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A new modified secant-like method for solving nonlinear equations

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ABSTRACT

In this paper, we present a new secant-like method for solving nonlinear equations. Analysis of the convergence shows that the asymptotic convergence order of this method is $1+\sqrt{3}$. Some numerical results are given to demonstrate its efficiency.

1. Introduction

Numerical methods for solving nonlinear equations is a popular and important research topic in numerical analysis. In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x)=0$, where $f: D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function.

Newton's method is an important and basic approach for solving nonlinear equations [1,2], and its formulation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This method converges quadratically.

To improve the local order of convergence, a number of modified methods have been studied and reported in the literature (for example, [3–21]). By employing a second-derivative evaluation, we can obtain some well-known third-order methods, such as Chebyshev's method, Halley's method and the super-Halley method [3,4]. In order to replace the second derivative, an evaluation of the function or first derivative is added, and then many third-order and higher-order methods are obtained (for example, [5–19]).

However, in many other cases, it is expensive to compute the first derivative, and the above methods are still restricted in practical applications. The well-known secant method is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n). \tag{2}$$

This method can be derived by finding the root of the linear polynomial function

$$L_1(x) = f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_n). \tag{3}$$

This method does not require any derivative, but its order is only 1.618.

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In order to improve this method, Zhang et al. [22] consider

$$L_2(x) = f(x_n) + \frac{f(y_n) - f(x_n)}{y_n - x_n}(x - x_n), \tag{4}$$

and then find x_{n+1} such that $L_2(x_{n+1}) = 0$. From this, they obtain

$$x_{n+1} = x_n - \frac{x_n - y_n}{f(x_n) - f(y_n)} f(x_n), \tag{5}$$

where y_{n+1} is defined by

$$y_{n+1} = x_{n+1} - \frac{x_n - y_n}{f(x_n) - f(y_n)} f(x_{n+1}). \tag{6}$$

This method is also a secant-like method, and the order is improved to 2.618.

In this paper, we attempt to improve the order of the method proposed in [22] by using previous information, and then we present a new iterative method for solving nonlinear equations. Analysis of the convergence shows that the asymptotic convergence order of this method is $1 + \sqrt{3}$. The practical utility is demonstrated by numerical results.

2. Notation and basic results

Let $f(x)$ be a real function with a simple root x^* and let $\{x_n\}_{n\in N}$ be a sequence of real numbers that converges to x^*. We say that the *order of convergence* is q if there exists a $q \in \mathbb{R}^+$ such that

$$\lim_{n\to+\infty} \frac{x_{n+1} - x^*}{(x_n - x^*)^q} = C \neq 0, \infty.$$

Let $e_n = x_n - x^*$ be the nth iterate error. We call

$$e_{n+1} = Ce_n^q + \cdots \tag{7}$$

the *error equation*, in which the higher-order terms are neglected. If we can obtain the error equation for the method, then the value of q is its order of convergence.

3. The method and its convergence

Here, in order to construct our method, we use the following the second-order polynomial function:

$$P(x) = f(x_n) + \upsilon_n^{-1}(x - x_n) + \frac{\left(\upsilon_{n-1}^{-1} - \upsilon_n^{-1}\right)(x - x_n)(x - y_n)}{\alpha_1 x_{n-1} + \alpha_2 y_{n-1} + (2 - \alpha_1 - \alpha_2) z_{n-1} - \beta_1 x_n - \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, and y_n, z_n and υ_n are defined by $y_n = x_n - \nu_{n-1} f(x_n)$, $\nu_n = (y_n - x_n)/(f(y_n) - f(x_n))$ and $z_n = x_n - \nu_n f(x_n)$, respectively.

