed, i.e., each point of $c_{q(t)}$ corresponds to an equivalence class of integrals $q^i(c_{q(t)})$. $\{q^i(c_{q(t)})\}$ can be recognized as pseudo-coordinates on manifold M. If the constraints are integrable, the above-noted integrals are independent of paths on M and $q^i(c_{q(t)})$ reduce to $q^i(t)$, the function of end point t of path $c_{q(t)}$. Then path $q^\mu(t)\in M$ corresponds to path $q^i(t)\in Q$ pointwise, i.e., $q^i=q^i(q^\mu)$.

It can be verified that the metric and connection on manifold M can be induced from the nonholonomic mappings (2) and (3) in the same way as in Ref. 23 by generalizing the Euclid space taken as auxiliary space to the Riemann manifold Q. First, the metric g_{ij} on Riemann manifold Q induces the metric $g_{\mu\nu}$ on manifold M. According to the mapping (2), $u^i=\varepsilon^i_\mu u^\mu$, $v^j=\varepsilon^j_\nu v^\nu$, for $u^i,v^j\in T_{i_q(q)}Q$ and $u^\mu,v^\nu\in T_qM$. Then $(u,v)=g_{ij}u^iv^j=g_{ij}\varepsilon^i_\mu\varepsilon^j_\nu u^\mu v^\nu=(\varepsilon_\mu,\varepsilon_\nu)u^\mu v^\nu$. Therefore, the induced metric on manifold M is

$$g_{\mu\nu}=(\varepsilon_\mu,\varepsilon_\nu)=g_{ij}\varepsilon^i_\mu\varepsilon^j_\nu. \tag{4}$$

Second, the mappings (2) and (3) induce a connection on manifold M,

$$\Gamma^\sigma_{\mu\nu}=g^{\sigma\rho}(\varepsilon_\rho,\partial_\mu\varepsilon_\nu)=g^{\sigma\rho}g_{ij}\varepsilon^i_\rho\partial_\mu\varepsilon^j_\nu. \tag{5}$$

It is easy to verify the metricity condition of the connection, i.e., compatible condition of the connection with metric, $D_\mu g_{\nu\sigma}=0$, which makes the length of a vector invariant while parallel-transporting it along a path on manifold M. But the connection is asymmetric, i.e., $\Gamma^\sigma_{\nu\mu}\neq\Gamma^\sigma_{\mu\nu}$, whose antisymmetric part is named torsion of the connection:

$$S^\sigma_{\mu\nu}=\Gamma^\sigma_{[\mu\nu]}=\tfrac{1}{2}(\Gamma^\sigma_{\mu\nu}-\Gamma^\sigma_{\nu\mu})=\tfrac{1}{2}g^{\sigma\rho}g_{ij}\varepsilon^i_\rho(\partial_\mu\varepsilon^j_\nu-\partial_\nu\varepsilon^j_\mu). \tag{6}$$

Obviously, $S^\sigma_{\nu\mu}=0$ if the integrability condition of constraints, $\partial_\mu\varepsilon^j_\nu-\partial_\nu\varepsilon^j_\mu=0$, is satisfied.

Such an asymmetric connection compatible with metric is referred to as Riemann–Cartan connection. The constraint manifold M is then a Riemann–Cartan manifold with torsion $S^\sigma_{\mu\nu}$ and curvature

$$R^{\rho\tau}_{\mu\nu}=\tfrac{1}{2}g^{\rho\lambda}g^{\tau\sigma}g_{ij}(f^i_{\mu\lambda}f^j_{\nu\sigma}-f^i_{\nu\lambda}f^j_{\mu\sigma}), \tag{7}$$

where

$$f^i_{\mu\nu}=D_\mu\varepsilon^i_\nu=\partial_\mu\varepsilon^i_\nu-\Gamma^\sigma_{\mu\nu}\varepsilon^i_\sigma. \tag{8}$$

On the Riemann–Cartan constraint manifold M there exist two kinds of special curves, geodesic and autoparallel trajectories, as follows:

$$\ddot{q}^\mu+\overline{\Gamma}^\mu_{\nu\sigma}\dot{q}^\nu\dot{q}^\sigma=0, \tag{9a}$$

$$\ddot{q}^\mu+\Gamma^\mu_{\nu\sigma}\dot{q}^\nu\dot{q}^\sigma=0, \tag{9b}$$

where $\overline{\Gamma}^\mu_{\nu\sigma}=\frac{1}{2}g^{\mu\lambda}(\partial_\sigma g_{\nu\lambda}+\partial_\nu g_{\sigma\lambda}-\partial_\lambda g_{\nu\sigma})$ is Riemann–Christoffel connection. It can be proved in the following that the difference of shortness from straightness on Riemann–Cartan manifold can geometrically characterize the "inexplicable" deviation of vakonomic dynamics from nonholonomic dynamics.

