Dimension of Besicovitch–Eggleston sets in countable symbolic space

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DIMENSION OF BESICOVITCH-EGGLESTON SETS IN COUNTABLE SYMBOLIC SPACE

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Abstract. This paper is mainly concerned with Hausdorff dimensions of Besicovitch-Eggleston subsets in countable symbolic space. A notable point is that, the dimension values posses a universal lower bound depending only on the underlying metric. As a consequence of the main results, we obtain Hausdorff dimension formulas for sets of real numbers with prescribed digit frequencies in their Lüroth expansions.

Keywords: Lüroth expansion, Countable symbolic space, Digit frequency Hausdorff dimension, Convergence exponent, Bernoulli dimension

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1. Introduction

Consider the piecewise linear mapping $T : [0, 1) \mapsto [0, 1)$ defined by

$$T(0) = 0 \text{ and } T(x) = n(n + 1)x - n, \quad x \in [1/(n + 1), 1/n).$$

Let $\phi(x) = n$ for $x \in [1/(n + 1), 1/n)$, and $x_k = \phi(T^{k-1}x)$ for $k \geq 1$. In this way, one obtains a finite or infinite sequence $\{x_1, x_2, \cdots\}$ called the Lüroth digits of x. Let $\mathbb{Q}^c$ denote the set of irrational numbers. Each $x \in [0, 1)\cap\mathbb{Q}^c$ can be represented as an infinite Lüroth series

$$x = \sum_{n=1}^{\infty} \frac{x_n}{\prod_{k=1}^{n} x_k\big(x_k + 1\big)},$$

where $x_k \in \mathbb{N} := \{1, 2, 3, \cdots\}$ for each $k \geq 1$ (see [6] p.36-41 and [14]).

Throughout this paper, we write

$$\tau_j(x, n) := \mathrm{Card}\{k : x_k = j, 1 \leq k \leq n\}$$

for the number of occurrences of "j" among the first n digits $\{x_1, x_2 \cdots , x_n\}$.

Let $\vec{p} = (p_1, p_2, \dots)$ be a probability vector, namely, $p_j \geq 0$ for all $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} p_j = 1$, which will be referred to as a *frequency vector* hereafter. Define the associated *Besicovitch-Eggleston set* by

$$\mathcal{E}_{\vec{p}} := \Big\{x \in [0, 1) \cap \mathbb{Q}^c : \lim_{n\to\infty} \frac{\tau_j(x, n)}{n} = p_j, \quad \forall j \geq 1\Big\}.$$

Let $\dim_H$ stand for the Hausdorff dimension. Our first result is the following

Theorem 1.1. *Given a frequency vector $\vec{p}$, we have*

$$\dim_H(\mathcal{E}_{\vec{p}}) = \max\left\{\frac{1}{2}, \quad \liminf_{n\to\infty} \frac{-\sum_{j=1}^{n} p_j \log p_j}{\sum_{j=1}^{n} p_j \log(j(j + 1))}\right\}.$$

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Apparently, this dimension formula differs significantly from those in the m-ary expansion cases studied by Besicovitch [3] and Eggleston [7]. Let us briefly explain where the "universal" lower bound $1/2$ comes from. Roughly speaking, it depends on the "geometric structure" of the transformation T which generates the Lüroth series. More precisely, we have a countable partition $\{[1/(n+1), 1/n) : n = 1, 2, \cdots\}$ on $[0, 1)$ and the transformation is piecewise linear on each partition element. The number $1/2$ is nothing but the convergent exponent of the sequence of interval lengths $\{1/n(n+1) : n = 1, 2, \cdots\}$, namely

$$\frac{1}{2} = \inf\left\{t \geq 0 : \sum_{n=1}^{\infty} \frac{1}{(n(n+1))^t} < \infty\right\}.$$

In the case of m-ary expansion, the partition associated to the transformation $x \mapsto mx \pmod 1$ is finite, and the corresponding convergent exponent is zero.

The above result may be extended to the generalized Lüroth expansions(see [6]), as well as to other expansions generated by piecewise linear interval transformations. However, for convenience of presentation, we choose to work in the setting of the countable symbolic space $\mathbb{N}^{\mathbb{N}} := \{x = (x_k)_{k\geq 1} :\ x_k \in \mathbb{N}\}$. The transference from the symbolic space to the interval will be illustrated in the last section.

Let us describe the general framework. Fixing a vector $\vec{q} = (q_1, q_2, \dots)$ such that $q_k > 0$ for all $k \geq 1$ and $\sum_{k=1}^{\infty} q_k = 1$, called a *metric vector*, we define a metric on $\mathbb{N}^{\mathbb{N}}$(called the *$\vec{q}$-metric*) by

$$\rho_{\vec{q}}(x, y) := \prod_{k=1}^{n} q_{x_k}, \quad \text{where } n = \min\{k \geq 0 :\ x_{k+1} \neq y_{k+1}\} \quad \forall x, y \in \mathbb{N}^{\mathbb{N}},$$

with the convention that $\rho_{\vec{q}}(x, y) = 1$ if $n = 0$ in the above.

