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A Semilinear Elliptic Equation with Double Resonance

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Abstract In this paper, the existence and multiplicity of a class of double resonant semilinear elliptic equations with the Dirichlet boundary value are studied.

Keywords Semilinear elliptic equation, double resonance

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1 Introduction

We consider the semilinear elliptic equation

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following linear growth condition: there exists a constant $C_0 > 0$ such that

$$|f(x,u)| \leq C_0(1+|u|), \quad \forall\, u \in \mathbb{R},\ x \in \Omega. \tag{1.2}$$

We look for the weak solutions of (1.1) which are the same as the critical points of the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,t)dt$, and $H_0^1(\Omega)$ is the Sobolev space with the norm

$$\|u\| = \|\nabla u\|_2 = \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}.$$

A special case of condition (1.2) is that f is asymptotically linear at infinity: there exists a continuous function $a(x)$ such that

$$f(x,u) = a(x)u + o(|u|) \quad \text{as } |u| \to \infty. \tag{f_1}$$

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Equation (1.1) is called non-resonant if

$$\begin{cases} -\Delta u - a(x)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no non-zero solution. Let $\lambda_1(a) < \lambda_2(a) \leq \cdots \leq \lambda_k(a) \leq \cdots$ be the eigenvalues of the equation

$$-\Delta u - a(x)u = \lambda u$$

with the Dirichlet boundary value, and let $\lambda_k(0) = \lambda_k$. Then the non-resonance of (1.1) means that $\lambda_k(a) \neq 0$ for all k. In some literatures, (1.1) is called non-resonant if there exists a constant $\epsilon > 0$ such that

$$\lambda_k + \epsilon \leq \frac{f(x,u)}{u} \leq \lambda_{k+1} - \epsilon, \quad \text{for } x \in \Omega, \ |u| \gg 1. \tag{1.3}$$

It is well known that if (1.1) is non-resonant, then (1.1) has a solution. This can be proved either by minimax argument or by Morse theory (see [1, 2]).

Let the condition (f_1) be satisfied. Equation (1.1) is called resonant if $\lambda_k(a) = 0$ for some k. In this case, it is more complicated for the solvability. There are a lot of literatures studying this problem. For instance, if

$$|f(x,u) - \lambda_k u| \leq C, \quad \forall\, u \in \mathbb{R}, \ x \in \Omega, \tag{$(f_1)'$}$$

for a constant $C > 0$ and the Landesman–Lazer condition

$$\lim_{\|u\| \to \infty} \int_\Omega \left[F(x,u) - \frac{\lambda_k}{2} u^2 \right] dx = +\infty \ (\text{or } \ -\infty), \quad \text{where } u \in \operatorname{Ker}(-\Delta - \lambda_k)$$

holds, then (1.1) has a solution (see [2]). For other versions of this condition and some multiplicity theorems by using the topological degree technique and variational methods, we refer to [3–9] and references therein.

Equation (1.1) is called double resonant at infinity if

$$\lambda_k \leq \frac{f(x,u)}{u} \leq \lambda_{k+1}, \quad \text{for } x \in \Omega, \ |u| \gg 1, \tag{1.4}$$

and for some $x \in \Omega$, either the equalities hold or

$$\underline{\lim}_{|u| \to \infty} \frac{f(x,u)}{u} = \lambda_k$$

and

$$\overline{\lim}_{|u| \to \infty} \frac{f(x,u)}{u} = \lambda_{k+1}.$$

This means that $\frac{f(x,u)}{u}$ oscillates between two consecutive eigenvalues. The existence of solutions of (1.1) with this condition has also been studied by many authors (see [10–12]). In particular, it is proved in [10] that if (1.4) and the Landesman–Lazer conditions

$$\lim_{\|v\| \to \infty} \int_\Omega \left[F(x,v) - \frac{\lambda_k}{2} v^2 \right] dx = +\infty, \quad \text{where } v \in \operatorname{Ker}(-\Delta - \lambda_k), \tag{1.5}$$

$$\lim_{\|v\| \to \infty} \int_\Omega \left[F(x,v) - \frac{\lambda_{k+1}}{2} v^2 \right] dx = -\infty, \quad \text{where } v \in \operatorname{Ker}(-\Delta - \lambda_{k+1}) \tag{1.6}$$

hold, then (1.1) has a solution. There are also many papers concerning the solvability of resonant (1.1) with the following non-quadratic condition:

$$\lim_{|u|\to\infty}[f(x,u)u-2F(x,u)]=\pm\infty,\quad \text{uniformly for } x\in\Omega,$$

see for example [13–15].