III. NONHOLONOMIC VARIATIONS ON RIEMANN–CARTAN CONSTRAINT MANIFOLD

Let c_q and $\overline{c_q}$ be smooth curves connecting any two fixed points q^i_1 and q^i_2 on n-dimensional Riemann manifold Q. Consider a function $q^i(t,\alpha)\in C^2$ of two parameters, satisfying $q^i(t,0)=q^i(t)$, $q^i(t,1)=\overline{q^i}(t)$; $q^i(t_1,\alpha)=q^i_1$, $q^i(t_2,\alpha)=q^i_2$. Denote the differential along any path by $\mathrm{d}q^\mu=\partial_t q^i(t,\alpha)\mathrm{d}t\doteq v^i\mathrm{d}t$ while the variation of the path is denoted by $\delta q^i=\partial_\alpha q^i(t,\alpha)\mathrm{d}\alpha\doteq w^i\mathrm{d}\alpha$ with fixed ends condition

$$\delta q^i\big|_{t_{1,2}}=0,\qquad w^i\big|_{t_{1,2}}=0. \tag{10}$$

The vector field $w^i(q^j)\in T_qQ$ is called variation vector field on manifold Q. As in Ref. 23, denote d_v and d_w the derivative along vector fields v and w, respectively. The above-noted definition leads to the following commutation relation of differential and variational operations:

$$\mathrm{d}_w v^i-\mathrm{d}_v w^i=0, \tag{11}$$

which simply determine the variation of velocity. To define a variation it is necessary to specify the variation of velocity as well as the variation vector field.

A variation vector field $w^{\mu}(q^{\nu}) \in T_q M$ can also be defined on manifold M with torsion. The commutation relation (11) on Riemann manifold Q, however, cannot be simply transplanted to the constraint manifold M. Relation (11) leads to

$$\mathrm{d}_w(v^i - \varepsilon^i_\mu v^\mu) + (\partial_\mu \varepsilon^i_\rho - \partial_\rho \varepsilon^i_\mu) v^\rho w^\mu + \varepsilon^i_\mu (\mathrm{d}_w v^\mu - \mathrm{d}_v w^\mu) - \mathrm{d}_v(w^i - \varepsilon^i_\mu w^\mu) = 0. \tag{12}$$

Let $\varepsilon^\mu_i \doteq g^{\mu\nu} g_{ij} \varepsilon^j_\nu$ with $\varepsilon^\mu_i \varepsilon^i_\nu = \delta^\mu_\nu$; $\varepsilon^\mu_i \varepsilon^j_\mu = \delta^j_i$. Using this condition and the connection coefficients (5) obtained in last section, we are led to

$$\partial_\rho \varepsilon^i_\mu = \varepsilon^i_\sigma \Gamma^\sigma_{\rho\mu}, \quad \partial_\rho \varepsilon^\mu_i = - \varepsilon^\nu_i \Gamma^\mu_{\rho\nu}. \tag{13}$$

By the definition (6), it follows that

$$(\partial_\mu \varepsilon^i_\rho - \partial_\rho \varepsilon^i_\mu) = 2 \varepsilon^i_\sigma S^\sigma_{\mu\rho}. \tag{14}$$

Therefore relation (12) can be transformed into

$$\mathrm{d}_w(v^i - \varepsilon^i_\mu v^\mu) + \varepsilon^i_\mu (\mathrm{d}_w v^\mu - \mathrm{d}_v w^\mu) - \mathrm{d}_v(w^i - \varepsilon^i_\mu w^\mu) = 2 \varepsilon^i_\sigma S^\sigma_{\rho\mu} v^\rho w^\mu. \tag{15}$$