For a frequency vector $\vec{p}$, we define a *Besicovitch-Eggleston set* by

$$E_{\vec{p}} := \left\{x \in \mathbb{N}^{\mathbb{N}} : \lim_{n\to\infty} \frac{\tau_j(x, n)}{n} = p_j \ \forall j \in \mathbb{N}\right\}. \tag{1.1}$$

Our purpose is to determine the Hausdorff dimension of $E_{\vec{p}}$ with respect to the metric $\rho_{\vec{q}}$. Let us recall the definition. Denote by $|\cdot|_{\vec{q}}$ the diameter of a set under the metric $\rho_{\vec{q}}$. For $E \subset \mathbb{N}^{\mathbb{N}}$ and $s \geq 0$, set

$$\mathcal{H}^s(E) := \lim_{\delta\to 0} \inf\left\{\sum |U_i|_{\vec{q}}^s : U_i \in \mathcal{C}, |U_i|_{\vec{q}} < \delta\right\},$$

where the infimum is taken over $\mathcal{C}$ which is a collection of cylinders with diameter smaller than δ such that $E \subset \bigcup_{U\in\mathcal{C}} U$. The *Hausdorff dimension* with respect to the metric $\rho_{\vec{q}}$ of E is defined by

$$\dim_H(E) = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = +\infty\}.$$

Now we introduce two exponents. We define the *convergence exponent* of $\vec{q}$ by

$$\alpha(\vec{q}) := \inf\left\{t \geq 0 : \sum_{j=1}^{\infty} q_j^t < \infty\right\}$$

and the *Bernoulli dimension* of $\vec{p}$ relative to the $\vec{q}$-metric by

$$\beta(\vec{p}, \vec{q}) := \liminf_{n\to\infty} \frac{-\sum_{j=1}^{n} p_j \log p_j}{-\sum_{j=1}^{n} p_j \log q_j}.$$

Our main result is the following

Theorem 1.2. *Given $\vec{q}$ and $\vec{p}$ as above, we have*

$$\dim_H(E_{\vec{p}}) = \max\left\{\alpha(\vec{q}), \beta(\vec{p}, \vec{q})\right\}.$$

The following are some remarks concerning this dimension formula.

(1) Since $\sum_{j=1}^{\infty} q_j = 1$, we have $\alpha(\vec{q}) \leq 1$. In Lemma 2.3 below, we will see that $\beta(\vec{p}, \vec{q}) \leq 1$. If $\vec{p} = \vec{q}$, then $\beta(\vec{p}, \vec{q}) = 1$. The Hausdorff dimension of the whole space is equal to 1.

(2) In the special case that $q_k \sim 1/k(\log k)^{1+\epsilon}$ for some $\epsilon > 0$, we have $\alpha(\vec{q}) = 1$. Then $\dim_H E_{\vec{q}} = 1$ for all $\vec{q}$.

(3) Look at the case where $q_k = 1/k(k+1)$ for all $k \geq 1$, which corresponds to the Lüroth expansion. Then $\alpha(\vec{q}) = 1/2$.

(4) Our results can be extended to the case that $\vec{p}$ is a sub-probability vector, namely, $\sum_{j=1}^{\infty} p_j < 1$. In particular, one has $\dim_H(E_{\vec{p}}) = 1/2$ when $p_1 = p_2 = \cdots = 0$. This would not happen in the m-ary expansion case.

The study of the Hausdorff dimensions of sets of numbers defined in terms of the frequencies of the digits in their m-ary expansions has experienced a revival in the mathematics literature (see for example [2, 10, 18]). The solution of such a problem or of general problems of such kind is provided by a so-called variational principle (see [12] for the most general setting of a compact topological system, see also [17, 21]), which can be obtained in many cases by the thermodynamical formalism. Here we consider a similar question for the Lüroth expansion (Theorem 1.1.) through a more general symbolic setting of infinite symbols (Theorem 1.2.). The difference with earlier works is the (countable) infinity of the alphabet, from which comes a particular phenomenon that the formal variational principle does not hold as in the case of compact dynamics and the thermodynamical formalism does not work verbatim. The case of Gauss dynamics related to continued fractions is worked out in [11]. Another example showing the difference between finite and infinite alphabet can be found in [13].

In the present paper, we have the will to give a simple and self-contained exposition on the Lüroth expansion and its generalization via the introduction of a metric, that avoids at most as possible the language of dynamical systems and the thermodynamic formalism. Actually, as we pointed out above, the thermodynamic formalism does not work as we wish and we do not know if there always exists a Gibbs measure, to be suitably defined, on the set in question, which has the dimension of the set. We also point out that in the similar case of continued fractions, Kifer, Peres and Weiss [15] showed that Bernoulli measures can not be maximizing, contrary to the case of m-ary expansion.

We should say that the lower bound $\beta(p, q)$ defined by the liminf in Theorem 1.1. is due to Kinney and Pitcher (see Theorem 6.1. [16] p. 307). This lower estimate is also obtained in [1]. One of our contribution is the observation that 1/2 is always a lower bound. This characterizes the particular phenomenon.

The paper is organized as follows. In Section 2, we prove the existence of some points whose digit sequence satisfying certain growth conditions, and discuss some properties of the exponents $\alpha(\vec{q})$ and $\beta(\vec{p}, \vec{q})$. Section 3 is devoted to the proof of Theorem 1.2. In Section 4, by comparing dimensions of sets on $[0, 1)$ with respect to different covering classes, we shall derive Theorem 1.1 from Theorem 1.2.

2. Preliminary

In this section, we focus on the symbolic space $\mathbb{N}^{\mathbb{N}}$. First, we verify the existence of some special points in the Besicovitch-Eggleston set $E_{\vec{p}}$, which will be called seeds. In Section 3, we will use such seeds to construct a Cantor set and show that the Hausdorff dimension of the Besicovitch-Eggleston set is always bounded from below by the convergent exponent $\alpha(\vec{q})$. Then we discuss certain properties of the convergence exponent $\alpha(\vec{q})$ and the Bernoulli dimension $\beta(\vec{p}, \vec{q})$.

2.1. Existence of seeds.

Lemma 2.1. *Given a sequence of positive integers $\{a_n\}_{n\geq 1}$ tending to the infinity, there exist points $z = (z_1, z_2, \dots) \in E_{\vec{p}}$ such that $z_n \leq a_n$ for all $n \geq 1$.*

Proof. For any $n \geq 1$, we construct a probability vector

$$(p_1^{(n)}, p_2^{(n)}, \dots, p_k^{(n)}, \dots)$$

such that $p_k^{(n)} > 0$ for all $1 \leq k \leq a_n$ and $p_k^{(n)} = 0$ for all $k > a_n$, and that

$$\lim_{n\to\infty} p_k^{(n)} = p_k \text{ for each } k \geq 1. \tag{2.1}$$

This sequence of probability vectors determine a product probability measure $\mathbb{P}$ on $\mathbb{N}^{\mathbb{N}}$, which is supported by $\prod_{n=1}^{\infty}\{1, \dots, a_n\}$.