In the case that f depends on x, comparing with (1.4), it is more natural to consider equation (1.1) with the condition: $\exists\, M_0>0$ such that

$$a(x)\le \frac{f(x,u)}{u}\le b(x),\quad \text{for } |u|\ge M_0,\ x\in\Omega, \tag{1.7}$$

where a and b are continuous functions. It is proved in [15] that equation (1.1) possesses a solution if

$$\lambda_k\preceq a(x)\le \frac{f(x,u)}{u}\le b(x)\preceq \lambda_{k+1},\quad |u|\gg 1 \tag{1.8}$$

uniformly for $x\in\Omega$, where $\lambda_k\preceq a(x)$ means that $\lambda_k\le a(x)$ and the inequality holds on a set of positive measure. This result can be generalized as follows: Let (1.7) be satisfied and

$$\lambda_k(a)<0,\quad \lambda_{k+1}(b)>0 \tag{1.9}$$

for some integer k; then equation (1.1) has a solution, see [16], and [17] for the ODE case. Clearly, (1.9) is a generalization of (1.3) and (1.8). It is easy to construct a function $f(x,u)$ such that (1.7) and (1.9) hold, but $\frac{f(x,u)}{u}$ can across many eigenvalues of $-\Delta$ for some $x\in\Omega$. As pointed out in [16] that (1.9) is indeed a type of non-resonant condition.

The aim of our paper is to find solutions of equation (1.1) with (1.7) and the following double resonant condition

$$\lambda_k(a)\le 0,\quad \lambda_{k+1}(b)\ge 0. \tag{1.10}$$

Our main result is the following:

Theorem 1.1 *Suppose that the conditions* (1.7), (1.10) *and the following generalized Landesman–Lazer conditions are satisfied*:

(f_2) $\lim_{\|v\|\to\infty}\int_\Omega[F(x,v)-\frac{a(x)}{2}v^2]dx=+\infty$, *where* $v\in\operatorname{Ker}(-\Delta-a)$,

(f_3) $\lim_{\|v\|\to\infty}\int_\Omega[F(x,v)-\frac{b(x)}{2}v^2]dx=-\infty$, *where* $v\in\operatorname{Ker}(-\Delta-b)$.

Then equation (1.1) *has a solution.*

Remark 1.2 The periodic boundary value problem of

$$-u''=f(x,u) \tag{1.11}$$

with

$$a(x)\le \liminf_{|u|\to\infty}\frac{f(x,u)}{u}\le \limsup_{|u|\to\infty}\frac{f(x,u)}{u}\le b(x) \tag{1.12}$$

uniformly for $x\in S^1=\mathbb{R}/2\pi\mathbb{Z}$ and the Landesman–Lazer conditions

$$\int_{u>0}\liminf_{t\to\infty}g_1(x,t)u(x)dx+\int_{u<0}\limsup_{t\to-\infty}g_1(x,t)u(x)dx>0, \tag{1.13}$$

$$\int_{v>0}\liminf_{t\to\infty}g_2(x,t)v(x)dx+\int_{v<0}\limsup_{t\to-\infty}g_2(x,t)v(x)dx>0, \tag{1.14}$$

where $u \in \mathrm{Ker}(-\Delta - a)\backslash\{0\}$, $v \in \mathrm{Ker}(-\Delta - b)\backslash\{0\}$, and $g_1(x,t) = f(x,t) - a(x)t$, $g_2(x,t) = b(x)t - f(x,t)$ has been considered in [18]. Under the condition $\lambda_k(a) \le 0 \le \lambda_{k+1}(b)$, using the topological degree methods it was proved that equation (1.11) possesses a 2π-periodic solution. Obviously, (1.12) is weaker than (1.7). However, there are functions that do not satisfy (1.13) and (1.14), but (f_2) and (f_3). An example of such function is

$$f(x,t) = \lambda_k t + \frac{1}{t}, \quad |t| > 0$$

with $a = \lambda_k$.

The proof of our theorem is based on the linking theorem. There are two main difficulties for the double resonant problem. One is the Palais–Smale condition for I and the other is to construct linking sets which rely on a decomposition of $H_0^1(\Omega)$. For the case $a = \lambda_k$ and $b = \lambda_{k+1}$, we have $H_0^1(\Omega) = E_1 \oplus E_2$, where $E_1 = \bigoplus_{j\le k} \mathrm{Ker}(-\Delta - \lambda_j)$ and E_2 is the orthogonal complement of E_1. This can not be used if (1.8) is replaced by (1.10). We need to give a decomposition of $H_0^1(\Omega)$ according to the eigenfunctions of different functions a and b. Such a decomposition has been used in [18] for the ODE.

The paper is organized as follow: In Section 2, the Palais–Smale condition for I is proved. In Section 3, a decomposition lemma for $H_0^1(\Omega)$ is recalled, which is the basis of linking theorem, and the proof of Theorem 1.1 is given. In Section 4, nontrivial solutions of (1.1) are studied. In the sequel, the letter C will be used to denote various positive constants whose exact value is irrelevant.

2 The Palais–Smale Condition

In this section, we will prove that the Palais–Smale condition holds for I.