Because of the existence of torsion tensor $S^\sigma_{\rho\mu}$ the variation vector field on the constraint manifold M cannot satisfy the following conditions simultaneously:

$$d_v(w^i - \varepsilon^i_\mu w^\mu) = 0, \tag{16a}$$

$$d_w v^\mu - d_v w^\mu = 0, \tag{16b}$$

$$d_w(v^i - \varepsilon^i_\mu v^\mu) = 0, \tag{16c}$$

which means that unlike the case of holonomic systems, there does not exist a free variation on the manifold M. The unfree variation vector field w and the corresponding differential d_w are named nonholonomic variations. The first condition imposed on the variation of coordinates is induced from the constraints and is called Chetaev's condition. The second is the commutation relation of differential and variational operations which leads to the existence of smooth local coordinate net formed by the integral curves of vector fields v^μ and w^μ. The third condition is an invariance of constraint conditions with respect to the variation operation, making the variation of velocities be unfree.

According to relation (15), the nonholonomic variation can be classified into the following three kinds.

(1) Hölder's variation. Choose the first two relations from the last equations, i.e.,

$$\mathrm{d}_v(w^i_{\mathrm{h}} - \varepsilon^i_\mu w^\mu_{\mathrm{h}}) = 0, \quad \mathrm{d}_{w_{\mathrm{h}}} v^\mu - \mathrm{d}_v w^\mu_{\mathrm{h}} = 0. \tag{17}$$

It follows from (15) that

$$\mathrm{d}_{w_{\mathrm{h}}}(v^i - \varepsilon^i_\mu v^\mu) = 2 \varepsilon^i_\sigma S^\sigma_{\rho\mu} v^\rho w^\mu_{\mathrm{h}}, \tag{18}$$

which indicates how the constraint conditions vary with respect to Hölder's variation due to the torsion of manifold M.

(2) Suslov's variation. Suppose that

$$\mathrm{d}_v(w^i_s - \varepsilon^i_\mu w^\mu_s) = 0, \quad \mathrm{d}_{w_s}(v^i - \varepsilon^i_\mu v^\mu) = 0. \tag{19}$$

It leads from the relation (15) to

$$\mathrm{d}_{w_\mathrm{s}} v^\sigma - \mathrm{d}_v w_\mathrm{s}^\sigma = 2S_{\rho\mu}^\sigma v^\rho w_\mathrm{s}^\mu, \tag{20}$$

which means that Suslov's variation does not commute with differentiation of coordinates. Making use of covariant differentiation,

$$D_v w_\mathrm{s}^\sigma = \mathrm{d}_v w_\mathrm{s}^\sigma + \Gamma_{\rho\mu}^\sigma v^\rho w_\mathrm{s}^\mu, \quad D_{w_\mathrm{s}} v^\sigma = \mathrm{d}_{w_\mathrm{s}} v^\sigma + \Gamma_{\rho\mu}^\sigma w_\mathrm{s}^\rho v^\mu. \tag{21}$$

The variation of velocity v^σ can be specified by

$$D_v w_\mathrm{s}^\sigma - D_{w_\mathrm{s}} v^\sigma = 0. \tag{22}$$

(3) Vakonomic variation. Let

$$\mathrm{d}_{w_\mathrm{v}} v^\mu - \mathrm{d}_v w_\mathrm{v}^\mu = 0, \quad \mathrm{d}_{w_\mathrm{v}} (v^i - \varepsilon_\mu^i v^\mu) = 0. \tag{23}$$

Then it is referred from (15) that the Chetaev's conditions cannot be satisfied and are replaced with corresponding conditions

$$\mathrm{d}_v (w_\mathrm{v}^i - \varepsilon_\mu^i w_\mathrm{v}^\mu) = 2\varepsilon_\sigma^i S_{\mu\rho}^\sigma v^\rho w_\mathrm{v}^\mu. \tag{24}$$

It should be pointed that all the three kinds of variations satisfy the fixed ends conditions

$$w^\mu\big|_{t_1} = w^\mu\big|_{t_2} = 0. \tag{25}$$

IV. NONHOLONOMIC VERSUS VAKONOMIC EQUATIONS ON CONSTRAINT MANIFOLD

We apply Suslov's variation and vakonomic variation to variational principle to get two kinds of dynamical equations for the Chaplygin's nonholonomic constrained systems.

