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A priori estimates for classical solutions of fully nonlinear elliptic equations

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Abstract For the fully nonlinear uniformly elliptic equation $F(D^2u) = 0$, it is well known that the viscosity solutions are $C^{2,\alpha}$ if the nonlinear operator F is convex (or concave). In this paper, we study the classical solutions for the fully nonlinear elliptic equation where the nonlinear operators F is locally $C^{1,\beta}$ almost everywhere for any $0 < \beta < 1$. We will prove that the classical solutions u are $C^{2,\alpha}$, moreover, the $C^{2,\alpha}$ norm of u depends on n, F and the continuous modulus of D^2u.

Keywords fully nonlinear, classical solutions, $C^{2,\alpha}$ estimates

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1 Introduction

In this paper, we consider the following fully non-linear uniformly elliptic equations:

$$F(D^2u) = 0. \tag{1}$$

Let now $\mathcal{S}_n$ denote the space of $n \times n$ symmetric matrices. Equation (1) is uniformly elliptic if there exist two constants $0 < \lambda \leqslant \Lambda < \infty$ such that

$$\lambda||N|| \leqslant F(M+N) - F(M) \leqslant \Lambda||N||, \tag{2}$$

for any $M, N \in \mathcal{S}_n$ with $N \geqslant 0$, where $N \geqslant 0$ means N is positive semi-definite.

Under the assumption that F is convex (or concave), it is well known that the viscosity solutions of (1) are $C^{2,\alpha}$. This result was proved by Evans and later Caffarelli simplified the proof(see [1-7]). In the proof of above result, the convex (or concave) hypothesis for F is essential since all the existing proofs depend on $\frac{\partial^2 u}{\partial e \partial e}$, the second derivative of u along any direction e, is a sub-solution of Pucci's extremal operator $\mathcal{M}^+$ (see [1] for the definition of $\mathcal{M}^+$).

Without the convex (or concave) hypothesis on F, Caffarelli and Yuan Yu proved that the viscosity solutions of (1) are $C^{2,\alpha}$ if the level set $\sum = \{M : F(M) = 0\}$ satisfies:(i) $\sum \bigcap \{M : tr(M) = t$ is strictly

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convex for all constants t; (ii) the angle between the identity matrix I and the normal F_{ij} to $\sum$ is strictly positive on the non-convex part of $\sum$(see [8]).

In this paper, we prove the interior $C^{2,\alpha}$ regularity of classical solutions for (1). Our hypothesis is that F is locally $C^{1,\beta}$ almost everywhere for $0 < \beta < 1$. Precisely, F is differentiable almost everywhere and for any bounded domain $\mathcal{D} \subset \mathcal{S}_n$, there exists a constant K such that for any $M, N \in \mathcal{D}$, if F is differentiable at N, then

$$|F(M) - F(N) - tr(F'(N)(M - N))| \leqslant K|M - N|^{1+\beta}. \tag{3}$$

(Recall that $tr(A)$ denotes the trace of the matrix A.) We point out that if F is convex (or concave), then according to Alexandroff-Buselman-Feller theorem ([1][9]), F is locally $C^{1,1}$ almost everywhere. Our main theorem is as following.

Theorem 1.1. Suppose F is locally $C^{1,\beta}$ almost everywhere for $0 < \beta < 1$ and $u \in C^2(B_1)$ is a solution of (1). If ρ is a non-decreasing function defined on R^+ such that $\lim\limits_{\delta\to 0} \rho(\delta) = 0$, and

$$|D^2u(x) - D^2u(y)| \leqslant \rho(|x - y|) \qquad \forall x, y \in B_1, \tag{4}$$

then there exist uniform constants $0 < \alpha < 1$ depending only on λ, Λ, n and C depending only on $\lambda, \Lambda, n, \rho$ such that

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \leqslant C\|u\|_{C^2(\overline{B_{3/4}})}. \tag{5}$$

Theorem 1.1 will be proven in Section 3 by an iteration. Before this, in Section 2, we demonstrate that the Hölder continuity can be measured by L^p norm.