For each $k \geq 1$. Consider the sequence of random variables $\{X_i(x) = 1_{\{k\}}(x_i)\}_{i=1}^{\infty}$. By Kolmogorov's strong law of large numbers (see [20] p.388), we have

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mathbb{E}(X_i)\right) = 0 \qquad \mathbb{P} - a.s.,$$

which implies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_{\{k\}}(x_i) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} p_k^{(i)} = p_k \qquad \mathbb{P} - a.s.. \tag{2.2}$$

Namely, for almost all $z \in \prod_{n=1}^{\infty}\{1, \dots, a_n\}$, each digit k has the required frequency p_k, this completes the proof. □

2.2. Convergence exponent $\alpha(\vec{q})$ and Bernoulli dimension $\beta(\vec{p}, \vec{q})$.

We first prove a lemma about the convergence exponent $\alpha := \alpha(\vec{q})$.

Lemma 2.2. *Let $\alpha = \alpha(\vec{q})$ be the convergence exponent of a metric vector $\vec{q} = (q_1, q_2, \dots)$ which is not necessarily decreasing. Consider a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $q_{\pi(1)} \geq q_{\pi(2)} \geq \cdots$. Then, there exists an increasing subsequence $\{m_k\}_{k\geq 1} \subset \mathbb{N}$ such that*

$$\sum_{k=1}^{n} \log m_k \gg n^2, \tag{2.3}$$

and for any $\epsilon > 0$ and $0 < \delta < 1$, there exists an integer $N = N(\epsilon, \delta)$ such that for $k \geq N$, we have

$$m_k - m_k^{\delta} > m_k^{1-\epsilon}, \tag{2.4}$$

and for $\pi^{-1}(n) \in (m_k - m_k^{\delta}, m_k]$ $(k \geq N)$, we have

$$(\pi^{-1}(n))^{1-\epsilon} < q_n^{-\alpha} < (\pi^{-1}(n))^{\frac{1+\epsilon}{1-\epsilon}}. \tag{2.5}$$

Proof. We rearrange the elements q_k's in a decreasing order. That is to say, we take a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $q_{\pi(1)} \geq q_{\pi(2)} \geq \cdots$. Since the convergence exponent of $\vec{q}$ is invariant under the permutation π, we have (see [19], p.26)

$$\alpha = \limsup_{n\to\infty} \frac{\log n}{\log q_{\pi(n)}^{-1}}. \tag{2.6}$$

Therefore, one may choose an increasing sequence $\{m_k\}_{k\geq 1}$ satisfying (2.3) and for any $0 < \epsilon, \delta < 1$, there exists an integer $N_1 = N_1(\epsilon, \delta)$ such that for all $k \geq N_1$, (2.4) is satisfied and

$$m_k^{1-\epsilon} < q_{\pi(m_k)}^{-\alpha} < m_k^{1+\epsilon}. \tag{2.7}$$

By (2.6), there exists an integer N_2 such that for all $n \geq N_2$

$$q_{\pi(n)}^{-\alpha} > n^{1-\epsilon}. \tag{2.8}$$

Thus if we take $N = \max\{N_1, N_2\}$ then by (2.8), (2.7) and (2.4), for all $n \in (m_k - m_k^\delta, m_k]$ $(k \geq N)$, we have

$$n^{1-\epsilon} < q_{\pi(n)}^{-\alpha} \leq q_{\pi(m_k)}^{-\alpha} < m_k^{1+\epsilon} < (m_k - m_k^\delta)^{\frac{1+\epsilon}{1-\epsilon}} < n^{\frac{1+\epsilon}{1-\epsilon}}.$$

In other words, for $\pi^{-1}(n) \in (m_k - m_k^\delta, m_k]$ $(k \geq N)$, we have (2.5). □

Then we prove three lemmas about the Bernoulli dimension $\beta(\vec{p}, \vec{q})$.

Lemma 2.3. *For a metric vector $\vec{q} = (q_1, q_2, \dots)$ and a frequency vector $\vec{p} = (p_1, p_2, \dots)$, we have*

$$-\sum_{j=1}^{\infty} p_j \log p_j \leq -\sum_{j=1}^{\infty} p_j \log q_j \tag{2.9}$$

Proof. Let us assume that $-\sum_{j=1}^\infty p_j \log q_j < \infty$. Then (2.9) is equivalent to

$$\sum_{j=1}^{\infty} p_j \log \frac{q_j}{p_j} \leq 0,$$

which follows from the concavity of the log function. □

Lemma 2.4. *If $-\sum_{j=1}^\infty p_j \log q_j = \infty$, then*

$$\beta(\vec{p}, \vec{q}) \leq \limsup_{n\to\infty} \frac{-\sum_{j=1}^n p_j \log p_j}{-\sum_{j=1}^n p_j \log q_j} \leq \alpha(\vec{q}).$$

Proof. The left-hand side inequality follows immediately from the definition of the Bernoulli dimension $\beta(\vec{p}, \vec{q})$. To prove the right-hand side inequality, we shall use the following result (see [22], p.217): Let $t_1, \cdots, t_m$ be given real numbers. If $s_j \geq 0$ and $\sum_{j=1}^m s_j = 1$ then

$$\sum_{j=1}^{m} s_j (t_j - \log s_j) \leq \log\Big(\sum_{j=1}^{m} e^{t_j}\Big). \tag{2.10}$$