First, we make use of Suslov's variation to check the recently discovered stationary action principle[25]

$$\mathrm{d}_{w_\mathrm{s}} S = \mathrm{d}_{w_\mathrm{s}} \int_{t_1}^{t_2} \mathcal{L}(q^\mu, v^\mu) \mathrm{d}t = 0. \tag{26}$$

Computing the variation directly and making use of the above variation condition (20) and fixed ends conditions (25), one can derive the equations of motion

$$\frac{\partial \mathcal{L}}{\partial q^\mu} - \mathrm{d}_v \left(\frac{\partial \mathcal{L}}{\partial v^\mu} \right) = 2S_{\mu\nu}^\rho \frac{\partial \mathcal{L}}{\partial v^\rho} v^\nu, \tag{27}$$

which describes nonholonomic dynamics on the constraint manifold M.

We concern the geometric property of nonholonomic equations (27). Substitute $\mathcal{L} = \frac{1}{2} g_{\mu\nu}(q) v^\mu v^\nu$ into Eq. (27), then

$$g_{\mu\nu} \dot{v}^\nu + (\overline{\Gamma}_{\mu\nu\rho} - 2S_{\rho\nu\mu}) v^\rho v^\nu = 0, \tag{28}$$

where $S_{\rho\nu\mu} = g_{\rho\lambda} S_{\nu\mu}^\lambda$, $\overline{\Gamma}_{\mu\nu\rho} = g_{\mu\lambda} \overline{\Gamma}_{\nu\rho}^\lambda$ and $\overline{\Gamma}_{\nu\rho}^\lambda$ is Christoffel symbols. Considering the geometric relation

$$\overline{\Gamma}_{\mu\nu\rho} - 2S_{\rho\nu\mu} = g_{\mu\lambda} \Gamma_{\nu\rho}^\lambda \tag{29}$$

and lift the index μ, then

$$D_v v^\lambda = \dot{v}^\lambda + \Gamma^\lambda_{\nu\rho} v^\rho v^\nu = 0. \tag{30}$$

Thus it can be seen that the equations of motion for nonholonomic dynamics describe the autoparallels of Riemann–Cartan constraint manifold M.

Second, apply the vakonomic variation to the action on the manifold M, the stationary action principle

$$\mathrm{d}_{w_\mathrm{v}} S = \mathrm{d}_{w_\mathrm{v}} \int_{t_1}^{t_2} \mathcal{L}(q^\mu, v^\mu)\mathrm{d}t = 0 \tag{31}$$

simply leads to vakonomic equations

$$\mathrm{d}_v\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right) - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0. \tag{32}$$

In fact, it is Euler–Lagrange equations. Substitute $\mathcal{L}=\frac{1}{2}g_{\mu\nu}(q)v^\mu v^\nu$ into Eq. (32), then

$$\overline{D}_v v^\lambda = \dot{v}^\lambda + \overline{\Gamma}^\lambda_{\nu\rho} v^\rho v^\nu = 0, \tag{33}$$

which are the geodesic equations on Riemann–Cartan constraint manifold M. In the following we verify that the geodesic equations (33) are just a geometrical representation of vakonomic equations for Chaplygin's nonholonomic constrained systems in Riemann–Cartan constraint manifold.

Let L be a Lagrangian of a dynamical system on the Riemann manifold Q. Rewrite $(n-m)$ nonholonomic constraints (1) the system is subject to as follows:

$$f^\alpha = v^\alpha - \varepsilon^\alpha_\mu v^\mu = 0. \tag{34}$$

Based on Hamilton's principle of least action, the vakonomic equations

$$[L]_i = -\lambda_\alpha [f^\alpha]_i - \mathrm{d}_v(\lambda_\alpha)\frac{\partial f^\alpha}{\partial v^i} \tag{35}$$

can be derived by the method of Lagrange multipliers. Using the notations $\lambda_i = \delta^\alpha_i \lambda_\alpha$, $f^i = v^i - \varepsilon^i_\mu v^\mu = 0$ in order to map the equations onto the constraint manifold M conveniently, the vakonomic equations are then equivalent to

$$[L]_i = -\lambda_j [f^j]_i - \mathrm{d}_v \lambda_j \frac{\partial f^j}{\partial v^i}. \tag{36}$$

