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# INITIAL BOUNDARY VALUE PROBLEM FOR THE SINGULARLY PERTURBED BOUSSINESQ-TYPE EQUATION

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Abstract. We are concerned with the singularly perturbed Boussinesq-type equation including the singularly perturbed sixth-order Boussinesq equation, which describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for bond number (surface tension parameter) less than but very close to 1/3. The existence and uniqueness of the global generalized solution and the global classical solution of the initial boundary value problem for the singularly perturbed Boussinesq-type equation are proved.

## 1. **Introduction.**

In the numerical study of the ill-posed Boussinesq equation

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}, \tag{1}$$

Daripa and Hua [2] proposed the singularly perturbed Boussinesq equation

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx} + \delta u_{xxxxxx} \tag{2}$$

as a dispersive regularization of the ill-posed classical Boussinesq equation (1), where $\delta > 0$ is small parameter. The authors use both filtering and regularization techniques to control growth of the errors and to provide better approximate solutions of this equation. Dash and Daripa [4] presented a formal derivation of equation (2) from two-dimensional potential flow equations for water waves through an asymptotic series expansion for small amplitude and long wave length. The physical relevance of equation (2) in the context of water waves was also addressed in [4], it was shown that equation (2) actually describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for bond number (surface tension parameter) less than but very close to $1/3$. On the basis of far-field analysis and heuristic arguments, Daripa and Dash [3] proved that the traveling wave solutions of equations (2) are weakly non-local solitary waves characterized by small amplitude fast oscillations in the far-field and obtained weakly non-local solitary wave solutions of equation (2). Feng [5] investigated the generalized Boussinesq equation including the singularly perturbed Boussinesq equation

$$u_{tt} = [Q(u)]_{xx} + \sum_{i=1}^{n} b_i u_{(2i+2)x}, \tag{3}$$

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where $Q(u) = u + b_0 u^r$, $u_{(2i+2)x} = \frac{\partial^{2i+2}u}{\partial x^{2i+2}}$, $r$ and $b_i$ $(i = 1, 2, \cdots, n)$ are all real constants. It is easily seen that the choices $b_0 = 1$, $r = 2$, $n = 2$, $b_1 = 1$ and $b_2 = \delta$ lead equation (3) to the singularly perturbed Boussinesq equation (2). By the means of two proper ansatzs, the author obtained explicit traveling solitary wave solutions of the generalized Boussinesq equation (3). To our best knowledge, however, there have not been any discussion on global solutions of the initial boundary value problem for the equation (2) in the literature.

It is well known that when $\delta = 0$ equation (2) becomes the "bad" Boussinesq equation which is ill-posed [7][8] due to the properties of the linear part that are so "bad" that the traditional mathematical methods cease to be effective. So, one can ask such a question, as a dispersive regularization of "bad" Boussinesq equation (1), does the initial boundary value problem for the equation (2) admit any global solutions? Furthermore, replacing nonlinear function $u^2$ by the more extensive nonlinear function $\sigma(u)$ in equation (2), does the above-problem admit any global solutions? In this paper, we consider the following generalized singularly perturbed Boussinesq-type equation

$$u_{tt} = u_{xx} + \sigma(u)_{xx} + \alpha u_{x^4} + \beta u_{x^6},\ x \in \Omega,\ t > 0, \tag{4}$$

with the initial boundary value conditions

$$\begin{aligned} &u_x(0,t) = u_x(1,t) = u_{x^3}(0,t) = u_{x^3}(1,t) = u_{x^5}(0,t) = u_{x^5}(1,t) = 0,\ t > 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ x \in \overline{\Omega}, \end{aligned} \tag{5}$$

or with

$$\begin{aligned} u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = u_{x^4}(0,t) = u_{x^4}(1,t) = 0,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ x \in \overline{\Omega}, \end{aligned} \tag{6}$$

where and in the sequel $u_{x^i} = \frac{\partial^i u}{\partial x^i}$, $\sigma(s)$ is a given nonlinear function, $\alpha > 0$ and $\beta > 0$ are real numbers, $u_0(x)$, $u_1(x)$ are given initial value functions, $\Omega = (0, 1)$. By virtue of the Galerkin method and prior estimates, under the assumption "$\sigma'(s)$ is bounded below and $\sigma(s)$ satisfies some smooth condition", we prove that the problem (4),(5) and (4),(6) admit a unique global generalized solution and a unique global classical solution, respectively.

The paper is organized as follows. In Section 2, the main results are stated. The existence and uniqueness of global generalized solution and global classical solution of the problem (4),(5) and (4),(6) are proved in section 3.