Now by the definition of the convergence exponent, for any $\xi > \alpha(\vec{q})$ we have $\sum_{j=1}^\infty q_j^\xi < \infty$. Applying (2.10) to $m = n+1$, $s_j = p_j$ for $1 \leq j \leq n$ and

$s_{n+1} = \sum_{j=n+1}^{\infty} p_j$, and $t_j = \xi \log q_j$ for $1 \le j \le n$ and $t_{n+1} = 0$, we get

$$\xi \sum_{j=1}^{n} p_j \log q_j - \sum_{j=1}^{n} p_j \log p_j - \Big(\sum_{j=n+1}^{\infty} p_j \Big) \log \Big(\sum_{j=n+1}^{\infty} p_j \Big) \le \log \Big(1 + \sum_{j=1}^{n} q_j^{\xi} \Big).$$

Therefore,

$$\frac{-\sum_{j=1}^{n} p_j \log p_j}{-\sum_{j=1}^{n} p_j \log q_j} \le \xi + \frac{(\sum_{j=n+1}^{\infty} p_j) \log(\sum_{j=n+1}^{\infty} p_j)}{-\sum_{j=1}^{n} p_j \log q_j} + \frac{\log(1 + \sum_{j=1}^{n} q_j^{\xi})}{-\sum_{j=1}^{n} p_j \log q_j}.$$

Recalling that $-\sum_{j=1}^{\infty} p_j \log q_j = \infty$, we finish the proof by letting $n \to \infty$. □

For $a_1, \cdots, a_n \in \mathbb{N}$, we call the set

$$I(a_1, \ldots, a_n) := \{x \in \mathbb{N}^{\mathbb{N}} : x_1 = a_1, \cdots, x_n = a_n\}$$

an n-cylinder. The n-cylinder containing x will be denoted by $I_n(x)$.

Given a frequency vector $\vec{p}$, we define the Bernoulli measure $\mu_{\vec{p}}$ on $\mathbb{N}^{\mathbb{N}}$ by

$$\mu_{\vec{p}}(I(a_1, \ldots, a_n)) = \prod_{j=1}^{n} p_{a_j}.$$

Under the condition $-\sum_{j=1}^{\infty} p_j \log q_j < \infty$, we can calculate the Hausdorff dimension of this Bernoulli measure (see [9] for the definition of the dimension of a measure).

Lemma 2.5. *If* $-\sum_{j=1}^{\infty} p_j \log q_j < \infty$*, then* $\dim_H \mu_{\vec{p}} = \beta(\vec{p}, \vec{q})$ *and*

$$\beta(\vec{p}, \vec{q}) = \frac{-\sum_{j=1}^{\infty} p_j \log p_j}{-\sum_{j=1}^{\infty} p_j \log q_j}.$$

Proof. Consider the sequence of random variables $\{\log q_{x_j}\}_{j=1}^{\infty}$, which are independent and identically distributed with respect to the Bernoulli measure $\mu_{\vec{p}}$. Direct computation yields

$$\mathbb{E}(-\log p_{x_1}) = -\sum_{j=1}^{\infty} p_j \log p_j \le -\sum_{j=1}^{\infty} p_j \log q_j = \mathbb{E}(-\log q_{x_1}) < \infty.$$

For any $x = (x_1, x_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$, we have

$$\log \mu_{\vec{p}}(I_n(x)) = \sum_{j=1}^{n} \log p_{x_j}, \qquad \log |I_n(x)|_{\vec{q}} = \sum_{j=1}^{n} \log q_{x_j}.$$

Hence, by the law of large numbers, we obtain

$$\lim_{n\to\infty} \frac{\log \mu_{\vec{p}}(I_n(x))}{\log |I_n(x)|_{\vec{q}}} = \frac{-\sum_{j=1}^{\infty} p_j \log p_j}{-\sum_{j=1}^{\infty} p_j \log q_j} \qquad \mu_{\vec{p}} - a.e..$$

By a result in [9], we obtain

$$\dim_H \mu_{\vec{p}} = \frac{-\sum_{j=1}^{\infty} p_j \log p_j}{-\sum_{j=1}^{\infty} p_j \log q_j}.$$

□

3. Proof of Theorem 1.2

For ease of notation, we shall write α and β instead of $\alpha(\vec{p})$ and $\beta(\vec{p}, \vec{q})$.

3.1. **Lower bound.** We first prove that $\dim_H(E_{\vec{p}}) \geq \alpha$ for any $\vec{p}$.

For any $0 < \epsilon, \delta < 1$, let $N, \{m_k\}_{k\geq 1}$ be the same as in Lemma 2.2. By Lemma 2.1 and (2.3), we can choose a "seed" $z = (z_1, z_2, \cdots) \in E_{\vec{p}}$ such that

$$\sum_{k=1}^{(n+1)^2} \log q_{z_k}^{-1} \ll \sum_{k=1}^{n} \log m_k. \tag{3.1}$$

We will use this "seed" to sprout a large enough Cantor set in $E_{\vec{p}}$.

We say $x \in \mathbb{N}^{\mathbb{N}}$ is a "bud" of z if $x_n = z_n$ for non square integer n. By collecting some buds of the given seed z, we define

$$F_z(\epsilon, \delta) := \left\{ x \in \mathbb{N}^{\mathbb{N}} : x \text{ is a bud of } z;\ x_{k^2} \begin{cases} = z_{k^2} & \text{if } k < N \\ \in \pi((m_k - m_k^\delta, m_k]) & \text{if } k \geq N \end{cases} \right\}.$$

By the definition, each $x \in F_z(\epsilon, \delta)$ has the same digit frequency as that of z, so $F_z(\epsilon, \delta) \subset E_{\vec{p}}$ for any $\epsilon > 0$ and $\delta > 0$. The following proposition immediately implies $\dim_H(E_{\vec{p}}) \geq \alpha$.