It can be derived by simple computation:

$$\frac{\partial f^j}{\partial v^i} = \delta^j_i - \delta^\mu_i \varepsilon^j_\mu, \quad \frac{\partial f^j}{\partial q^i} = -\delta^\nu_i \varepsilon^j_\sigma \Gamma^\sigma_{\nu\mu} v^\mu, \quad [f^j]_i = 2\delta^\nu_i \varepsilon^j_\sigma S^\sigma_{\mu\nu} v^\nu. \tag{37}$$

Substitute it into the vakonomic equations, then

$$[L]_i = -2\delta^\nu_i \lambda_j \varepsilon^j_\sigma S^\sigma_{\mu\nu} v^\nu + \mathrm{d}_v \lambda_i + \mathrm{d}_v \lambda_j \varepsilon^j_\mu \delta^\mu_i. \tag{38}$$

Supposing the Lagrangian L on Riemann manifold Q is independent of coordinates q^α and neglecting any integral constants, the Lagrange multipliers can be found out from the above equations

$$\lambda_i = -\varepsilon^\mu_i \frac{\partial L}{\partial v^\mu}, \quad \mathrm{d}_v \lambda_i = -\varepsilon^\mu_i D_v\left(\frac{\partial L}{\partial v^\mu}\right) \tag{39}$$

along with the reduction of vakonomic equations onto the constraint manifold M:

$$[L]_\mu = 2S^{\sigma}_{\mu\nu}v^{\nu}\frac{\partial L}{\partial v^{\sigma}} - D_v\left(\frac{\partial L}{\partial v^{\mu}}\right). \tag{40}$$

For a mechanical system, $L=\frac{1}{2}g_{\mu\nu}v^{\mu}v^{\nu}$, the above equations can be transformed into

$$g_{\mu\nu}\overline{D}_v v^{\nu} + g_{\mu\nu}D_v v^{\nu} = 2g_{\nu\sigma}S^{\nu}_{\mu\rho}v^{\rho}v^{\sigma}. \tag{41}$$

Expand further, then

$$2g_{\mu\nu}\overline{D}_v v^{\nu} + (K_{\mu\rho\sigma} - S_{\sigma\mu\rho})v^{\rho}v^{\sigma} = 0, \tag{42}$$

where $K_{\mu\rho\sigma}=g_{\mu\nu}K^{\nu}_{\rho\sigma}=g_{\mu\nu}(\Gamma^{\nu}_{\rho\sigma}-\overline{\Gamma}^{\nu}_{\rho\sigma})$, $S_{\sigma\mu\rho}=g_{\nu\sigma}S^{\nu}_{\mu\rho}$. Substitute the geometric relation

$$K_{\mu\rho\sigma} - S_{\sigma\mu\rho} = K_{\rho\mu\sigma} = -K_{\sigma\mu\rho} \tag{43}$$

into the above equation, we finally obtain Eq. (33), a geodesic representation of vakonomic equations on the constraint manifold M.

Remark 1: We have not applied Hölder's variation to the action because the result is the same as that of the vakonomic variation if we did for the particular action, which does not imply that Hölder's variation cannot give any new result for general nonholonomic systems.

Remark 2: On the Riemann–Cartan manifold M satisfying the metricity of connection the autoparallel trajectories will coincide with the geodesic ones if the torsion vanishes. In this case the nonholonomic equations also coincide with the vakonomic ones. However, it does not mean that the constraints are integrable in the sense of Frobenius theorem although inverse proposition is certainly true. This fact will be illustrated by example 3 of Sec. V.

V. ILLUSTRATIVE EXAMPLES

Now we are going to show by the following simple examples how the interrelation between the nonholonomic and vakonomic equations can be geometrically characterized on a Riemann–Cartan manifold.

Example 1: We illustrate the above result by the following example of a nonholonomically constrained particle with the Lagrangian $L=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ and the nonholonomic constraint $\dot{z}=y\dot{x}$.