## 2. Main theorems.

Throughout this paper, we use the abbreviations $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. In the following we state the main results of this paper.

**Theorem 2.1.** *Assume that* $u_0 \in H^6(\Omega)$, $u_1 \in H^3(\Omega)$, $\int_0^1 u_0(x)dx = \int_0^1 u_1(x)dx = 0$, $u_{0x^{2k+1}}(0,t) = u_{0x^{2k+1}}(1,t) = u_{1x^{2k+1}}(0,t) = u_{1x^{2k+1}}(1,t) = 0$ $(k = 0, 1, 2)$, $\sigma \in C^5(\mathbf{R})$ *and* $\sigma'(s)$ *is bounded below, namely there exists a constant* $C_0$ *such that* $\sigma'(s) \geq C_0$, *for any* $s \in \mathbf{R}$. *Then, for any* $T > 0$ *the initial boundary value problem (4),(5) admits a unique global generalized solution* $u(x,t)$ *with*

$$u \in C([0,T]; H^6(\Omega)) \cap C^1([0,T]; H^3(\Omega)) \cap C^2([0,T]; L^2(\Omega)).$$

**Theorem 2.2.** *Assume that the assumptions of Theorem 2.1 hold,* $u_0 \in H^{10}(\Omega)$, $u_1 \in H^7(\Omega)$, $\sigma \in C^9(\mathbf{R})$. *Then, the initial boundary value problem (4),(5) admits a unique global classical solution* $u(x,t)$.

**Theorem 2.3.** *Assume that* $u_0 \in H^6(\Omega),\ u_1 \in H^3(\Omega),\ u_{0x^{2k}}(0,t) = u_{0x^{2k}}(1,t) = u_{1x^{2k}}(0,t) = u_{1x^{2k}}(1,t) = 0\ (k = 0,1,2),\ \sigma \in C^5(\mathbf{R}),\ \sigma^{(2i)}(0) = 0\ (i = 1,2)$ *and* $\sigma'(s)$ *is bounded below. Then, for any* $T > 0$ *the initial boundary value problem (4),(6) admits a unique global generalized solution* $u(x,t)$ *with*

$$u \in C([0,T]; H^6(\Omega)) \cap C^1([0,T]; H^3(\Omega)) \cap C^2([0,T]; L^2(\Omega)).$$

**Theorem 2.4.** *Assume that the assumptions of Theorem 2.3 hold,* $u_0 \in H^{10}(\Omega)$, $u_1 \in H^7(\Omega)$, $\sigma \in C^9(\mathbf{R})$ *and* $\sigma^{(2i)}(0) = 0\ (i = 3,4)$. *Then, the initial boundary value problem (4),(6) admits a unique global classical solution* $u(x,t)$.

3. **Global solution of the problem (4),(5) and (4),(6).** We first discuss the initial boundary value problem (4),(5).

Integrating both sides of Eq.(4) over (0,1) and using (5) and the assumption of Theorem 2.1, we obtain $\int_0^1 u(x,t)dx = 0,\ t \geq 0$. Let $v(x,t) = \int_0^x u(\xi,t)d\xi$, then $u = v_x$ and $v$ satisfies

$$v_{tt} = v_{xx} + \sigma(v_x)_x + \alpha v_{x^4} + \beta v_{x^6},\ x \in \Omega,\ t > 0, \tag{7}$$

$$v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = v_{x^4}(0,t) = v_{x^4}(1,t) = 0,\ t > 0, \tag{8}$$

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x),\ x \in \overline{\Omega}, \tag{9}$$

where $v_0(x) = \int_0^x u_0(\xi)d\xi,\ \ v_1(x) = \int_0^x u_1(\xi)d\xi$.

In the following, we first prove the initial boundary value problem (7)-(9) admits a unique global generalized and classical solution. For this goal, we introduce an orthogonal basis in $L^2(\Omega)$. Let $\{w_j(x)\}$ be the orthogonal basis in $L^2(\Omega)$ composed of the eigenfunctions of the eigenvalue problem

$$w''(x) + \lambda w(x) = 0,\ x \in \Omega,\quad w(0) = w(1) = 0,$$

corresponding to eigenvalue $\lambda_j\ (j = 1,2,\cdots)$. And let $v^n(x,t) = \sum_{j=1}^n T_{jn}(t)w_j(x)$ be Galerkin approximate solution of the problem (7)-(9), where $T_{jn}(t)\ (j = 1,\cdots)$ are functions to be determined, $n$ is a nature number. Assume that the initial data $v_0(x)$ and $v_1(x)$ can be expressed by