It is easy to obtain that

$$\left(\upsilon_{n-1}^{-1} - \upsilon_n^{-1}\right)(y_n - x_n)(z_n - y_n) = \upsilon_n^{-1}(y_n - z_n)^2.$$

In order to eliminate the nonlinearity, we replace x_{n+1} in the terms $(x_{n+1} - x_n)$ and $(x_{n+1} - y_n)$ of $P(x_{n+1})$ with y_n and z_n, respectively, and then finding its solution, we use the following new method:

$$\begin{cases} y_n = x_n - \nu_{n-1} f(x_n), \\ \nu_n = (y_n - x_n)/(f(y_n) - f(x_n)), \\ z_n = x_n - \nu_n f(x_n), \\ x_{n+1} = z_n - \dfrac{(y_n - z_n)^2}{\alpha_1 x_{n-1} + \alpha_2 y_{n-1} + (2 - \alpha_1 - \alpha_2) z_{n-1} - \beta_1 x_n - \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n}, \end{cases} \tag{8}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.

The method defined by (8) can be viewed as an iterative method with three substeps. The first two substeps are a variant of the method proposed in [22]. The third substep is an acceleration by using the values computed previously.

At the beginning of the process, the value of ν_{-1} needs to be given by some approaches. One choice of ν_{-1} is given by

$$\nu_{-1} = \varepsilon,$$

while another choice is

$$\nu_{-1} = \frac{\varepsilon f(x_0)}{f(x_0 + \varepsilon f(x_0)) - f(x_0)}.$$

Here, ε is a nonzero real number. The latter requires one more evaluation of the function than the former. However, the choice of ν_{-1} cannot affect the asymptotic convergence order of the method defined by (8).

Since the values of x_{-1}, y_{-1} are not available, the first iteration cannot carry out the third substep, and hence we let $x_1 = z_0$.

The following theorem indicates the best choice of parameters in the new method.

Theorem 1. *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $x^* \in D$. Let $f(x)$ have first, second and third derivatives in the interval D; then the asymptotic convergence order of the method defined by (8) is $1+\sqrt{3}$ when $\alpha_1 = \alpha_2 = 1$.*

Proof. Let $d_n = y_n - x^*$ and $\omega_n = z_n - x^*$. Using Taylor expansion and taking into account that $f(x^*) = 0$, we get

$$f(x_n) = f'(x^*)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots\right], \tag{9}$$

where $c_k = (1/k!) f^{(k)}(x^*)/f'(x^*)$, $k = 2, 3, \ldots$. Furthermore, we have

$$f(y_n) = f'(x^*)\left[d_n + c_2 d_n^2 + c_3 d_n^3 + \cdots\right]. \tag{10}$$

From (9) and (10), we have

$$\begin{aligned}
\nu_n &= (y_n - x_n)/(f(y_n) - f(x_n)),\\
&= \frac{1}{f'(x^*)} \frac{(y_n - x^*) - (x_n - x^*)}{(d_n - e_n) + c_2(d_n^2 - e_n^2) + c_3(d_n^3 - e_n^3) + \cdots}\\
&= \frac{1}{f'(x^*)} \frac{d_n - e_n}{(d_n - e_n) + c_2(d_n^2 - e_n^2) + c_3(d_n^3 - e_n^3) + \cdots}\\
&= \frac{1}{f'(x^*)} \frac{1}{1 + c_2(e_n + d_n) + c_3(e_n^2 + e_n d_n + d_n^2) + \cdots}\\
&= \frac{1}{f'(x^*)}\left[1 - c_2(e_n + d_n) - c_3(e_n^2 + e_n d_n + d_n^2) + c_2^2(e_n + d_n)^2 + \cdots\right].
\end{aligned} \tag{11}$$