Proposition 3.1. *For any $0 < \epsilon < 1$ and $0 < \delta < 1$,*

$$\dim_H(F_z(\epsilon, \delta)) \geq \frac{\alpha\delta(1-\epsilon)}{1+\epsilon}.$$

By Billinsley's theorem (see [4] and [23]), in order to prove Proposition 3.1 we need only to prove the following lemma.

Lemma 3.2. *For any $0 < \epsilon < 1$ and $0 < \delta < 1$, there exists a measure μ supported by $F_z(\epsilon, \delta)$ such that for any $x \in F_z(\epsilon, \delta)$,*

$$\liminf_{m\to\infty} \frac{\log \mu(I_m(x))}{\log |I_m(x)|_{\vec{q}}} \geq \frac{\alpha\delta(1-\epsilon)}{1+\epsilon}.$$

Proof. We construct a measure μ on Cantor set $F_z(\epsilon, \delta)$. For $x \in F_z(\epsilon, \delta)$ and for $n^2 \leq m < (n+1)^2$ $(n \geq N)$, set

$$\mu(I_m(x)) = \prod_{k=N}^{n} \frac{1}{m_k^\delta}.$$

This measure is well defined on $F_z(\epsilon, \delta)$. Notice that

$$\left|I_m(x)\right|_{\vec{q}} = \prod_{k=1}^{m} q_{x_k} = \prod\nolimits_{k=1}^{*m} q_{x_k} \prod_{k=N}^{n} q_{x_{k^2}} = \prod\nolimits_{k=1}^{*m} q_{z_k} \prod_{k=N}^{n} q_{x_{k^2}}, \tag{3.2}$$

where $*$ signifies the absence of the square numbers in $[N, m]$ in the product. Then

$$\frac{\log \mu(I_m(x))}{\log |I_m(x)|_{\vec{q}}} = \frac{\delta \sum_{k=N}^{n} \log m_k}{\sum_{k=1}^{*m} \log q_{z_k}^{-1} + \sum_{k=N}^{n} \log q_{x_{k^2}}^{-1}},$$

where $*$ signifies the absence of the square numbers in $[N, m]$ in the summation.

Since for $x \in F_z(\epsilon, \delta)$, $x_{k^2} \in \pi((m_k - m_k^\delta, m_k])$, we get $\pi^{-1}(x_{k^2}) \in (m_k - m_k^\delta, m_k]$. Thus by (2.5), we have

$$(\pi^{-1}(x_{k^2}))^{1-\epsilon} < q_{x_{k^2}}^{-\alpha} < (\pi^{-1}(x_{k^2}))^{\frac{1+\epsilon}{1-\epsilon}}.$$

Hence

$$\frac{1}{\alpha}(1-\epsilon) \log(m_k - m_k^\delta) < \log q_{x_{k^2}}^{-1} < \frac{1}{\alpha} \frac{1+\epsilon}{1-\epsilon} \log m_k. \tag{3.3}$$

Therefore by the right hand inequality of (3.3),

$$\frac{\log \mu(I_m(x))}{\log |I_m(x)|_{\vec{q}}} > \frac{\delta \sum_{k=N}^{n} \log m_k}{\sum_{k=1}^{(n+1)^2} \log q_{z_k}^{-1} + \frac{1}{\alpha}\frac{1+\epsilon}{1-\epsilon}\sum_{k=N}^{n} \log m_k}.$$

Letting $m \to \infty$ and by (3.1), we get, for any $x \in F_z(\epsilon, \delta)$,

$$\liminf_{m\to\infty} \frac{\log \mu(I_m(x))}{\log |I_m(x)|_{\vec{q}}} \geq \frac{\alpha\delta(1-\epsilon)}{1+\epsilon}.$$

□

Now we prove that $\dim_H(E_{\vec{p}}) \geq \beta$. We shall distinguish two cases according to the convergence or divergence of the series $-\sum_{j=1}^{\infty} p_j \log q_j$.

In the case of $-\sum_{j=1}^{\infty} p_j \log q_j = \infty$, we have proved that $\dim_H(E_{\vec{p}}) \geq \alpha$. Using $\alpha \geq \beta$ by Lemma 2.4, we get $\dim_H(E_{\vec{p}}) \geq \beta$.

For the case of $-\sum_{j=1}^{\infty} p_j \log q_j < \infty$, we consider the Bernoulli measure $\mu_{\vec{p}}$ on $E_{\vec{p}}$. By Lemma 2.5, we have

$$\dim_H(E_{\vec{p}}) \geq \dim_H \mu_{\vec{p}} \geq \beta.$$

3.2. **Upper bound.** We will distinguish two cases: $\beta \leq \alpha$ and $\beta > \alpha$.

Case $\beta \leq \alpha$. We want to show that $\dim_H(E_{\vec{p}}) \leq \gamma := \alpha + 3\delta$, for any $\delta > 0$.

For any fixed integer N and any real number $\epsilon > 0$, we have

$$E_{\vec{p}} \subset \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} H_m(\epsilon, N),$$

where

$$H_m(\epsilon, N) := \left\{ x \in \mathbb{N}^{\mathbb{N}} : \left| \frac{\tau_j(x,m)}{m} - p_j \right| < \epsilon, 1 \leq j \leq N \right\}.$$

For any integer $k \in \mathbb{N}$ and $n \geq k$, we have

$$\begin{aligned}
&\mathcal{H}^{\gamma}\Big(\bigcap_{m=k}^{\infty} H_m(\epsilon, N)\Big) \leq \mathcal{H}^{\gamma}(H_n(\epsilon, N)) \\
\leq\ & \sum_{|\frac{\tau_j(x,n)}{n} - p_j| < \epsilon, 1\leq j\leq N} |I_n(x)|_{\vec{q}}^{\gamma} \\
=\ & \sum_{n(p_j-\epsilon)<m_j<n(p_j+\epsilon), 1\leq j\leq N} \ \sum_{\tau_j(x,n)=m_j, 1\leq j\leq N} |I_n(x)|_{\vec{q}}^{\gamma} \\
\leq\ & \sum_{n(p_j-\epsilon)<m_j<n(p_j+\epsilon), 1\leq j\leq N} \frac{n!}{m_1! m_2! \cdots m_N! m_{N+1}!} \prod_{j=1}^{N} q_j^{m_j \gamma} \Big(\sum_{j=N+1}^{\infty} q_j^{\gamma} \Big)^{m_{N+1}},
\end{aligned}$$

where $m_{N+1} = n - m_1 - \cdots - m_N$.