Throughout this paper, we will use the following notations. $B_r(x)$ denotes the open ball in R^n centered at x with radius r; $B_r := B_r(0), D_r(x) := B_1 \cap B_r(x)$; $\mathcal{B}_r$ denotes the open ball in $\mathcal{S}_n$ centered at the origin with radius r; $|E|$ denotes the Lebesgue measure of any measurable set E; $\overline{u_D}$ denotes the average of the function u over the set D; $osc_D u$ denotes the oscillation of the function u in the set $D \subset R^n$, exactly,

$$osc_D u := \sup_{x,y\in D} |u(x) - u(y)|.$$

2 Hölder continuity

A function u defined on $\overline{B}_1$ is $C^\alpha(\overline{B}_1)$ (or Hölder continuous with exponent α) if there exists a constant C such that

$$[u]_{\alpha,\overline{B}_1} := \sup_{x,y\in\overline{B}_1, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leqslant C,$$

where $0 < \alpha < 1$. This is the usual definition of Hölder continuity of u, where the continuity is measured by L^∞ norm. The following theorem claims that it can also be measured by L^p norm for any $p \geqslant 1$.

Theorem 2.1. Suppose $u \in L^p(B_1)$ with $p \geqslant 1$. If there exist constants $0 < \alpha < 1$ and $A > 0$ such that for any $x \in \overline{B}_1$ and any $r > 0$,

$$\frac{1}{|D_r(x)|} \int_{D_r(x)} |u - \overline{u_{D_r(x)}}|^p \leqslant Ar^{\alpha p}, \tag{6}$$

then

$$[u]_{\alpha,\overline{B}_1} \leqslant C_0 A^{\frac{1}{p}}, \tag{7}$$

where C_0 is a constant depending only on n and α. (see [p70 of 10,Theorem 1.2]

Corollary 2.1. *Suppose $u \in L^p(B_1)$ with $p \geqslant 1$. If there exist constants $0 < \alpha < 1, 0 < r_1 \leqslant \frac{1}{2}$ and $A_1 > 0$ such that for any $x \in \overline{B_{\frac{1}{2}}}$ and any $m = 1, 2, \cdots$, there exists a constant $a_{x,m}$ satisfying*

$$\frac{1}{|B_{r_1^m}(x)|} \int_{B_{r_1^m}(x)} |u - a_{x,m}|^p \leqslant A_1 r_1^{\alpha m p}, \tag{8}$$

then

$$[u]_{\alpha,\overline{B}_{1/2}} \leqslant C_1 r_1^{-\frac{n}{p}-\alpha}\big(A_1^{\frac{1}{p}} + \|u\|_{L^p(B_1)}\big), \tag{9}$$

where C_1 *is a constant depending only on* n, α *and* $||u||_{L^p(B_1)}$.

Proof. Let $A = \max\{\frac{2^{p+1}||u||^p_{L^p(B_1)}}{|B_{r_1}|r_1^{p\alpha}}, 2^p A_1 r_1^{-n-p\alpha}\}$, then we only need to show (6) holds for any $x \in \overline{B_{\frac{1}{2}}}$ and any $r > 0$. Since (6) holds clearly for $r > r_1$, we only need to show the case of $r \leqslant r_1$. If $r \leqslant r_1$, then $D_r(x) = B_r(x)$ since $r_1 \leqslant \frac{1}{2}$ and $x \in \overline{B_{\frac{1}{2}}}$. Let m satisfy $r_1^{m+1} < r \leqslant r_1^m$, from

$$|\overline{u_{B_r(x)}} - a_{x,m}| \leqslant \frac{1}{|B_r(x)|}\int_{B_r(x)} |u - a_{x,m}| \leqslant \Big(\frac{1}{|B_r(x)|}\int_{B_r(x)} |u - a_{x,m}|^p\Big)^{\frac{1}{p}},$$