Thus it follows from (9), (10), (11) and $y_n - x_n = -\upsilon_{n-1} f(x_n)$ that

$$\begin{aligned}
d_n &= e_n - \left[1 - c_2(e_{n-1} + d_{n-1}) - c_3(e_{n-1}^2 + e_{n-1}d_{n-1} + d_{n-1}^2) + c_2^2(e_{n-1} + d_{n-1})^2 + \cdots\right]\left(e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots\right)\\
&= c_2 e_n(e_{n-1} + d_{n-1}) + (c_3 - c_2^2) e_n (e_{n-1} + d_{n-1})^2 - c_3 e_n e_{n-1} d_{n-1} - c_2 e_n^2 + c_2^2 e_n^2 (e_{n-1} + d_{n-1}) + \cdots,
\end{aligned} \tag{12}$$

and hence, we obtain

$$\begin{aligned}
\omega_n &= x_n - x^* - \nu_n f(x_n)\\
&= e_n - \left[1 - c_2(e_n + d_n) - c_3(e_n^2 + e_n d_n + d_n^2) + c_2^2(e_n + d_n)^2 + \cdots\right]\left[e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots\right]\\
&= c_2 e_n d_n + (c_3 - c_2^2)(e_n^2 d_n + e_n d_n^2) + \cdots.
\end{aligned} \tag{13}$$

From (8), we obtain

$$e_{n+1} = \omega_n - \frac{(d_n - \omega_n)^2}{\alpha_1 e_{n-1} + \alpha_2 d_{n-1} + (2 - \alpha_1 - \alpha_2)\omega_{n-1} - \beta_1 e_n - \beta_2 d_n - (2 - \beta_1 - \beta_2)\omega_n}. \tag{14}$$

We first consider the case $\alpha_1 = 0$, $\alpha_2 \neq 0$, and in this case we have

$$\begin{aligned}
e_{n+1} &= c_2 e_n d_n - \frac{1}{\alpha_2} \frac{c_2 e_n\left(\frac{e_{n-1}}{d_{n-1}} + 1\right) + \cdots}{1 + \frac{2-\alpha_2}{\alpha_2}\frac{\omega_{n-1}}{d_{n-1}} + \cdots}(d_n - \omega_n) + \cdots\\
&= c_2 e_n d_n - \frac{e_n d_n}{\alpha_2 e_{n-2}} + \cdots\\
&= -\frac{c_2 e_n^2 e_{n-1}}{\alpha_2 e_{n-2}} + \cdots.
\end{aligned} \tag{15}$$

From (13) and (15), we can see that, from z_n to x_{n+1}, the order is not improved when $\alpha_1 = 0$, $\alpha_2 \neq 0$. Similarly, it is obtained that the case $\alpha_1 = \alpha_2 = 0$ also cannot improve the order. We now turn to consider the case $\alpha_1 \neq 0$, and using (12)–(14), we obtain

$$\begin{aligned}
e_{n+1} &= c_2e_nd_n - \frac{1}{\alpha_1}\frac{c_2e_n\left(1+\frac{d_{n-1}}{e_{n-1}}\right)+\cdots}{1+\frac{\alpha_2}{\alpha_1}\frac{d_{n-1}}{e_{n-1}}+\cdots}(d_n-\omega_n)+\cdots \\
&= \left(1-\frac{1}{\alpha_1}\right)c_2e_nd_n - \frac{1}{\alpha_1}\left(1-\frac{\alpha_2}{\alpha_1}\right)c_2e_nd_n\frac{d_{n-1}}{e_{n-1}}+\cdots \\
&= \left(1-\frac{1}{\alpha_1}\right)c_2e_nd_n - \frac{1}{\alpha_1}\left(1-\frac{\alpha_2}{\alpha_1}\right)c_2^2e_nd_ne_{n-2}+\cdots \\
&= \left(1-\frac{1}{\alpha_1}\right)c_2^2e_n^2e_{n-1} + \left[1-\frac{1}{\alpha_1}\left(2-\frac{\alpha_2}{\alpha_1}\right)\right]c_2^3e_n^2e_{n-1}e_{n-2}+\cdots.
\end{aligned} \tag{16}$$