$$v_0(x) = \sum_{j=1}^{\infty} \alpha_j w_j(x), \quad v_1(x) = \sum_{j=1}^{\infty} \beta_j w_j(x),$$

where $\alpha_j,\ \beta_j\ (j = 1,2,\cdots)$ are constants. Substituting the approximate solution $v^n(x,t)$ into (7)-(9), we have

$$v^n_{tt} = v^n_{xx} + \sigma(v^n_x)_x + \alpha v^n_{x^4} + \beta v^n_{x^6}, \tag{10}$$

$$v^n(0,t) = v^n(1,t) = v^n_{xx}(0,t) = v^n_{xx}(1,t) = v^n_{x^4}(0,t) = v^n_{x^4}(1,t) = 0, \tag{11}$$

$$v^n(x,0) = v^n_0(x), \quad v^n_t(x,0) = v^n_1(x). \tag{12}$$

Multiplying both sides of (10) and (12) by $w_j(x)$,respectively, and integrating on $\Omega$, we obtain

$$(v^n_{tt} - v^n_{xx} - \alpha v^n_{x^4} - \beta v^n_{x^6} - \sigma(v^n_x)_x, w_j) = 0, \tag{13}$$

$$T_{jn}(0) = \alpha_j, \quad \dot{T}_{jn}(0) = \beta_j, \quad j = 1,2,\cdots,n, \tag{14}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$, and "$\cdot$" denotes $\frac{d}{dt}$.

**Lemma 3.1.** *(Adams[1]) There exist constants* $\varepsilon > 0$ *and* $C(\varepsilon) > 0$ *such that for any integers* $j$ *and* $m$, $0 \leq j \leq m$, *the following inequality holds*

$$\|D^j_x u(t)\|^2 \leq C(\varepsilon)\|u(t)\|^2 + \varepsilon\|D^m_x u(t)\|^2,$$

**Lemma 3.2.** *Assume that $\sigma \in C^1(\mathbf{R})$, $\sigma'(s)$ is bounded below, $v_0 \in H^3(\Omega)$, $v_1 \in L^2(\Omega)$ and $v_{0x^{2k}}(0) = v_{0x^{2k}}(1) = v_{1x^{2k}}(0) = v_{1x^{2k}}(1) = 0$, $(k = 0, 1, 2)$. Then for any $n$, the Cauchy problem (13),(14) admits a global classical solution $T_{jn} \in C^2[0,T]$ $(j = 1, 2, \cdots, n)$. Moreover, we have the following estimate*

$$\|v^n(t)\|^2_{H^3(\Omega)} + \|v^n_t\|^2 \le C_1(T), \quad t \in [0,T], \tag{15}$$

*here and in the sequel $C_i(T)$ $(i = 1, 2, \cdots)$ are constants depending on $T$ but independent of $n$.*

*Proof.* Let

$$\sigma_1(s) = \sigma(s) - k_0 s - \sigma(0), \; k_0 = \min\{C_0, 0\} \le 0. \tag{16}$$

(16) implies that $\sigma_1(s)$ is a monotonically increasing function, and thus $\int_0^s \sigma_1(\tau)d\tau \ge 0$. By (16), we know that equation (13) is equivalent to the following system

$$(v^n_{tt} - (1+k_0)v^n_{xx} - \alpha v^n_{x^4} - \beta v^n_{x^6} - \sigma_1(v^n_x)_x, w_j) = 0, \; j = 1, 2, \cdots, n. \tag{17}$$

Multiplying both sides of equation (17) by $2\dot{T}_{jn}$, summing up for $j = 1, 2, \cdots, n$, and integrating by parts, we obtain

$$\frac{d}{dt}\Big(\|v^n_t(t)\|^2 + (1+k_0)\|v^n_x(t)\|^2 - \alpha\|v^n_{xx}(t)\|^2 + \beta\|v^n_{xxx}(t)\|^2 + 2\int_\Omega\int_0^{v^n_x}\sigma_1(s)dsdx\Big) = 0. \tag{18}$$

When $k_0 < -1$, by virtue of Lemma 3.1, there is constants $C_1 > 0$ and $C_2 > 0$ such that

$$\|v^n_x(t)\|^2 \le C_1\|v^n(t)\|^2 - \frac{\beta}{4(1+k_0)}\|v^n_{xxx}(t)\|^2, \; t > 0, \tag{19}$$

$$\|v^n_{xx}(t)\|^2 \le C_2\|v^n(t)\|^2 + \frac{\beta}{4\alpha}\|v^n_{xxx}(t)\|^2, \; t > 0. \tag{20}$$