Definition 2.1 *A functional I is said to satisfy Palais–Smale* ((P.S.) *for short*) *condition, if every sequence $\{u_n\} \subset H_0^1(\Omega)$ with*

$$I(u_n) \text{ being bounded, } (\|u_n\| + 1)I'(u_n) \to 0 \text{ as } n \to \infty, \tag{2.1}$$

possesses a convergent subsequence.

This Palais–Smale type condition was introduced by Cerami, and it was shown that this condition suffices to get a deformation theorem and the critical point theorem (see [19]).

Proposition 2.2 *Under the assumptions of Theorem* 1.1*, the functional I satisfies the* (P.S.) *condition.*

We will prove the proposition by several lemmas following the argument in [10] with slight modification. Let $\{u_n\} \subset H_0^1(\Omega)$ be a (P.S.) sequence. In order to prove the proposition, it suffices to show that u_n is bounded in $H_0^1(\Omega)$ by the condition (1.2).

By contradiction, we assume $\|u_n\| \to \infty$ as $n \to \infty$. Then $\|u_n\|_2 \to \infty$ since

$$\|u_n\| \le C(1 + \|u_n\|_2)$$

by (1.2) and (2.1). Let $z_n = \frac{u_n}{\|u_n\|_2}$, then there exist $z \in H_0^1(\Omega)$ and a subsequence such that $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$, and $z_n \to z$ strongly in $L^2(\Omega)$, $\|z\|_2 = 1$.

By condition (1.7), let

$$c_n(x) = \frac{f(x, u_n(x))}{u_n(x)} \text{ if } |u_n(x)| \geq M_0, \quad c_n(x) = a(x) \text{ if } |u_n(x)| < M_0.$$

Then $\{c_n\}$ is bounded in $L^\infty(\Omega)$. Hence, after a subsequence, there is a $c(x) \in L^q(\Omega)$ such that $c_n(x) \rightharpoonup c(x)$ weakly in $L^q(\Omega)$ as $n \to \infty$, where $2 < q < \frac{2N}{N-2}$ is fixed. Since the set

$$\{u \in L^q(\Omega) \,|\, a(x) \leq u(x) \leq b(x),\ x \in \Omega\}$$

is weakly closed, we get that

$$a(x) \leq c(x) \leq b(x), \quad \text{for } x \in \Omega.$$

Let $I'(u_n) = \varepsilon_n$. Using (2.1), for $\phi \in C_0^1(\Omega)$, we have

$$\int_\Omega \nabla z_n \nabla \phi dx - \int_\Omega \frac{f(x, u_n)}{\|u_n\|_2} \phi dx = (\varepsilon_n', \phi),$$

where $\varepsilon_n' = \varepsilon_n / \|u_n\|_2 \to 0$ as $n \to \infty$ in $(H_0^1(\Omega))^*$. Standard computations (cf. [16]) show that

$$\int_\Omega \nabla z \nabla \phi dx = \int_\Omega c(x) z \phi dx,$$

which implies that $z \neq 0$ is a solution of

$$\begin{cases} -\Delta u = c(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Case 1 If $a(x) < c(x)$ and $c(x) < b(x)$ on a subset of positive measure, then by strict monotonicity we have

$$\lambda_k(c) < \lambda_k(a) \leq 0, \quad \lambda_{k+1}(c) > \lambda_{k+1}(b) \geq 0,$$

which contradicts that equation (2.2) has a nonzero solution.

Case 2 $a(x) \equiv c(x)$.

We write

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2} \int_\Omega a(x) u^2 dx - \int_\Omega G(x, u) dx,$$

where $G(x, u) = \int_0^u g(x, t) dt,\ g(x, t) = f(x, t) - a(x) t$. Let $c_n^1(x) = c_n(x) - a(x)$, then $c_n^1(x) \geq 0$ and $c_n^1 \rightharpoonup 0$ weakly in $(L^{\frac{q}{2}}(\Omega))^* = L^{\frac{q}{q-2}}(\Omega)$. Let

$$\begin{aligned} H^- &= \bigoplus_{\lambda_i(a) < 0} \operatorname{Ker}(-\Delta - a - \lambda_i(a)), \\ H^0 &= \operatorname{Ker}(-\Delta - a), \\ H^+ &= \overline{\bigoplus_{\lambda_j(a) > 0} \operatorname{Ker}(-\Delta - a - \lambda_j(a))}, \end{aligned}$$

then we have

$$H_0^1(\Omega) = H^- \oplus H^0 \oplus H^+.$$

Lemma 2.3 *There exists a constant $\delta > 0$ such that*

$$\int_\Omega [|\nabla w|^2 - a(x)w^2]dx \geq \delta\|w\|^2, \quad \text{for } w \in H^+, \tag{2.3}$$

$$\int_\Omega [|\nabla v|^2 - a(x)v^2]dx \leq -\delta\|v\|^2, \quad \text{for } v \in H^-. \tag{2.4}$$