By means of the usual method, the nonholonomic and vakonomic equations are given by

$$\ddot{x} + \frac{y\dot{x}\dot{y}}{1+y^2} = 0, \quad \ddot{y} = 0, \tag{44a}$$

$$\ddot{x} + \frac{2y\dot{x}\dot{y}}{1+y^2} = 0, \quad \ddot{y} - y\dot{x}^2 = 0, \tag{44b}$$

respectively. We will illustrate that they describe autoparallel and geodesic trajectories on a two-dimensional Riemann–Cartan constraint manifold M with local coordinates (x,y).

Let x,y; $\dot{x},\dot{y}$ play the role of the q^{μ}, $\dot{q}^{\mu}$ and $z,\dot{z}$ the role of the q^{α}, $\dot{q}^{\alpha}$ in our discussion of the general theory. Obviously, $i,j=1,2,3$; $\mu,\nu=1,2$; $\alpha=3$ and

$$\varepsilon^1_1 = 1, \varepsilon^1_2 = 0, \varepsilon^2_1 = 0, \varepsilon^2_2 = 1; \quad \varepsilon^3_1 = y, \varepsilon^3_2 = 0.$$

The metric $g_{\mu\nu}$ and $g^{\mu\nu}$ take the form

$$(g_{\mu\nu}) = \begin{pmatrix} 1+y^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad (g^{\mu\nu}) = \begin{pmatrix} \frac{1}{1+y^2} & 0 \\ 0 & 1 \end{pmatrix}.$$

The nonvanishing coefficients of the corresponding Christoffel symbols are given by

$$\overline{\Gamma}^2_{11} = -y, \quad \overline{\Gamma}^1_{12} = \overline{\Gamma}^1_{21} = \frac{y}{1+y^2}.$$

Then the geodesic equations on the manifold M,

$$\ddot{q}^1 + 2\overline{\Gamma}^1_{12}\dot{q}^1\dot{q}^2 = 0, \quad \ddot{q}^2 + \overline{\Gamma}^2_{11}\dot{q}^1\dot{q}^1 = 0,$$

take the form of vakonomic equations (44b) after a replacement of q^1,q^2 with x,y.

It can be verified that the only one nonvanishing coefficient of Riemann–Cartan connection on the manifold M is

$$\Gamma^1_{21} = \frac{y}{1+y^2}.$$

Then the autoparallel equations

$$\ddot{q}^1 + \Gamma^1_{21}\dot{q}^2\dot{q}^1 = 0, \quad \ddot{q}^2 = 0$$

are simply the nonholonomic equations (44a) if the coordinates q^1,q^2 are replaced with x,y.

Example 2: A special Chaplygin sleigh. Let us consider the free motion of a sleigh on a horizontal plane in the case when the projection of the center of mass coincides with the point of contact of a sharp wheel and the plane. We suppose the simplified sleigh has unit mass and unit moment of inertia in $R^2 \times T^1$ with coordinates (x,y,φ), subjected to the nonholonomic constraint $\dot{y} = \dot{x} \tan \varphi$. Then the regular Lagrangian is given by $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2)$.

We discuss the two kinds of differential equations on the Riemann–Cartan submanifold M of $R^2 \times T^1$ in the following. As is well known, the reduced nonholonomic and vakonomic equations for the system on M are given by

$$\ddot{x} + \dot{x}\dot{\varphi} \tan \varphi = 0, \quad \ddot{\varphi} = 0, \tag{45a}$$

$$\ddot{x} + 2\dot{x}\dot{\varphi} \tan \varphi = 0, \quad \ddot{\varphi} - \dot{x}^2 \tan \varphi \sec^2 \varphi = 0, \tag{45b}$$

respectively.