Adding $2[1 - C_1(1+k_0) + C_2\alpha](v^n, v^n_t)$ to the both sides of (18), integrating the product over $[0,t]$, making use of (19),(20) and the Cauchy inequality, we get

$$\begin{aligned}
&\|v^n(t)\|^2 + \|v^n_t(t)\|^2 + \frac{\beta}{2}\|v^n_{xxx}(t)\|^2 + 2\int_\Omega\int_0^{v^n_x}\sigma_1(s)dsdx \\
\le\; & [1 - C_1(1+k_0) + C_2\alpha]\|v_0\|^2 + \|v_1\|^2 + \frac{\beta}{2}\|v_0'''\|^2 + 2\int_\Omega\int_0^{v_0'}\sigma_1(s)dsdx \\
&+[1 - C_1(1+k_0) + C_2\alpha]\int_0^t(\|v^n(\tau)\|^2 + \|v^n_t(\tau)\|^2)d\tau, \; t \in [0,T].
\end{aligned} \tag{21}$$

Applying the Gronwall inequality to (21), we can obtain the estimate (15).

When $-1 \le k_0 \le 0$, adding $2(1 + C_2\alpha)(v^n, v^n_t)$ to the both sides of (18), integrating the product over $[0,t]$, making use of (20) and the Cauchy inequality and Gronwall inequality, we get (15) immediately.

Using (15) and the Leray-Schauder fixed point theorem [6], we can employ the standard process to prove that the problem (10),(11) admits a solution $T_{jn} \in C^2[0,T]$ $(j = 1, 2, \cdots, n)$. Lemma 3.2 is proved. □

**Lemma 3.3.** *(Zhou and Fu [9]). Assume that $G(z)$ is a $k$-times continuously differentiable function with respect to variables $z$ and $z \in L_\infty([0,T]; H^k(\Omega))$. Then*

$$\Big\|\frac{\partial^k}{\partial x^k}G(z)\Big\|^2 \le C(M,k)\|z(t)\|^2_{H^k(\Omega)},$$

*where* $M = \max_{(x,t)\in[0,T]\times\overline{\Omega}} |z(x,t)|$, $C(M,k)$ *is a positive constant depending only on* $M$ *and* $k$.

**Lemma 3.4.** *Assume that the assumptions of Lemma (3.2) hold,* $\sigma \in C^5(\mathbf{R})$, $v_0 \in H^7(\Omega)$, $v_1 \in H^4(\Omega)$. *Then, the approximate solution* $v^n(x,t)$ *satisfies the following inequality*

$$\|v^n_{tt}(t)\|^2_{H^1(\Omega)} + \|v^n_t(t)\|^2_{H^4(\Omega)} + \|v^n(t)\|^2_{H^7(\Omega)} \le C_2(T), \quad 0 \le t \le T. \tag{22}$$

*Proof.* Multiplying both sides of equation (13) by $2\lambda_j^4 \dot{T}_j n(t)$, summing up for $j = 1, 2, \cdots, n$, integrating by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\|v^n_{x^4t}(t)\|^2 + \|v^n_{x^5}(t)\|^2 - \alpha\|v^n_{x^6}(t)\|^2 + \beta\|v^n_{x^7}(t)\|^2\Big)\\ &= 2(\sigma(v^n_x)_{x^5}, v^n_{x^4t}),\end{aligned} \tag{23}$$

where the fact"$\sigma(v^n_x)_{x^{2k+1}}\Big|_0^1 = 0$ $(k = 0, 1)$" has been used. By Lemma 3.1, there is a constant $C_3 > 0$ such that

$$\|v^n_{x^6}(t)\|^2 \le C_3\|v^n(t)\|^2 + \frac{\beta}{2\alpha}\|v^n_{x^7}(t)\|^2, \ t > 0. \tag{24}$$

Adding $2(1 + C_3\alpha)(v^n, v^n_t)$ to the both sides of (23), integrating the product over [0,t] and using the Cauchy inequality, Lemma 3.3, (15) and (24), we have

$$\begin{aligned}&\|v^n(t)\|^2 + \|v^n_{x^4t}(t)\|^2 + \|v^n_{x^5}(t)\|^2 + \frac{\beta}{2}\|v^n_{x^7}(t)\|^2\\ &\le \|v_0\|^2 + \|v_1^{(4)}\|^2 + \|v_0^{(5)}\|^2 + \frac{\beta}{2}\|v_0^{(7)}\|^2 + (1 + C_3\alpha)\int_0^t (\|v^n(\tau)\|^2 + \|v^n_\tau(\tau)\|^2)d\tau\\ &+ \int_0^t (\|\sigma(v^n_x)_{x^5}\|^2 + \|v^n_{x^4\tau}(\tau)\|^2)d\tau\\ &\le \|v_0\|^2 + \|v_1^{(4)}\|^2 + \|v_0^{(5)}\|^2 + \frac{\beta}{2}\|v_0^{(7)}\|^2 + (1 + C_3\alpha)TC_1(T)\\ &+ (1 + C_3\alpha + C(M))\int_0^t (\|v^n(\tau)\|^2 + \|v^n_{x^4\tau}(\tau)\|^2 + \|v^n_{x^7}(\tau)\|^2)d\tau.\end{aligned} \tag{25}$$