Proof Since $a \in C(\overline{\Omega})$, from the definition of H^+ we have

$$\begin{aligned}
\int_\Omega |\nabla w|^2 dx &= \int_\Omega [|\nabla w|^2 - a(x)w^2]dx + \int_\Omega a(x)w^2 dx \\
&\leq \int_\Omega [|\nabla w|^2 - a(x)w^2]dx + C\int_\Omega w^2 dx \\
&\leq \int_\Omega [|\nabla w|^2 - a(x)w^2]dx + \frac{C}{\lambda_+}\int_\Omega [|\nabla w|^2 - a(x)w^2]dx \\
&\leq C\int_\Omega [|\nabla w|^2 - a(x)w^2]dx,
\end{aligned}$$

where $\lambda_+ = \min\{\lambda_i(a) \,|\, \lambda_i(a) > 0\}$. This proves (2.3), and the proof of (2.4) is trivial. □

Let $u_n = v_n + u_n^0 + w_n$, where $v_n \in H^-, u_n^0 \in H^0$ and $w_n \in H^+$.

Lemma 2.4 *If $a(x) = c(x)$, then $\|v_n\|^2 + \|w_n\|^2$ is bounded.*

Proof Using the assumption (2.1), we have

$$(\|u_n\| + 1)\langle I'(u_n), w_n - u_n^0 - v_n\rangle = o(1)\|w_n - u_n^0 - v_n\|. \tag{2.5}$$

Lemma 2.3 gives

$$\begin{aligned}
\langle I'(u_n), w_n - u_n^0 - v_n\rangle &= \int_\Omega [|\nabla w_n|^2 - a(x)|w_n|^2]dx - \int_\Omega [|\nabla v_n|^2 - a(x)|v_n|^2]dx \\
&\quad - \int_\Omega g(x, u_n)(w_n - u_n^0 - v_n)dx \\
&\geq \delta(\|v_n\|^2 + \|w_n\|^2) - \int_\Omega g(x, u_n)(w_n - u_n^0 - v_n)dx.
\end{aligned} \tag{2.6}$$

This together with (2.5) implies that

$$\delta(\|v_n\|^2 + \|w_n\|^2) \leq C\frac{\|w_n - u_n^0 - v_n\|}{\|u_n\| + 1} + \int_\Omega g(x, u_n)(w_n - u_n^0 - v_n)dx. \tag{2.7}$$

Since $w_n - u_n^0 - v_n = 2w_n - u_n$, simple computations show that

$$\frac{\|w_n - u_n^0 - v_n\|}{\|u_n\| + 1} \leq C\|w_n\| + C, \quad \text{for } n \text{ large enough.} \tag{2.8}$$

Similarly, (1.2) and the Poincaré inequality imply that

$$\begin{aligned}
\int_{|u_n| \leq M_0} g(x, u_n)(w_n - u_n^0 - v_n)dx &= \int_{|u_n| \leq M_0} g(x, u_n)(2w_n - u_n)dx \\
&\leq \int_{|u_n| \leq M_0} |g(x, u_n)| \cdot |(2w_n - u_n)|dx \\
&\leq C(\int_{|u_n| \leq M_0} |2w_n - u_n|^2 dx)^{1/2} \\
&\leq C\|w_n\| + C.
\end{aligned} \tag{2.9}$$

By the embedding theorem, after a subsequence, we may assume $\frac{w_n^2}{\|w_n\|^2} \to w_0$ strongly in $L^{\frac{q}{2}}(\Omega)$. Then using $c_n^1 \rightharpoonup 0$ weakly in $(L^{\frac{q}{2}}(\Omega))^*$, we get

$$\begin{aligned}
\int_{|u_n|\geq M_0} g(x,u_n)(w_n - u_n^0 - v_n)dx &= \int_{|u_n|\geq M_0} \frac{g(x,u_n)}{u_n}[w_n^2 - (u_n^0 + v_n)^2]dx \\
&\leq \int_{|u_n|\geq M_0} \frac{g(x,u_n)}{u_n} w_n^2 dx \\
&= \int_{|u_n|\geq M_0} \frac{g(x,u_n)}{u_n} \frac{w_n^2}{\|w_n\|^2} dx \cdot \|w_n\|^2 \\
&= \int_{|u_n|\geq M_0} c_n^1(x) \frac{w_n^2}{\|w_n\|^2} dx \cdot \|w_n\|^2 \\
&\leq \frac{\delta}{2}\|w_n\|^2, \quad \text{for } n \text{ large enough.}
\end{aligned} \tag{2.10}$$

Then this lemma follows from (2.7)–(2.10). □

Lemma 2.5 *If $\|v_n\|^2 + \|w_n\|^2$ is bounded, then $I(u_n) \to -\infty$ as $n \to \infty$.*