Let $\phi(t) := -t \log t$ for $t \in [0, 1]$ with $\phi(0) = 0$. We have the following elementary inequality whose proof is postponed to the end of this section.

Lemma 3.3. *Let $N \geq 1$ and $\epsilon > 0$. Then, for n sufficiently large, one has*

$$\frac{n!}{m_1! m_2! \cdots m_N! m_{N+1}!} \prod_{j=1}^{N} q_j^{m_j \gamma} \Big(\sum_{j=N+1}^{\infty} q_j^{\gamma} \Big)^{m_{N+1}} \leq \exp\{(A+B)n\},$$

where

$$A := \sum_{j=1}^{N} \phi(p_j) + \phi\Big(\sum_{j=N+1}^{\infty} p_j \Big) + A_{\epsilon,N} + O\Big(\frac{\log n}{n}\Big),$$

and

$$B := \gamma \sum_{j=1}^{N} p_j \log q_j + \Big(\sum_{j=N+1}^{\infty} p_j \Big) \log \Big(\sum_{j=N+1}^{\infty} q_j^{\gamma} \Big) + B_{\epsilon,N},$$

with

$$\lim_{\epsilon \to 0} A_{\epsilon,N} = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} B_{\epsilon,N} = 0.$$

Since $\beta \leq \alpha < \gamma = \alpha + 3\delta$, by the definition of β, there exist infinite many integers N such that

$$\sum_{j=1}^{N} \phi(p_j) < -(\alpha + \delta) \sum_{j=1}^{N} p_j \log q_j. \tag{3.4}$$

Since $\sum_{j=1}^{\infty} p_j = \sum_{j=1}^{\infty} q_j = 1$ and $\sum_{j=1}^{\infty} q_j^{\gamma} < \infty$ for $\gamma > \alpha$, the infinite many N's can be chosen to satisfy

$$\phi\Big(\sum_{j=N+1}^{\infty} p_j \Big) + \Big(\sum_{j=N+1}^{\infty} p_j \Big) \log \Big(\sum_{j=N+1}^{\infty} q_j^{\gamma} \Big) \leq -\delta \sum_{j=1}^{N} p_j \log q_j. \tag{3.5}$$

We fix N and let ϵ be small enough such that

$$A_{\epsilon,N} + B_{\epsilon,N} \leq -\delta \sum_{j=1}^{N} p_j \log q_j.$$

Then by (3.4) and (3.5), we get

$$\begin{aligned} A + B \quad &< \quad -(\alpha + \delta) \sum_{j=1}^{N} p_j \log q_j + \gamma \sum_{j=1}^{N} p_j \log q_j - 2\delta \sum_{j=1}^{N} p_j \log q_j + O\left(\frac{\log n}{n}\right) \\ &= \quad (\gamma - \alpha - 3\delta) \sum_{j=1}^{N} p_j \log q_j + O\left(\frac{\log n}{n}\right) = O\left(\frac{\log n}{n}\right). \end{aligned}$$

Letting $n \to \infty$, we have

$$\mathcal{H}^{\gamma}\Big(\bigcap_{n=k}^{\infty} H_n(\epsilon, N) \Big) = 0.$$

Thus we obtain $\dim_H(E_{\vec{p}}) \leq \gamma$.

Case $\beta > \alpha$. We want to show $\dim_H(E_{\vec{p}}) \leq \gamma := \beta + 3\delta$, for any $\delta > 0$. By the definition of β, there exist infinite many N's such that

$$\sum_{j=1}^{N} \phi(p_j) < -(\beta + \delta) \sum_{j=1}^{N} p_j \log q_j.$$

Using this in place of (3.4), we can complete the proof in the same way as in the case of $\beta \leq \alpha$.

3.3. **Proof of Lemma 3.3.**

First, by the Stirling formula, we have

$$\frac{1}{n}\log\frac{n!}{m_1!m_2!\cdots m_{N+1}!}=\sum_{j=1}^{N+1}\phi\Big(\frac{m_j}{n}\Big)+O\Big(\frac{\log n}{n}\Big). \tag{3.6}$$

Recall that $|\frac{m_j}{n}|<\epsilon$ for $1\le j\le N$ and $m_{N+1}=n-m_1-\cdots-m_N$, by the uniform continuity of the function $\phi(t)=-t\log t$ on $[0,1]$, we have

$$\begin{aligned}&\frac{1}{n}\log\frac{n!}{m_1!m_2!\cdots m_N!m_{N+1}!}\\ \le\ &\sum_{j=1}^{N}\phi(p_j)+\phi\Big(\sum_{j=N+1}^{\infty}p_j\Big)+C_{\epsilon,N}+O\left(\frac{\log n}{n}\right),\end{aligned}$$

where $\lim_{\epsilon\to 0}C_{\epsilon,N}=0$.