where $M = \max_{(x,t)\in[0,T]\times\overline{\Omega}} |v^n_x(x,t)|$ ( from (15) and Sobolev embedding theorem we know that $v^n_x(x,t)$ is bounded ), It follows from (15),(25) and the Gronwall inequality that

$$\|v^n_t(t)\|^2_{H^4(\Omega)} + \|v^n(t)\|^2_{H^7(\Omega)} \le C_3(T), \quad t \in [0,T]. \tag{26}$$

Multiplying both sides of (13) by $\ddot{T}_{jn}$, summing up for $j = 1, 2, \cdots, n$, using the Cauchy inequality, we get

$$\|v^n_{tt}(t)\|^2 \le (\|v^n_{xx}(t)\| + \alpha\|v^n_{x^4}(t)\| + \beta\|v^n_{x^6}(t)\| + \|\sigma(v^n_x)_x\|)\|v^n_{tt}(t)\|. \tag{27}$$

Combining (26) with (27) leads to

$$\|v^n_{tt}(t)\|^2 \le C_4(T), \quad t \in [0,T]. \tag{28}$$

Similarly, multiplying both sides of (13) by $-\lambda_j \ddot{T}_{jn}$, summing up for $j = 1, 2, \cdots, n$, integrating by parts, using (26) and the Cauchy inequality we have

$$\|v^n_{xtt}(t)\|^2 \le C_5(T), \quad t \in [0,T]. \tag{29}$$

By (26), (28) and (29) yields (22). Lemma 3.4 is proved. □

**Theorem 3.5.** *Assume that the assumptions of Lemma 3.4 hold, then the initial boundary value problem (7)-(9) admits a unique global generalized solution $v(x,t)$, namely, $v(x,t)$ satisfies the identity*

$$\int_0^T \int_\Omega \Big[v_{tt} - v_{xx} - \alpha v_{x^4} - \beta v_{x^6} - \sigma(v_x)_x\Big] g(x,t)dxdt = 0, \quad g(x,t) \in L^2(\Omega \times (0,T))$$

*and the initial boundary value conditions (8) and (9) in the classical sense. The solution $v(x,t)$ has the continuous derivatives $v_{x^i}(x,t)$ $(i = 1,2,3)$ and $v_t(x,t)$ and the generalized derivatives $v_{x^i}(x,t)$ $(i = 4,5,6,7)$, $v_{x^it}(i = 1,2,3,4)$ and $v_{x^itt}(i = 0,1)$.*

*Proof of Theorem 3.5.* It follows from Lemma 3.4 and Sobolev embedding theorem that

$$\|v^n(t)\|_{C^{6,\lambda}(\overline{\Omega})} \le C_6(T), \quad \|v_t^n(t)\|_{C^{3,\lambda}(\overline{\Omega})} \le C_7(T),$$
$$\|v_{tt}^n(t)\|_{C^{0,\lambda}(\overline{\Omega})} \le C_8(T), \ t \in [0,T], \tag{30}$$

where $0 < \lambda \le 1/2$. Combining (30) with Ascoli– Arzelá theorem, we conclude that there exists a function $v(x,t)$ and a subsequence of $\{v^n(x,t)\}$, still denoted by $\{v^n(x,t)\}$ such that when $n \to \infty$, $\{v^n_{x^i}(x,t)\}$ $(i = 0,1,2,3)$ and $\{v^n_t(x,t)\}$ uniformly converge to $v_{x^i}(x,t)$ $(i = 0,1,2,3)$ and $v_t(x,t)$ on $\overline{\Omega} \times [0,T]$ respectively. By virtue of the estimate (22), we obtain that subsequences $\{v^n_{x^i}(x,t)\}$ $(i = 0,1,\cdots,7)$, $\{v^n_{x^it}(x,t)\}$ $(i = 0,1,2,3,4)$ and $\{v^n_{x^itt}(x,t)\}$ $(i = 0,1)$ weakly converge to $v_{x^i}(x,t)$ $(i = 0,1,\cdots,7)$, $v_{x^it}(x,t)$ $(0,1,2,3,4)$ and $v_{x^itt}(x,t)$ $(i = 0,1)$ in $L^2(\Omega \times (0,T))$ respectively. Making use of the weakly compact theorem of the space $L^2(\Omega \times (0,T))$, we can deduce that the initial boundary value problem (7)-(9) admits a global generalized solution.