Proof Since $\|v_n\|^2 + \|w_n\|^2$ is bounded, we get

$$\begin{aligned}
I(u_n) &= \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{1}{2}\int_\Omega a(x)u_n^2 dx - \int_\Omega G(x,u_n)dx \\
&\leq C + \int_\Omega \Big[G\Big(x, \frac{u_n^0}{2}\Big) - G(x,u_n)\Big]dx - \int_\Omega G\Big(x,\frac{u_n^0}{2}\Big)dx.
\end{aligned}$$

By the mean value theorem, we have

$$\int_\Omega \Big[G\Big(x,\frac{u_n^0}{2}\Big) - G(x,u_n)\Big]dx = \int_\Omega \int_0^1 g\Big(x, t\frac{u_n^0}{2} + (1-t)u_n\Big)\Big(\frac{u_n^0}{2} - u_n\Big)dtdx.$$

Assume that $h_n = t\frac{u_n^0}{2} + (1-t)u_n$, then by (1.2) it follows that

$$\begin{aligned}
&\int_{|h_n|\leq M_0} \Big[G\Big(x,\frac{u_n^0}{2}\Big) - G(x,u_n)\Big]dx \\
&= \int_{|h_n|\leq M_0} \int_0^1 g\Big(x, t\frac{u_n^0}{2} + (1-t)u_n\Big)\Big(\frac{u_n^0}{2} - u_n\Big)dtdx \\
&\leq \int_{|h_n|\leq M_0} \int_0^1 \Big|g\Big(x, t\frac{u_n^0}{2} + (1-t)u_n\Big)\Big| \cdot \Big|\Big(\frac{u_n^0}{2} - u_n\Big)\Big|dtdx \\
&\leq C\int_{|h_n|\leq M_0} \int_0^1 \Big|\frac{u_n^0}{2} - u_n\Big|dtdx \\
&= C\int_{|h_n|\leq M_0} \int_0^1 \Big|\frac{1}{t-2}h_n + \frac{1}{t-2}(u_n - u_n^0)\Big|dtdx \\
&\leq C\Big(\int_{|h_n|\leq M_0} |h_n + (v_n + w_n)|^2 dx\Big)^{1/2} \\
&\leq C.
\end{aligned} \tag{2.11}$$

On the other hand, by (1.7) and the elementary inequality

$$\Big(\frac{p}{2} - q\Big)^2 + \Big(\frac{p}{2} - q\Big)q \leq (p-q)^2, \quad \forall\, p,q \in \mathbb{R},$$

we deduce that

$$
\begin{aligned}
&\int_{|h_n|\geq M_0}\Big[G\Big(x,\frac{u_n^0}{2}\Big)-G(x,u_n)\Big]dx\\
&\quad=\int_{|h_n|\geq M_0}\int_0^1 g\Big(x,t\frac{u_n^0}{2}+(1-t)u_n\Big)\Big(\frac{u_n^0}{2}-u_n\Big)dtdx\\
&\quad=\int_{|h_n|\geq M_0}\int_0^1 \frac{g(x,h_n)}{h_n}\Big[t\Big(\frac{u_n^0}{2}-u_n\Big)^2+\Big(\frac{u_n^0}{2}-u_n\Big)u_n\Big]dtdx\\
&\quad\leq\int_{|h_n|\geq M_0}\int_0^1 \frac{g(x,h_n)}{h_n}(w_n+v_n)^2dtdx\\
&\quad\leq C\int_{|h_n|\geq M_0}|w_n+v_n|^2dx\leq C. \qquad (2.12)
\end{aligned}
$$

Therefore, (2.11) and (2.12) imply that

$$\int_\Omega\Big[G\Big(x,\frac{u_n^0}{2}\Big)-G(x,u_n)\Big]dx\leq C.$$

This together with the condition (f_2) and Lemma 2.4 gives

$$I(u_n)\to-\infty,\quad \text{as } n\to\infty.$$

We complete the proof. □

This lemma contradicts the assumption (2.1). Hence Case 2 is not possible.

Case 3 $c(x)\equiv b(x)$.

This case is similar to Case 2. Now, we set

$$I(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx-\frac{1}{2}\int_\Omega b(x)u^2dx-\int_\Omega F_1(x,u)dx,$$

where $F_1(x,u)=\int_0^u f_1(x,t)dt$, $f_1(x,t)=f(x,t)-b(x)t$, and

$$
\begin{aligned}
H_-&=\bigoplus_{\lambda_i(b)<0}\mathrm{Ker}(-\Delta-b-\lambda_i(b)),\\
H_0&=\mathrm{Ker}(-\Delta-b),\\
H_+&=\overline{\bigoplus_{\lambda_j(b)>0}\mathrm{Ker}(-\Delta-b-\lambda_j(b))}.
\end{aligned}
$$

Then

$$H_0^1(\Omega)=H_-\oplus H_0\oplus H_+.$$

Similarly to Case 2, we have

Lemma 2.6 *There exists a constant $\delta>0$ such that*

$$
\begin{aligned}
&\int_\Omega[|\nabla V|^2-b(x)V^2]dx\leq-\delta\|V\|^2,\quad \textit{for } V\in H_-,\\
&\int_\Omega[|\nabla W|^2-b(x)W^2]dx\geq\delta\|W\|^2,\quad \textit{for } W\in H_+.
\end{aligned}
$$

Let $u_n=V_n+U_n^0+W_n$, where $V_n\in H_-$, $U_n^0\in H_0$ and $W_n\in H_+$.