we have

$$\begin{aligned}\Big(\tfrac{1}{|B_r(x)|}\textstyle\int_{B_r(x)} |u - \overline{u_{B_r(x)}}|^p\Big)^{\frac{1}{p}} &\leqslant \Big(\tfrac{1}{|B_r(x)|}\textstyle\int_{B_r(x)} |u - a_{x,m}|^p\Big)^{\frac{1}{p}} + |\overline{u_{B_r(x)}} - a_{x,m}| \\ &\leqslant 2\Big(\tfrac{1}{|B_r(x)|}\textstyle\int_{B_r(x)} |u - a_{x,m}|^p\Big)^{\frac{1}{p}}.\end{aligned} \tag{10}$$

By $\frac{|B_{r_1^k}(x)|}{|B_{r_1^{k+1}}(x)|} = \frac{1}{r_1^n}$ and (8), we deduce

$$\begin{aligned}\Big(\tfrac{1}{|B_r(x)|}\textstyle\int_{B_r(x)} |u - a_{x,m}|^p\Big)^{\frac{1}{p}} &\leqslant \Big(\tfrac{1}{|B_r(x)|}\textstyle\int_{B_{r_1^m}(x)} |u - a_{x,m}|^p\Big)^{\frac{1}{p}} \leqslant \\ &\leqslant \Big(A_1 r_1^{\alpha m p}\tfrac{|B_{r_1^m}(x)|}{|B_r(x)|}\Big)^{\frac{1}{p}} \leqslant \Big(A_1 r_1^{\alpha m p} r_1^{-n}\Big)^{\frac{1}{p}} \leqslant A_1^{\frac{1}{p}} r_1^{-\frac{n}{p}-\alpha} r^{\alpha}.\end{aligned}$$

Combining with (10), we have (6) holds with $A \geqslant 2^p A_1 r_1^{-n-p\alpha}$.

3 The Proof of Theorem 1.1

Lemma 3.1. Let u be a harmonic function defined in B_1 and φ be a continuous function defined on ∂B_1. If $u = \varphi$ on ∂B_1, then

$$\sup_{x\in B_{\frac{1}{2}}} |D^k u(x)| \leqslant C\|\varphi\|_{L^2(\partial B_1)}, \tag{11}$$

where C is a constant depending only on n and k.

Proof. Let v be the harmonic function satisfying $v = |\varphi|$ on ∂B_1, the maximum principle now imply that $v(x) \geqslant 0$ and $-v(x) \leqslant u(x) \leqslant v(x)$ in B_1. By Mean Value equalities and Hölder inequality, we have that

$$v(0) = \frac{1}{|\partial B_1|}\int_{\partial B_1} |\varphi| ds \leqslant \frac{1}{|\partial B_1|}\Big(\int_{\partial B_1} |\varphi|^2 ds\Big)^{1/2} |\partial B_1|^{1/2} \leqslant C\|\varphi\|_{L^2(\partial B_1)}; \tag{12}$$

where C is a constant depending only on n. On the other hand, we have that

$$\sup_{x\in B_{\frac{1}{2}}} |D^k u(x)| \leqslant C_1 \sup_{x\in B_{\frac{3}{4}}} |u(x)| \leqslant C_1 \sup_{x\in B_{\frac{3}{4}}} v(x) \leqslant C_1 v(0); \tag{13}$$

The last inequality is the Harnack inequality. Here we have used the same letter C_1 to denote constants depending only on n and k. By (12) and (13), it is easy to see (11) holds.

Remark 3.1. Suppose that $u(x)$ is the solution of the following problem

$$\begin{cases} -a_{ij}D_{ij}u = 0 \quad \texttt{in} \quad B_1, \\ u = \varphi \quad \texttt{on} \quad \partial B_1, \end{cases}$$

where $(a_{ij})_{n\times n}$ is a constant matrix satisfying $\lambda I \leqslant (a_{ij})_{n\times n} \leqslant \Lambda I$, then the conclusion of lemma 3.1 will be still true.

We prove Theorem 1.1 through the following key lemma.