In the following, we prove the uniqueness of the solution of the initial boundary value problem (7)-(9).

Assume that $v_1(x,t)$ and $v_2(x,t)$ are two generalized solutions of the problem (7)-(9). Let $w(x,t) = v_1(x,t) - v_2(x,t)$. Thus $w(x,t)$ satisfies the initial boundary value problem

$$w_{tt} = w_{xx} + \sigma(v_{1x})_x - \sigma(v_{2x})_x + \alpha w_{x^4} + \beta w_{x^6}, \ x \in \Omega, \ t > 0, \tag{31}$$

$$w(0,t) = w(1,t) = w_{xx}(0,t) = w_{xx}(1,t) = w_{x^4}(0,t) = w_{x^4}(1,t) = 0, t > 0, \tag{32}$$

$$w(x,0) = 0, \quad w_t(x,0) = 0, \ x \in \overline{\Omega} \tag{33}$$

Multiplying both sides of equation (31) by $2w_t$ and integrating over $\Omega$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|w_t(t)\|^2 + \|w_x(t)\|^2 - \alpha\|w_{xx}(t)\|^2 + \beta\|w_{xxx}(t)\|^2\Big)\\
&= 2\int_\Omega [\sigma'(v_{1x})v_{1xx} - \sigma'(v_{2x})v_{2xx}]w_t dx\\
&= 2\int_\Omega \sigma'(v_{1x})w_{xx}w_t dx + 2\int_\Omega \sigma''(v_{1x} + \theta(v_{2x} - v_{1x}))v_{2xx}w_x w_t dx\\
&\le C_9(T)(\|w_t(t)\|^2 + \|w_x(t)\|^2 + \|w_{xx}(t)\|^2),
\end{aligned} \tag{34}$$

where $0 < \theta < 1$ and the fact "$\sigma'(v_{1x})$ and $\sigma''(v_{1x} + \theta(v_{2x} - v_{1x}))v_{2xx}$ are bounded on $[0,T] \times \Omega$ " has been used.

By Lemma 3.1, there is a constant $C_4 > 0$ such that

$$\|w_{xx}(t)\|^2 \le C_4\|w(t)\|^2 + \frac{\beta}{2\alpha}\|w_{xxx}(t)\|^2. \tag{35}$$

Adding $2(1 + C_4\alpha) \int_\Omega w w_t dx$ to the both sides of (34), integrating over $[0, t]$, using the Cauchy inequality and observing the estimate (35), we have

$$\|w(t)\|^2 + \|w_t(t)\|^2 + \|w_x(t)\|^2 + \frac{\beta}{2}\|w_{xxx}(t)\|^2$$
$$\leq C_{10}(T) \int_0^t (\|w(t)\|^2 + \|w_t(t)\|^2 + \|w_x(t)\|^2 + \|w_{xxx}(t)\|^2) dt. \tag{36}$$

Applying the Gronwall inequality to (36) leads to

$$\|w(t)\|^2 + \|w_t(t)\|^2 + \|w_x(t)\|^2 + \frac{\beta}{2}\|w_{xxx}(t)\|^2 = 0. \tag{37}$$

(37) implies the uniqueness. □

In order to show that the initial boundary value problem (7)-(9) admits a global classical solution, we make further estimates for the approximate solution $v^n(x, t)$.

**Lemma 3.6.** *Assume that the assumptions of Lemma 3.4 and the following conditions hold: $\sigma \in C^9(\mathbf{R})$, $v_0 \in H^{11}(\Omega)$, $v_1 \in H^8(\Omega)$. Then the following estimate holds*

$$\|v^n(t)\|^2_{H^{11}(\Omega)} + \|v^n_t(t)\|^2_{H^8(\Omega)} + \|v^n_{tt}(t)\|^2_{H^2(\Omega)} + \|v^n_{ttt}(t)\|^2_{H^2(\Omega)} \leq C_{11}(T). \tag{38}$$

*Proof.* Multiplying both sides of equation(13) by $2\lambda_j^8 \dot{T}_{jn}(t)$, summing up the products for $j = 1, 2, \cdots, n$, integrating by parts, making use of Lemma 3.1, Lemma 3.3 and the Gronwall inequality, we deduce that

$$\|v^n(t)\|^2_{H^{11}(\Omega)} + \|v^n_t(t)\|^2_{H^8(\Omega)} \leq C_{12}(T), \quad t \in [0, T]. \tag{39}$$