Lemma 2.7 *If $c(x)=b(x)$, then $\|V_n\|^2+\|W_n\|^2$ is bounded and $I(u_n)\to\infty$ as $n\to\infty$.*

This lemma implies that Case 3 is also impossible. Hence Proposition 2.2 is proved.

3 Proof of Theorem 1.1

In this section, we first recall a decomposition of $H_0^1(\Omega)$ from [16]. Such a decomposition has been used in [18] for the ODE case. The following proof is different from that in [18]. Let $\{\phi_j\}$, $\{\psi_j\}$ be the eigenfunctions of

$$\begin{cases} -\Delta u - a(x)u = \lambda_j(a)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and

$$\begin{cases} -\Delta u - b(x)u = \lambda_j(b)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

and we also assume that

$$\int_\Omega |\phi_j|^2 dx = \int_\Omega |\psi_j|^2 dx = 1, \quad \forall\, j.$$

Lemma 3.1 *Let $a, b \in C(\overline{\Omega})$ satisfying $a(x) \le b(x)$ for $x \in \Omega$, and let $k \ge 1$. If $\lambda_k(a) \le \lambda_{k+1}(b)$, then*

$$H_0^1(\Omega) = X_1 \oplus X_2,$$

where

$$X_1 = \text{span}\{\phi_1, \phi_2, \dots, \phi_k\}, \quad X_2 = \overline{\text{span}\{\psi_{k+1}, \psi_{k+2}, \dots\}}.$$

Proof Without loss of generality, we assume that $\Omega_0 = \{x \in \Omega \,|\, a(x) \ne b(x)\}$ is not empty. For $u \in H_0^1$, we have the following decompositions according to the eigenfunction expansions of (3.1) and (3.2), respectively,

$$u = \sum_{i=1}^k a_i\phi_i + \sum_{i=k+1}^\infty a_i\phi_i, \tag{3.3}$$

$$u = \sum_{i=1}^k b_i\psi_i + \sum_{i=k+1}^\infty b_i\psi_i. \tag{3.4}$$

Let $a_{ij} = \int_\Omega \phi_i(x)\psi_j(x)dx$, $1 \le i, j \le k$.

Claim $\det(a_{ij}) \ne 0$.

If not, there exists a non-zero vector $(c_1, \dots, c_k)$ such that

$$\int_\Omega \Phi(x)\psi_j(x)dx = 0, \ \ j = 1, 2, , \dots, k,$$

where $\Phi(x) = \sum_1^k c_i\phi_i \ne 0$ by the property of eigenfunctions (see [20]) and we also assume that

$$\int_\Omega |\Phi(x)|^2 dx = 1.$$

Then from the minimax characterization of the eigenvalues, this implies

$$\int_\Omega (|\nabla\Phi(x)|^2 - b(x)|\Phi(x)|^2)dx \ge \lambda_{k+1}(b).$$

On the other hand, because $\Phi \in X_1$, we have

$$\int_\Omega (|\nabla\Phi(x)|^2 - a(x)|\Phi(x)|^2)dx \le \lambda_k(a).$$

Since $\Phi \neq 0$ and $a(x) < b(x)$ on Ω_0, we get

$$\int_\Omega (|\nabla\Phi(x)|^2 - b(x)|\Phi(x)|^2)dx < \int_\Omega (|\nabla\Phi(x)|^2 - a(x)|\Phi(x)|^2)dx,$$

which implies that

$$\lambda_{k+1}(b) < \lambda_k(a).$$

This is a contradiction.

By the claim, we know that (a_{ij}) is invertible; let (b_{ij}) be the inverse of (a_{ij}). Then we have

$$\begin{aligned} u &= \sum_{i=1}^k b_i\psi_i + \sum_{i=k+1}^\infty b_i\psi_i \\ &= \sum_{i=1}^k \left(\sum_{j=1}^k b_j b_{ji}\right)\phi_i + \sum_{i=1}^k b_i\psi_i - \sum_{i=1}^k \left(\sum_{j=1}^k b_j b_{ji}\right)\phi_i + \sum_{i=k+1}^\infty b_i\psi_i \\ &= u_1 + u_2, \end{aligned}$$

where $u_1 = \sum_{i=1}^k(\sum_{j=1}^k b_j b_{ji})\phi_i \in X_1$. Next we show that

$$u_2 = \sum_{i=1}^k b_i\psi_i - \sum_{i=1}^k \left(\sum_{j=1}^k b_j b_{ji}\right)\phi_i + \sum_{i=k+1}^\infty b_i\psi_i \in X_2,$$

that is equivalent to

$$\int_\Omega u_2(x)\psi_l(x) = 0$$

for $1 \leq l \leq k$. This follows from

$$\int_\Omega u_2(x)\psi_l(x)dx = b_l - \sum_{i,j=1}^k b_j b_{ji} \int_\Omega \phi_i(x)\psi_l(x)dx = b_l - \sum_{j=1}^k b_j\delta_{jl} = 0.$$