From (16), we can see that the order will be improved by taking $\alpha_1 = \alpha_2 = 1$. In the following, by letting $\alpha_1 = \alpha_2 = 1$, then from (12), (13) and (14), we obtain

$$\begin{aligned}
e_{n+1} &= \omega_n - \frac{d_n-\omega_n}{e_{n-1}+d_{n-1}-\beta_1e_n-\beta_2d_n-(2-\beta_1-\beta_2)\omega_n}(d_n-\omega_n) \\
&= c_2e_nd_n - (d_n-\omega_n)\frac{c_2e_n(e_{n-1}+d_{n-1})+(c_3-c_2^2)e_n(e_{n-1}+d_{n-1})^2-c_3e_ne_{n-1}d_{n-1}+\cdots}{e_{n-1}+d_{n-1}-\beta_1e_n-\beta_2d_n-(2-\beta_1-\beta_2)\omega_n}+\cdots \\
&= c_2e_nd_n - \frac{c_2e_n+(c_3-c_2^2)e_n(e_{n-1}+d_{n-1})-\frac{c_3e_ne_{n-1}d_{n-1}}{e_{n-1}+d_{n-1}}+\cdots}{1-\frac{\beta_1e_n+\beta_2d_n+(2-\beta_1-\beta_2)\omega_n}{e_{n-1}+d_{n-1}}}(d_n-\omega_n)+\cdots \\
&= c_2e_nd_n - \left[c_2e_n+(c_3-c_2^2)e_n(e_{n-1}+d_{n-1})-\frac{c_3e_ne_{n-1}d_{n-1}}{e_{n-1}+d_{n-1}}+\cdots\right] \\
&\quad\times\left[1+\frac{\beta_1e_n+\beta_2d_n+(2-\beta_1-\beta_2)\omega_n}{e_{n-1}+d_{n-1}}+\cdots\right]\times(d_n-\omega_n)+\cdots \\
&= c_2e_nd_n - \left[c_2e_n+(c_3-c_2^2)e_n(e_{n-1}+d_{n-1})-\frac{c_3e_nd_{n-1}}{1+\frac{d_{n-1}}{e_{n-1}}}+\cdots\right]\left[1+\frac{\beta_1e_n+\cdots}{e_{n-1}+d_{n-1}}+\cdots\right]\times(d_n-\omega_n)+\cdots \\
&= c_2e_nd_n - \left[c_2e_n+(c_3-c_2^2)e_n(e_{n-1}+d_{n-1})-\frac{c_3e_nd_{n-1}}{1+\frac{d_{n-1}}{e_{n-1}}}+\cdots\right]\left[1+\frac{\beta_1\frac{e_n}{e_{n-1}}}{1+\frac{d_{n-1}}{e_{n-1}}}+\cdots\right]\times(d_n-\omega_n)+\cdots \\
&= c_2e_nd_n - \left[c_2e_n+(c_3-c_2^2)e_n(e_{n-1}+d_{n-1})-c_3e_nd_{n-1}\left(1-\frac{d_{n-1}}{e_{n-1}}\right)+\cdots\right] \\
&\quad\times\left[1+\beta_1\frac{e_n}{e_{n-1}}\left(1-\frac{d_{n-1}}{e_{n-1}}\right)+\cdots\right]\times(d_n-\omega_n)+\cdots \\
&= c_2e_nd_n - \left[c_2e_n+(c_3-c_2^2)e_ne_{n-1}+c_2\beta_1\frac{e_n^2}{e_{n-1}}+\cdots\right]\times(d_n-\omega_n)+\cdots \\
&= (c_2^2-c_3)e_ne_{n-1}d_n - c_2\beta_1\frac{e_n^2}{e_{n-1}}d_n+\cdots \\
&= c_2(c_2^2-c_3)e_n^2e_{n-1}^2 - c_2^2\beta_1e_n^3+\cdots \\
&= c_2(c_2^2-c_3)e_n^2e_{n-1}^2+\cdots.
\end{aligned} \tag{17}$$

Let $C = c_2(c_2^2-c_3)$; then (17) becomes

$$e_{n+1} = Ce_n^2e_{n-1}^2+\cdots. \tag{18}$$

Suppose that the order of (8) is q when $\alpha_1 = \alpha_2 = 1$; then from (7) we have

$$e_n = Ce_{n-1}^q+\cdots, \tag{19}$$

and

$$e_{n+1} = Ce_n^q+\cdots = C^{q+1}e_{n-1}^{q^2}+\cdots. \tag{20}$$

Table 1
Test functions and their roots.