Lemma 3.2. Suppose $u \in C^2(B_1)$ is a solution of (1) with $D^2u(x) \in \mathcal{B}_1$ for any $x \in B_1$, F satisfies (3) for any $M \in \mathcal{B}_2$ and almost every $N \in \mathcal{B}_2$. There exist positive constants $0 < \epsilon_0 < 1$ depending only on $\lambda, \Lambda, n, \beta, K$ and $0 < \delta_0 \leqslant \frac{1}{2}$ depending only on λ, Λ, n such that if

$$osc_{B_1} D^2 u \leqslant \epsilon_0 \tag{14}$$

and there exists $M_1 \in \mathcal{B}_2$ satisfying

$$\Big(\frac{1}{|B_1|}\int_{B_1} |D^2u - M_1|^2\Big)^{\frac{1}{2}} = 2\epsilon \leqslant 2\epsilon_0, \tag{15}$$

then there exists $M_2 \in \mathcal{B}_2$ satisfying

$$\Big(\frac{1}{|B_{\delta_0}|}\int_{B_{\delta_0}} |D^2u - M_2|^2\Big)^{\frac{1}{2}} \leqslant \epsilon. \tag{16}$$

Proof. By (15) and F is differentiable in $\mathcal{B}_2$ a.e., we can and we do choose $\tilde{M} = (\tilde{m}_{ij})_{n\times n} \in \mathcal{B}_2$ such that

$$\Big(\frac{1}{|B_1|}\int_{B_1} |D^2u - \tilde{M}|^2\Big)^{\frac{1}{2}} \leqslant 3\epsilon \tag{17}$$

and that F is differentiable at $\tilde{M}$ with $F'(\tilde{M}) = (a_{ij})_{n\times n} \in \mathcal{S}_n$. (2) implies

$$\lambda I \leqslant (a_{ij})_{n\times n} \leqslant \Lambda I \tag{18}$$

in the sense of positive semi-definite. We claim that

$$||D^2u - \tilde{M}||_{L^\infty(B_1)} \leqslant 3\epsilon_0 \tag{19}$$

In fact, if (19) is false, then by (14), $|D^2u(x) - \tilde{M}| > 2\epsilon_0$ for any $x \in B_1$. This contradicts with (17).

Set $P_1(x) = \frac{1}{2}x^T\tilde{M}x$ for any $x \in R^n$. According to Poincare's inequality, there exists a constant C depending only on n such that

$$||(u-P_1) - \overline{D(u-P_1)}_{B_1}\cdot x - \overline{(u-P_1)}_{B_1}||_{L^2(B_1)} \leqslant C||D(u-P_1) - \overline{D(u-P_1)}_{B_1}||_{L^2(B_1)}$$

and

$$||D(u-P_1) - \overline{D(u-P_1)}_{B_1}||_{L^2(B_1)} \leqslant C||D^2(u-P_1)||_{L^2(B_1)}.$$

It follows that

$$||(u-P_1) - \overline{D(u-P_1)}_{B_1}\cdot x - \overline{(u-P_1)}_{B_1}||_{W^{1,2}(B_1)} \leqslant C||D^2(u-P_1)||_{L^2(B_1)}.$$

By trace theorem and (17), we conclude

$$\begin{aligned}
&||u-P_1 - \overline{D(u-P_1)}_{B_1}\cdot x - \overline{(u-P_1)}_{B_1}||_{L^2(\partial B_1)}\\
&\leqslant C||u-P_1 - \overline{D(u-P_1)}_{B_1}\cdot x - \overline{(u-P_1)}_{B_1}||_{W^{1,2}(B_1)}\\
&\leqslant C||D^2u - \tilde{M}||_{L^2(B_1)} \leqslant C\epsilon,
\end{aligned} \tag{20}$$

where C depends only on n.