If $u = 0$, then $b_j = 0$ for each j, hence $u_1 = 0$ and $u_2 = 0$. Thus we have shown that for each $u \in H_0^1(\Omega)$, there is a unique decomposition $u = u_1 + u_2$ with $u_1 \in X_1$ and $u_2 \in X_2$. It is easy to see from the definition of u_1 and u_2 that

$$\|u_1\| \leq C\|u\|, \quad \|u_2\| \leq C\|u\|$$

for some constant $C > 0$. Hence $H_0^1(\Omega)$ is the direct sum of X_1 and X_2. □

Proof of Theorem 1.1 (1) $I(v) \to -\infty$ as $\|v\| \to \infty,\ v \in X_1$.

If not, then there exist a constant A and a sequence $u_n = v_n + u_n^0 \in H^- \oplus H^0$ such that $\|u_n\| \to \infty$ as $n \to \infty$ and $I(u_n) \geq A$. On the other hand, using (1.7) and (2.4), we have

$$I(u_n) = \frac{1}{2}\int_\Omega (|\nabla u_n|^2 - a(x)u_n^2)dx + \int_\Omega \left(\frac{1}{2}a(x)u_n^2 - F(x,u_n)\right)dx \leq -\delta\|v_n\|^2 + C.$$

Therefore, $\|v_n\|^2$ is bounded and by Lemma 2.5 we obtain

$$I(u_n) \to -\infty, \quad \text{as } n \to \infty.$$

This is a contradiction.

(2) $I(w) \to \infty$ as $\|w\| \to \infty$, $w \in X_2$.

By contradiction, suppose that there exist a constant B and a sequence $u_n = u_n^0 + w_n \in H_0 \oplus H_+$ with $\|u_n\| \to \infty$ as $n \to \infty$ such that

$$I(u_n) \le B.$$

Then (1.7) and Lemma 2.6 imply that $\|w_n\|^2$ is bounded. This is impossible by Lemma 2.7.

(3) We fix an R such that

$$\sup_{u \in \partial B(R) \cap X_1} I(u) \le \beta - 1, \tag{3.5}$$

where $\beta = \inf_{u \in X_2} I(u)$, and $B(R) = \{u \in H_0^1(\Omega) \,|\, \|u\| \le R\}$. Set

$$\Gamma = \{\gamma : B(R) \cap X_1 \to H_0^1(\Omega) \,|\, \gamma(u) = u \text{ if } u \in X_1, \|u\| = R\},$$
$$c = \inf_{\gamma \in \Gamma} \max_{u \in B(R)} I(u).$$

Since $\partial B(R) \cap X_1$ and X_2 are linking and the (P.S.) condition holds for I, $c \ge \beta$ is a critical value of I. So there is a critical point $u \in H_0^1(\Omega)$ such that $I(u) = c$. The proof is finished. $\square$

4 Multiplicity of Solutions

In this section, we discuss multiplicity of nontrivial solutions of equation (1.1). First, let us recall some results of Morse theory that will be used below, for details, we refer to [2].

Let X be a real Banach space and $\Phi \in C^1(X, \mathbb{R})$ and satisfies the Palais–Smale condition. Let $K = \{u \in X \,|\, \Phi'(u) = 0\}$ be the critical set of Φ. Let $u \in K$ be an isolated critical point with $\Phi(u) = c \in \mathbb{R}$, and U be an isolated neighborhood of u, i.e., $K \cap U = \{u\}$. The group

$$C_*(\Phi, u) = H_*(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad * = 0, 1, 2, \ldots,$$

is called the $*$-th critical group of Φ at u, where $\Phi^c = \{u \in X \,|\, \Phi(u) \le c\}$, $H_*(\cdot,\cdot)$ are the singular relative homology groups with a coefficient group G.