Multiplying both sides of equation (13) by $\lambda_j^2 \ddot{T}_{jn}(t)$, summing up the products for $j = 1, 2, \cdots, n$, integrating by parts, by virtue of the Cauchy inequality, we obtain

$$\|v^n_{tt}(t)\|^2_{H^2(\Omega)} \leq C_{13}(T), \quad t \in [0, T]. \tag{40}$$

Differentiating (13) with respect to $t$, we have

$$(v^n_{ttt} - v^n_{xxt} - \alpha v^n_{x^4t} - \beta v^n_{x^6t} - \sigma(v^n_x)_{xt}, w_j) = 0, \ j = 1, 2, \cdots, n. \tag{41}$$

Multiplying both sides of (41) by $T_{jnttt}(t)$, summing up for $j = 1, 2, \cdots, n$, using the Cauchy inequality and the estimate (39), we get

$$\|v^n_{ttt}(t)\|^2 \leq C_{14}(T), \quad t \in [0, T]. \tag{42}$$

Multiplying both sides of (41) by $\lambda_j^2 T_{jnttt}(t)$, summing up for $j = 1, 2, \cdots, n$, integrating the products by parts, using the Cauchy inequality and the estimate (39), we conclude that

$$\|v^n_{ttt}(t)\|^2_{H^2(\Omega)} \leq C_{15}(T), \quad t \in [0, T]. \tag{43}$$

Combining the estimates (22), (39) with (43), we obtain (38). Lemma 3.6 is proved. □

**Theorem 3.7.** *Under the conditions of Lemma 3.6, the initial boundary value problem (7)-(9) admits a unique global classical solution $v(x, t)$ with*

$$v \in C([0, T]; C^7(\overline{\Omega})) \cap C^1([0, T]; C^1(\overline{\Omega})) \cap C^2([0, T]; C^1(\overline{\Omega})). \tag{44}$$

*Proof.* It follows from (38) and Sobolev embedding theorem that

$$\begin{aligned}&\|v^n(t)\|_{C^{10}(\overline{\Omega})} \le C_{16}(T),\ \|v_t^n(t)\|_{C^7(\overline{\Omega})} \le C_{17}(T),\\ &\|v_{tt}^n(t)\|_{C^1(\overline{\Omega})} \le C_{18}(T),\ \|v_{ttt}^n(t)\|_{C^1(\overline{\Omega})} \le C_{19}(T), \quad t \in [0,T].\end{aligned} \tag{45}$$

By using the (45) and Ascoli-Arzelá, we can prove that the initial boundary value problem (7)-(9) admits a global classical solution. Uniqueness of the generalized solution ensures that of classical solution. Theorem 3.7 is proved. □

*Proof of Theorem 2.1.* Differentiating (10) with respect to $x$, let $u^n(x,t) = v_x^n(x,t)$, using the equation (10) and boundary condition (11), we obtain that $u^n(x,t)$ satisfies the following problem

$$u_{tt}^n = u_{xx}^n + \sigma(u^n)_{xx} + \alpha u_{x^4}^n + \beta u_{x^6}^n, \tag{46}$$

$$u_x^n(0,t) = u_x^n(1,t) = u_{x^3}^n(0,t) = u_{x^3}^n(1,t) = u_{x^5}^n(0,t) = u_{x^5}^n(1,t) = 0, \tag{47}$$

$$u^n(x,0) = u_0^n(x), \quad u_t^n(x,0) = u_1^n(x), \tag{48}$$

where $u_0^n(x) = \sum_{j=1}^n a_j w_j(x)$ and $u_1^n = \sum_{j=1}^n b_j w_j(x)$ are the approximation of the

$$u_0(x) = \sum_{j=1}^{\infty} a_j w_j(x), \qquad u_1(x) = \sum_{j=1}^{\infty} b_j w_j(x),$$

here $a_j$ and $b_j$ are constants.