On the other hand,

$$\begin{aligned}&\frac{1}{n}\log\prod_{j=1}^{N}q_j^{m_j\gamma}\Big(\sum_{j=N+1}^{\infty}q_j^{\gamma}\Big)^{m_{N+1}}\\ =\ &\gamma\sum_{j=1}^{N}\frac{m_j}{n}\log q_j+\frac{m_{N+1}}{n}\log\Big(\sum_{j=N+1}^{\infty}q_j^{\gamma}\Big)\\ \le\ &\gamma\sum_{j=1}^{N}p_j\log q_j-\epsilon\gamma\sum_{j=1}^{N}\log q_j+\Big(\sum_{j=N+1}^{\infty}p_j\Big)\log\Big(\sum_{j=N+1}^{\infty}q_j^{\gamma}\Big)\\ &+N\epsilon\Big|\log\Big(\sum_{j=N+1}^{\infty}q_j^{\gamma}\Big)\Big|.\end{aligned}$$

Combing the last two inequalities, the proof is completed.

4. Dimension of sets determined by the Lüroth expansion

Theorem 1.1 can be proved directly in a similar way. However, we prefer to derive it from Theorem 1.2. This approach also serves to illustrate a general method of transferring dimensional result from the symbolic space to the unit interval $(0,1)$, which is of independent interest.

For each $x\in(0,1)$, let $(x_1,x_2\dots)$ denote the sequence of Lüroth digits. Let

$$\triangle(a_1,a_2,\dots,a_n):=\{x\in[0,1):x_1=a_1,x_2=a_2,\dots,x_n=a_n\}$$

which is called a rank-n basic interval.

Let $A\subset(0,1)$. Recall that, in the definition of the Hausdorff measure of A, if we use coverings by arbitrary intervals, the dimension index is the usual Hausdorff dimension $\dim_H(A)$; if we use coverings by the basic intervals, then we get another dimension index, which will be denoted by $\dim_\triangle(A)$.

It is clear that $\dim_H(A)\le\dim_\triangle(A)$. By a result of Wegmann([23], see also [5] pp.36), the equality holds in the following situation.

Proposition 4.1. *One has* $\dim_H(A)=\dim_\triangle(A)$ *for* $A\subset(0,1)$ *if*

$$\lim_{n\to\infty}\frac{\log|\triangle_n(x)|}{\log|\triangle_{n+1}(x)|}=1\quad\forall x\in A, \tag{4.1}$$

where $\triangle_n(x) := \triangle(x_1, x_2, \ldots, x_n)$ is the rank-n basic interval containing x.

In the present context, sets in the symbolic space $\mathbb{N}^{\mathbb{N}}$ are related to sets in the interval $(0,1)$ via the mapping $\Gamma : \mathbb{N}^{\mathbb{N}} \to (0,1)$ defined by

$$\Gamma(x_1, x_2, \cdots) = \sum_{n=1}^{\infty} \frac{x_n}{\prod_{k=1}^{n} x_k(x_k+1)}.$$

Let $I(a_1, \ldots, a_n) \subset \mathbb{N}^{\mathbb{N}}$ be an n-cylinder, then

$$\Gamma(I(a_1, \ldots, a_n)) = \triangle(a_1, a_2, \ldots, a_n).$$

This establishes a one-to-one correspondence between the cylinders in $\mathbb{N}^{\mathbb{N}}$ and the basic intervals in $(0,1)$.

Recall that, the Hausdorff dimension in $\mathbb{N}^{\mathbb{N}}$ is defined by using cylinder-coverings, and the dimension index $\dim_\Delta$ in $(0,1)$ is defined by using basic interval coverings. Now we specify the metric vector $\vec{q}$ by letting $q_k = 1/k(k+1)$ for all $k \geq 1$. Then the diameter of the cylinder $I(a_1, \ldots, a_n)$ under the $\vec{q}$-metric is

$$|I(a_1, \ldots, a_n)|_{\vec{q}} = \prod_{j=1}^{n} q_{a_j} = |\triangle(a_1, a_2, \ldots, a_n)|.$$

Therefore, we have the following "transferring" result.

Proposition 4.2. *Let $E \subset \mathbb{N}^{\mathbb{N}}$, then*

$$\dim_\Delta(\Gamma(E)) = \dim_H(E).$$

Now we are in a position to prove Theorem 1.1. Let $\vec{p}$ be a frequency vector, and $E_{\vec{p}}$ the associated Besicovitch-Eggleston set in the symbolic space. Since the convergent exponent is equal to $1/2$, Theorem 1.2 asserts that

$$\dim_H(E_{\vec{p}}) = \max\left\{\frac{1}{2}, \quad \liminf_{n\to\infty} \frac{-\sum_{j=1}^{n} p_j \log p_j}{\sum_{j=1}^{n} p_j \log(j(j+1))}\right\}.$$

Let $\mathcal{E}_{\vec{p}} = \Gamma(E_{\vec{p}}) \subset (0,1)$ be the corresponding Besicovitch-Eggleston set in Theorem 1.1, we shall prove that $\dim_H(\mathcal{E}_{\vec{p}}) = \dim_H(E_{\vec{p}})$.

Since $\dim_\Delta(\mathcal{E}_{\vec{p}}) = \dim_H(E_{\vec{p}})$ by Proposition 4.2, that $\dim_H(\mathcal{E}_{\vec{p}}) \leq \dim_H(E_{\vec{p}})$ is clear. It remains to show the converse inequality.

Recall that, we have used two subsets for the lower bound estimation of $\dim_H(E_{\vec{p}})$. Firstly, by Lemma 2.1, there exists $z = (z_n)_{n\geq 1} \in E_{\vec{p}}$ such that

$$z_n \leq n, \quad \text{for all} \quad n \geq 1. \tag{4.2}$$

For a positive number $a > 1$, set

$$F := \left\{x \in \mathbb{N}^{\mathbb{N}} : x_{k^2} \in (a^{k^2}, 2a^{k^2}]; \quad x_k = z_k \text{ if } k \text{ is nonsquare}\right\}.$$

It is clear that $F \subset E_{\vec{p}}$, and $dim_H(F) \geq 1/2$ by the proof of Proposition 3.1.