Our main result in this section is the following:

Theorem 4.1 *Let the condition* (1.7) *be satisfied and let* $k \ge 2$. *In addition, we assume that*

(f_4) $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x, 0) = 0$,

(f_5) *there exists a continuous function* $m(x)$ *with* $\lambda_1(m) > 0$ *such that*

$$\lim_{|u| \to 0} \frac{2F(x,u)}{u^2} \le m(x), \quad \textit{for } x \in \Omega. \tag{4.1}$$

Then equation (1.1) *has at least* 3 *nontrivial solutions if one of the following conditions is satisfied*:

(i) $\lambda_k(a) < 0 < \lambda_{k+1}(b)$,

(ii) $\lambda_k(a) \le 0 \le \lambda_{k+1}(b)$ *and* (f_2), (f_3) *hold.*

Remark 4.2 There are many papers dealing with the multiplicity of nontrivial solutions of equation (1.1) with (f_4) and

$$f'(x, 0) < \lambda_1. \tag{4.2}$$

For example, the paper [10] uses the conditions (1.5) and (1.6); in [21], the authors assume that $\lim_{|t| \to \infty} \frac{f(x,t)}{t} = \lambda_k$ and the Landesman–Lazer condition holds; and in [22], the case of jumping nonlinearities is considered and the existence of a sign changing solution is proved.

Let

$$f_{\pm}(x,u)=\begin{cases} f(x,u), & \pm u\geq 0,\\ 0, & \pm u<0\end{cases}$$

and

$$I_{\pm}(u)=\frac{1}{2}\int_{\Omega}|\nabla u|^2dx-\int_{\Omega}F_{\pm}(x,u)dx,$$

where $F_{\pm}(x,u)=\int_0^u f_{\pm}(x,t)dt$. By (f_4), we get $I_{\pm}\in C^{2-0}$, and similarly to the Proposition 2.2, $I_{\pm}$ satisfy the (P.S.) condition.

Lemma 4.3 *Under the conditions* (1.2) *and* (f_5), 0 *is a local minimum of* $I, I_{\pm}$.

Proof Since $\lambda_1(m)>0$, there exists a constant $\varepsilon>0$ such that $\lambda_1(m+\varepsilon)>0$. From (4.1), there exists $\delta=\delta(\varepsilon)$ such that

$$F(x,t)\leq\frac{1}{2}(m(x)+\varepsilon)|t|^2,\quad \text{for } |t|\leq\delta,\ x\in\Omega.$$

On the other hand, using (1.2) for $2<p\leq\frac{2N}{N-2}$ we can find $C>0$ such that

$$F(x,t)\leq C|t|^p,\quad \text{for } |t|>\delta,\ x\in\Omega.$$

Then we get

$$F(x,t)\leq\frac{1}{2}(m(x)+\varepsilon)|t|^2+C|t|^p,\quad \text{for } t\in\mathbb{R},\ x\in\Omega. \tag{4.3}$$

Combining (4.3), Lemma 2.3 and the embedding theorem, we have

$$\begin{aligned} I(u)&=\frac{1}{2}\int_{\Omega}|\nabla u|^2dx-\int_{\Omega}F(x,u)dx\\ &\geq\frac{1}{2}\int_{\Omega}|\nabla u|^2dx-\frac{1}{2}\int_{\Omega}(m(x)+\varepsilon)u^2dx-\int_{\Omega}C|u|^pdx\\ &\geq C\|u\|^2-C\|u\|_p^p\\ &\geq C\|u\|^2-C\|u\|^p\\ &>0,\quad \text{as } 0<\|u\|\ll 1,\end{aligned}$$

which implies that 0 is a local minimum of I.

The case for $I_+(I_-)$ is similar. □

Lemma 4.4 *There exist* $v_1,v_2\neq 0$ *such that* $I_+(v_1)\leq 0$ *and* $I_-(v_2)\leq 0$.

Proof Let $u^+=\max\{u,0\}$ and

$$I_+(u)=\frac{1}{2}\int_{\Omega}|\nabla u|^2dx-\frac{1}{2}\int_{\Omega}a(x)(u^+)^2dx-\int_{\Omega}G_+(x,u)dx,$$

where $G_+(x,u)=F_+(x,u)-\frac{1}{2}a(x)(u^+)^2$. Let $\varphi>0$ be the eigenfunction of $\lambda_1(a)<0$. From (1.7), we know that

$$I_+(t\varphi)\leq\frac{\lambda_1(a)}{2}\|t\varphi\|_2^2+C\to-\infty,\quad \text{as } t\to\infty.$$

Then we prove the case of I_+. The case for I_- is similar. □

Proof of Theorem 4.1 From Lemmas 4.3 and 4.4, the mountain pass lemma in [23] guarantees that I_+ has a critical point $u_1>0$. Using the results in [24], we obtain that

$$C_*(I,u_1)=C_*(I_+,u_1)=\delta_{*1}G.$$

Similarly, I_- has a critical point $u_2 < 0$ with

$$C_*(I, u_2) = C_*(I_-, u_2) = \delta_{*1}G.$$

The solution u_0 obtained by Theorem 1.1 satisfies

$$C_\mu(I, u_0) \neq 0,$$

where $\mu = \dim(H^- \oplus H^0) \geq 2$, since $k \geq 2$ (see [25]). Hence u_0, u_1, u_2 are three nontrivial critical points of I. This completes the proof. □

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