It follows from (22) that

$$\|u_{tt}^n(t)\|^2 + \|u_t^n(t)\|_{H^3(\Omega)}^2 + \|u^n(t)\|_{H^6(\Omega)}^2 \le C_{20}(T), \quad t \in [0,T]. \tag{49}$$

Therefore, by (49) we can employ the same method as in Theorem 3.5 to prove that $u(x,t)$ satisfies the identity

$$\int_0^T \int_\Omega \Big[u_{tt} - u_{xx} - \alpha u_{x^4} - \beta u_{x^6} - \sigma(u)_{xx}\Big] g(x,t)dxdt = 0, \quad g(x,t) \in L^2(\Omega \times (0,T))$$

and the initial boundary value conditions (5) in the classical sense. Hence $u(x,t)$ is a global generalized solution. Using the same method as in Theorem 3.5 we can obtain the uniqueness.Theorem 2.1 is proved. □

*Proof of Theorem 2.2.* From (38) and Sobolev embedding theorem, we have

$$\|u^n(t)\|_{C^9(\overline{\Omega})} + \|u_t^n(t)\|_{C^6(\overline{\Omega})} + \|u_{tt}^n(t)\|_{C(\overline{\Omega})} + \|u_{ttt}^n(t)\|_{C(\overline{\Omega})} \le C_{21}(T),\ t \in [0,T]. \tag{50}$$

Differentiating equation (7) with respect to $x$, substituting $v_x(x,t) = u(x,t)$ into the product equation, since $v(x,t)$ is the classical global solution of the problem (7)-(9), by (50), we know that $u(x,t)$ is the classical global solution of initial boundary value problem (4),(5). The uniqueness is obvious. Theorem 2.2 is proved. □

In the following we study the initial boundary value problem (4),(6).

*Proof of Theorem 2.3.* Let $v(x,t) = \int_0^x u(\xi,t)d\xi$, then $u = v_x$ and $v$ satisfies

$$v_{tt} = v_{xx} + \sigma(v_x)_x + \alpha v_{x^4} + \beta v_{x^6},\ x \in \Omega,\ t > 0, \tag{51}$$

$$\begin{aligned}&v_x(0,t) = v_x(1,t) = v_{x^3}(0,t) = v_{x^3}(1,t) = v_{x^5}(0,t) = v_{x^5}(1,t) = 0,\ t > 0,\\ &v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x),\ x \in \overline{\Omega},\end{aligned} \tag{52}$$

where $v_0(x) = \int_0^x u_0(\xi)d\xi, \quad v_1(x) = \int_0^x u_1(\xi)d\xi$.

We first prove the initial boundary value problem (51),(52) admits a unique global generalized and classical solution. For this goal, we also introduce an orthonormal base in $L^2(\Omega)$. Let $\{w_j(x)\}$ be the orthonormal base in $L^2(\Omega)$ composed of the eigenfunctions of the eigenvalue problem

$$w''(x) + \lambda w(x) = 0,\ x \in \Omega, \quad w'(0) = w'(1) = 0,$$

corresponding to eigenvalue $\lambda_j$ $(j = 1, 2, \cdots)$. And let $v^n(x,t) = \sum_{j=1}^{n} T_{jn}(t) w_j(x)$ be Galerkin approximate solution of the problem (51),(52), where $T_{jn}(t)$ $(j = 1, \cdots)$ are functions to be determined, $n$ is a nature number. Assume that the initial data $v_0(x)$ and $v_1(x)$ can be expressed by

$$v_0(x) = \sum_{j=1}^{\infty} \alpha_j w_j(x), \quad v_1(x) = \sum_{j=1}^{\infty} \beta_j w_j(x),$$

where $\alpha_j,\ \beta_j$ $(j = 1, 2, \cdots)$ are constants. Substituting the approximate solution $v^n(x,t)$ into (51),(52), we have

$$v^n_{tt} = v^n_{xx} + \sigma(v^n_x)_x + \alpha v^n_{x^4} + \beta v^n_{x^6}, \tag{53}$$

$$v^n_x(0,t) = v^n_x(1,t) = v^n_{x^3}(0,t) = v^n_{x^3}(1,t) = v^n_{x^5}(0,t) = v^n_{x^5}(1,t) = 0, \tag{54}$$

$$v^n(x,0) = v^n_0(x), \quad v^n_t(x,0) = v^n_1(x). \tag{55}$$

Multiplying both sides of (53) and (55) by $w_j(x)$, respectively, and integrating on $\Omega$, we obtain

$$(v^n_{tt} - v^n_{xx} - \alpha v^n_{x^4} - \beta v^n_{x^6} - \sigma(v^n_x)_x, w_j) = 0, \tag{56}$$

$$T_{jn}(0) = \alpha_j, \quad \dot{T}_{jn}(0) = \beta_j, \quad j = 1, 2, \cdots, n. \tag{57}$$

In the following, we can employ a similar method as deriving Theorem 2.1 to complete the proof of Theorem 2.3. Theorem 2.3 is proved. □

*Proof of Theorem 2.4.* We can employ the same method as deriving Theorem 2.2 to complete the proof of Theorem 2.4. □

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