Let $x \in \Gamma(F) \subset \mathcal{E}_{\vec{p}}$, then one can show that

$$\lim_{n\to\infty} \frac{\log|\triangle_n(x)|}{n^{3/2}} = -\frac{2}{3}\log a$$

which implies (4.1), so by Proposition 4.1 and 4.2, we have

$$\dim_H(\mathcal{E}_{\vec{p}}) \geq \dim_H(\Gamma(F)) = \dim_\Delta(\Gamma(F)) = \dim_H(F) \geq 1/2.$$

Secondly, in the case of $-\sum_{j=1}^{\infty} p_j \log q_j < \infty$, let

$$E = \left\{ x \in E_{\vec{p}} : \lim_{n\to\infty} \frac{\sum_{j=1}^{n} \log q_{x_j}}{n} = \sum_{j=1}^{\infty} p_j \log q_j \right\},$$

then Lemma 2.5 implies that $\dim_H(E) \geq \frac{-\sum_{j=1}^{\infty} p_j \log p_j}{-\sum_{j=1}^{\infty} p_j \log q_j}$.

Let $x \in \Gamma(E) \subset \mathcal{E}_{\vec{p}}$, since $|\triangle_n(x)| = \prod_{j=1}^{n} q_{x_j}$, we have

$$\lim_{n\to\infty} \frac{\log |\triangle_n(x)|}{n} = \sum_{j=1}^{\infty} p_j \log q_j < \infty,$$

which implies (4.1), so by Proposition 4.1 and 4.2, we have

$$\dim_H(\mathcal{E}_{\vec{p}}) \geq \dim_H(\Gamma(E)) = \dim_\Delta(\Gamma(E)) = \dim_H(E) \geq \frac{-\sum_{j=1}^{\infty} p_j \log p_j}{-\sum_{j=1}^{\infty} p_j \log q_j}.$$

Combining these two lower bounds, we have shown that $\dim_H(\mathcal{E}_{\vec{p}}) \geq \dim_H(E_{\vec{p}})$. The proof of Theorem 1.1 is completed now.

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References

[1] L. Barreira and G. Iommi, *Frequency of digits in the Lüroth expansion*, J. Number Theory, 129 (6) (2009), 1479-1490.

[2] L. Barreira, B. Saussol and J. Schmeling, *Distribution of frequencies of digits via multifractal analysis*, J. Number Theory, 97 (2002), no. 2, 410-438.

[3] A. S. Besicovitch, *On the sum of digits of real numbers represented in the dyadic system*, Math. Ann., 110 (1) (1935), 321-330.

[4] P. Billingsley, *Ergodic theory and information*, John Wiley and Sons, Inc., New York-London-Sydney, 1965

[5] H. Cajar, *Billingsley Dimension in Probability Spaces*, Springer-Verlag, Berlin. Heidelberg. 1981.

[6] K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs, 29. Mathematical Association of America, Washington DC, 2002.

[7] H. G. Eggleston, *The fractional dimension of a set defined by decimal properties*, Quart. J. Math., 20 (1949), 31-36.

[8] K. J. Falconer, *Fractal Geometry, Mathematical Foundations and Application*, John Wiley & Sons, Ltd., Chichester, 1990.

[9] A. H. Fan, *Sur les dimensions de mesures*, Studia Math., 111 (1) (1994), 1-17.

[10] A. H. Fan, D. J. Feng and J. Wu, *Recurrence, dimension and entropy*, J. London Math. Soc., 64 (1) (2001), 229-244.

[11] A. H. Fan, L. M. Liao and J. H. Ma, *On the frequency of partial quotients of regular continued fractions*, Math. Proc. Camb. Phil. Soc., 148 (2010), 179-192.

[12] A. H. Fan, L. M. Liao and J. Peyrière, *Generic points in systems of specification and Banach valued Birkhoff ergodic average*, Discrete Contin. Dyn. Syst., 21 (2008), 1103-1128.

[13] A. H. Fan, L. M. Liao, B. W. Wang and J. Wu, *On Khintchine exponents and Lyapunov exponents of continued fractions*, Ergod. Th. Dynam. Sys., 29 (2009), 73-109.

[14] J. Galambos, *Representations of real numbers by infite series*, Lecture Notes in Mathematics 502, Springer-Verlag, Berlin-New York, 1976.

[15] Y. Kifer, Y. Peres and B. Weiss, *A dimension gap for continued fractions with independent digits*, Israel J. Math., 124 (2001), no. 1, 61-76.

[16] J. R. Kinney and T. S. Pitcher, *The dimension of some sets defined in terms of f-expansions*, Z. Wahrscheinlichkeitstheorie verw. Geb, 4 (1966), 293-315.

[17] E. Olivier, *Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures*, Nonlinearity, 12 (1999), 1571-1585.

[18] L. Olsen, *Applications of multifractal divergence points to sets of numbers defined by their N-adic expansion*, Math. Proc. Cambridge Philos. Soc., 136 (2004), no. 1, 139-165.

[19] G. Pólya and G. Szegö, *Problems and theorems in analysis. Vol. I: Series, integral calculus, theory of functions*, Springer-Verlag, New York-Berlin, 1972.

[20] A. N. Shiryaev, *Probability*, Second Edition, Springer Verlag, GTM 95, Springer-Verlag, New York, 1996.

[21] F. Takens and E. Verbitzkiy, *On the variational principle for the topological entropy of certain non-compact sets*, Ergod. Th. Dynam. Sys., 23 (2003), 317-348.

[22] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, New York-Berlin, 2001.

[23] H. Wegmann, *Über den Dimensionsbegriff in Wahrscheinlichkeitsrumen. II.*, Z. Wahrscheinlichkeitstheorie und Verw. Geb., 9 (1968), 222-231.

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