Take $i,\ j = 1,2,3$; $\mu,\nu = 1,2$; $\alpha = 3$. Denote $q^\mu,\dot{q}^\mu$ by x,φ; $\dot{x},\dot{\varphi}$ and $q^\alpha,\dot{q}^\alpha$ by $y,\dot{y}$. Obviously,

$$\varepsilon^1_1 = 1, \varepsilon^1_2 = 0, \varepsilon^2_1 = 0, \varepsilon^2_2 = 1; \quad \varepsilon^3_1 = \tan \varphi, \varepsilon^3_2 = 0.$$

The metric $g_{\mu\nu}$ and $g^{\mu\nu}$ take the form

$$(g_{\mu\nu}) = \begin{pmatrix} \sec^2\varphi & 0 \\ 0 & 1 \end{pmatrix}, \quad (g^{\mu\nu}) = \begin{pmatrix} \cos^2\varphi & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the nonvanishing coefficients of the corresponding Christoffel symbols are given by

$$\overline{\Gamma}^2_{11} = -\tan \varphi \sec^2 \varphi, \quad \overline{\Gamma}^1_{12} = \overline{\Gamma}^1_{21} = \tan \varphi.$$

The Riemann–Cartan connection is simple to compute with only one nonvanishing coefficient

$$\Gamma^1_{21} = \tan \varphi.$$

It is very easy to verify that the following autoparallel and geodesic equations,

$$\ddot{x} + \Gamma^1_{21}\dot{\varphi}\dot{x} = 0, \quad \ddot{\varphi} = 0,$$

$$\ddot{x} + 2\Gamma^1_{12}\dot{x}\dot{\varphi} = 0, \quad \ddot{\varphi} + \Gamma^2_{11}\dot{x}\dot{x} = 0,$$

are just the nonholonomic and vakonomic equations (45a) and (45b), respectively, by substituting the above nonvanishing connection coefficients into them.

Example 3: Consider the problem of a vertically rolling disk which is another paradigm of nonholonomic systems. Choose the following appropriate generalized coordinates: the coordinates (x,y) of the center of mass of the disk, the azimuthal angle ψ which determines the position of the disk, and angle ϕ describing its internal rotation.

Setting the mass of the disk equal to 1 for simplicity, the Lagrangian is given by $L=\frac{1}{2}(\dot{x}^2+\dot{y}^2)+\frac{1}{2}(I_1\dot{\phi}^2+I_2\dot{\psi}^2)$ where I_1 and I_2 are moments of inertia. This regular Lagrangian leads to Hessian metric g_{ij} with nonvanishing diagonal elements: $g_{11}=I_1$, $g_{22}=I_2$, $g_{33}=1$, $g_{44}=1$. The nonholonomic constraints are given by the condition of rolling without slipping

$$\dot{x} = (R\cos\psi)\dot{\phi}, \quad \dot{y} = (R\sin\psi)\dot{\phi},$$

where R is the radius of the disk. Following the usual procedure for setting up the nonholonomic and vakonomic equations, we can consider the two kinds of equations of motion as the same and simply read

$$(R^2 + I_1)\ddot{\phi} = 0, \quad I_2\ddot{\psi} = 0. \tag{46}$$

Making use of the following notational identifications: $(q^1,q^2,q^3,q^4)=(\phi,\psi;x,y)$, the above constraints leads to

$$\varepsilon^1_1 = 1, \varepsilon^1_2 = \varepsilon^2_1 = 0, \varepsilon^2_2 = 1; \quad \varepsilon^3_1 = R\cos\psi, \varepsilon^3_2 = 0, \varepsilon^4_1 = R\sin\psi, \varepsilon^4_2 = 0$$

from which the metrics are given by

$$(g_{\mu\nu}) = \begin{pmatrix} R^2 + I_1 & 0 \\ 0 & I_2 \end{pmatrix}, \quad (g^{\mu\nu}) = \begin{pmatrix} \frac{1}{R^2+I_1} & 0 \\ 0 & \frac{1}{I_2} \end{pmatrix}.$$

It is straightforward to compute in the same way as the above examples that all of the coefficients of Riemann–Christoffel and Riemann–Cartan connections vanish,

$$\overline{\Gamma}^\mu_{\nu\sigma} = 0, \quad \Gamma^\mu_{\nu\sigma} = 0 \quad (\mu,\nu,\sigma = 1,2),$$

which means that the autoparallel coincides with the geodesic and satisfies the same equations

$$\ddot{\phi} = 0, \quad \ddot{\psi} = 0.$$

They are equivalent to Eq. (46).

This example indicates that similar to the autoparallel and geodesic, the difference between nonholonomic and vakonomic dynamics for Chaplygin's nonholonomic constrained systems is determined by the torsion of the corresponding Riemann–Cartan constraint manifold. The integrability of the constraints is an efficient but not a necessary condition for the coincidence of the two dynamics.