Example	Test functions	Root
1	$xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	-1.2076478271309
2	$\sin^2(x) - x^2 + 1$	1.4044916482153
3	$e^{x^2+7x-30} - 1$	3

Table 2
Comparison of various iterative methods for Example 1.

		Secant	ZLLM	New
$x_0 = -1$	n	8	6	5
	$\|f(x_n)\|$	1.72e−16	8.93e−32	1.18e−24
$x_0 = -1.5$	n	9	6	6
	$\|f(x_n)\|$	3.19e−21	1.83e−24	1.15e−43

Table 3
Comparison of various iterative methods for Example 2.

		Secant	ZLLM	New
$x_0 = 1.5$	n	6	4	4
	$\|f(x_n)\|$	1.61e−20	8.55e−25	8.52e−36
$x_0 = 2.5$	n	9	6	5
	$\|f(x_n)\|$	5.35e−25	2.17e−37	7.48e−23

Substituting (19) and (20) into (18) gives

$$C^{q+1}e_{n-1}^{q^2} = C^3 e_{n-1}^{2q+2} + \cdots, \tag{21}$$

which implies that

$$q^2 = 2q + 2. \tag{22}$$

It is obtained from (22) that the asymptotic convergence order $q = 1 + \sqrt{3}$. □

By Theorem 1, we take $\alpha_1 = \alpha_2 = 1$ in (8), and obtain the present method given by

$$\begin{cases} y_n = x_n - \nu_{n-1} f(x_n), \\ \nu_n = (y_n - x_n)/(f(y_n) - f(x_n)), \\ z_n = x_n - \nu_n f(x_n), \\ x_{n+1} = z_n - \dfrac{(y_n - z_n)^2}{x_{n-1} + y_{n-1} - \beta_1 x_n - \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n}, \end{cases} \tag{23}$$

where $\beta_1, \beta_2 \in \mathbb{R}$.

Theorem 1 shows that the asymptotic convergence order of the present method (23) is $1+\sqrt{3}$. We note that this method does not require any derivatives, which is efficient especially when the computational cost of the derivative is expensive.

4. Numerical examples

Now, we employ the new method given by (23) with $\beta_1 = \beta_2 = 0$ and $\nu_{-1} = 1$ to solve some nonlinear equations. The performance of the present method with the secant method given by (2) and the method given by (5) and (6) [22] (ZLLM) is compared. For the ZLLM, we take $y_0 = x_0 - f(x_0)$. For the secant method, we take $x_{-1} = x_0 - f(x_0)$.

Table 1 shows the expression of the test functions and the root with 14 significant digits. All computational results are displayed in Tables 2–4.

In these methods it is necessary to begin with one initial approximation, x_0. In the first column of Tables 2–4 we present the initial approximation, which is the same for all methods.

The iterative method is stopped when $|f(x_n)| < 1\text{e}{-15}$. In Tables 2–4, we show the number of iterations costed by each method and the evaluations of f at the final approximate roots computed by each method.

The results in Tables 2–4 show that the present method is efficient.

Table 4
Comparison of various iterative methods for Example 3.

		Secant	ZLLM	New
$x_0 = 2.95$	n	9	6	5
	$\lvert f(x_n) \rvert$	5.48e−25	9.41e−31	6.99e−27
$x_0 = 3.05$	n	29	11	6
	$\lvert f(x_n) \rvert$	5.89e−16	6.13e−18	1.17e−24

5. Conclusions

We present a new iterative method for solving nonlinear equations. Theorem 1 shows that the asymptotic convergence order of this method is $1 + \sqrt{3}$. This method requires no derivatives, so it is especially efficient when the computational cost of the derivative is expensive.

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