Concluding remark. If a system is subject to Chaplygin's nonholonomic constraints, its configuration space is no longer a Riemann manifold but a Riemann–Cartan manifold with torsion, on which the free variation operation does not exist. The stationary action principles on the constraint manifold with respect to Suslov's variation and vakonomic variation lead to autoparallel equations and geodesic equations on the manifold, respectively. This result accords with principle of inertia and principle of control theory.

Similar to the geometrization of gravitational fields in general relativity and gravitational gauge theories, a system subject to Chaplygin's nonholonomic constraints in an Euclidean or a Riemann space is equivalent to a free system in a Riemann–Cartan space. By means of this geometrization the seeming inconsistency between nonholonomic and vakonomic dynamics can be replaced by harmonious geometric relations: straightness and shortness on the same Riemann–Cartan manifold. The result are applicable to most autonomous nonholonomic constrained systems, which can be generalized to nonautonomous ones by means of the theory of connection on a contact manifold or a one-jet bundle in the forthcoming contribution.

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[1] F. X. Mei, Appl. Mech. Rev. **53**, 283 (2000).
[2] H. Cendra, A. Ibort, M. de León, and D. M. de Diego, J. Math. Phys. **45**, 2785 (2004).
[3] M. Pavon, J. Math. Phys. **46**, 032902 (2005).
[4] A. Ramos, J. Phys. A **37**, 4821 (2004).
[5] X. Gràcia and R. Martín, J. Phys. A **38**, 1071 (2005).
[6] X. Gràcia, J. Marín-Solano, and M.-C. Muñoz-Lecanda, Rep. Math. Phys. **51**, 127 (2003).
[7] T. Mestdag and B. Langerock, J. Phys. A **38**, 1097 (2005).
[8] Z. P. Li, *Constrained Hamiltonian Systems and Their Symmetries* (Press of Beijing Industrial University, Beijing, 1999) (in Chinese).
[9] Y. X. Guo, S. K. Luo, and F. X. Mei, Adv. Mech. **34**, 477 (2004) (in Chinese).
[10] Y. X. Guo, M. Shang, S. K. Luo, and F. X. Mei, Int. J. Theor. Phys. **40**, 1197 (2001).
[11] Y. X. Guo, S. K. Luo, M. Shang, and F. X. Mei, Rep. Math. Phys. **47**, 313 (2001).
[12] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and R. Murray, Arch. Ration. Mech. Anal. **136**, 21 (1996).
[13] E. Massa, S. Vignolo, and D. Bruno, J. Phys. A **35**, 6713 (2002).
[14] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*, Encyclopedia of Mathematical Science Vol. III (Springer, Berlin, 1988).
[15] F. Cardin and M. Favretti, J. Geom. Phys. **18**, 295 (1996).
[16] M. de León, J. C. Marrero, and M. D. de Diego, J. Geom. Phys. **35**, 126 (2002).
[17] G. Zampieri J. Diff. Eqns. **163**, 335 (2000).
[18] L. F. Liang and H. C. Hu, Adv. Mech. **30**, 358 (2000) (in Chinese).
[19] A. D. Lewis and R. M. Murray, Int. J. Non-Linear Mech. **30**, 793 (1995).
[20] Ju. I. Neimark and N. A. Fufaev, *Dynamics of Nonholonomic Systems* (American Mathematical Society, Providence, RI, 1972).
[21] I. L. Shapiro, Phys. Rep. **357**, 113 (2002).
[22] F. W. Hehl, J. D. McCrea, E. W. Mielke *et al.*, Phys. Rep. **258**, 1 (1995).
[23] S. V. Shabanov, J. Phys. A **31**, 5177 (1998).
[24] P. Fiziev and H. Kleinert, Europhys. Lett. **35**, 241 (1996).
[25] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 3rd ed. (World Scientific, Singapore, 2002).
[26] H. Kleinert and S. V. Shabanov, Phys. Lett. B **428**, 315 (1998).
[27] H. Kleinert and A. Pelster, Gen. Relativ. Gravit. **31**, 1439 (1999).
[28] H. Kleinert and A. Pelster, Acta Physiol. Pol. **29**, 1015 (1998).