Let $h(x)$ be the solution of the following problem

$$\begin{cases} -a_{ij}D_{ij}h = 0 \quad \texttt{in} \quad B_1;\\ h = u - P_1 - \overline{D(u-P_1)}_{B_1}\cdot x - \overline{(u-P_1)}_{B_1} \quad \texttt{on} \quad \partial B_1. \end{cases}$$

It is clear that $(a_{ij})_{n\times n}$ is a constant matrix satisfying (18) (Recall $(a_{ij})_{n\times n} = F'(\tilde{M})$). By remark 3.1 we have that for any $x \in B_{\frac{1}{2}}$,

$$|D^3h(x)| \leqslant C||u - P_1 - \overline{D(u-P_1)}_{B_1} \cdot x - \overline{(u-P_1)}_{B_1}||_{L^2(\partial B_1)}; \tag{21}$$

where C is a constant depending only on λ, Λ and n. In view of (20), then for any $x \in B_{1/2}$,

$$|D^3h(x)| \leqslant \hat{C}\epsilon,$$

where $\hat{C}$ depends on λ, Λ and n. Let

$$\delta_0 = \min\{\frac{1}{2}, \frac{1}{2\hat{C}}\}. \tag{22}$$

Therefore for any $x \in B_{\delta_0}$,

$$|D^2h(x) - D^2h(0)| \leqslant ||D^3h||_{L^\infty(B_{\delta_0})}\delta_0 \leqslant \hat{C}\delta_0\epsilon \leqslant \frac{1}{2}\epsilon. \tag{23}$$

Set

$$f(x) = -F(\tilde{M}) - a_{ij}(D_{ij}u(x) - \tilde{m}_{ij}) \tag{24}$$

and from (3), it follows that $|f(x)| \leqslant K|D^2u(x) - \tilde{M}|^{1+\beta}$ for any $x \in B_1$. Then by (17) and (19),

$$\begin{aligned} ||f||_{L^2(B_1)} &\leqslant K\Big(\int_{B_1} |D^2u - \tilde{M}|^{2(1+\beta)}\Big)^{\frac{1}{2}} \\ &\leqslant K(3\epsilon_0)^\beta\Big(\int_{B_1}|D^2u - \tilde{M}|^2\Big)^{\frac{1}{2}} \leqslant K\sqrt{|B_1|}(3\epsilon_0)^\beta 2\epsilon. \end{aligned} \tag{25}$$

Let $v \in W^{2,2}(B_1)$ be the solution of the following problem

$$\begin{cases} -a_{ij}D_{ij}v = f \quad \texttt{in} \quad B_1, \\ v = 0 \quad \texttt{on} \quad \partial B_1, \end{cases}$$

then there exists a constant C depending only on λ, Λ and n such that

$$||v||_{W^{2,2}(B_1)} \leqslant C||f||_{L^2(B_1)} \leqslant \sqrt{|B_1|}CK(3\epsilon_0)^\beta 2\epsilon.$$

Therefore we can and we do choose $0 < \epsilon_0 < 1$ such that

$$\Big(\frac{1}{|B_{\delta_0}|}\int_{B_{\delta_0}} |D^2v|^2\Big)^{\frac{1}{2}} \leqslant \frac{1}{2}\epsilon, \tag{26}$$

where δ_0 is given by (22).

Let P_2 be the solution of the following problem

$$\begin{cases} -a_{ij}D_{ij}P_2 = F(\tilde{M}) \quad \texttt{in} \quad B_1, \\ P_2 = 0 \quad \texttt{on} \quad \partial B_1, \end{cases}$$

Since $(a_{ij})_{n\times n}$ is a constant matrix, we have P_2 is a second order polynomial .

Set $w = P_1 + P_2 + v + h + \overline{D(u-P_1)}_{B_1} \cdot x + \overline{(u-P_1)}_{B_1}$ and then by (24),

$$-a_{ij}D_{ij}w = -a_{ij}\tilde{m}_{ij} + F(\tilde{M}) + f = -a_{ij}D_{ij}u \quad \texttt{in} \quad B_1.$$

Since $w|_{\partial B_1} = u|_{\partial B_1}$, we have $w = u$ in B_1 and then

$$D^2u = D^2w = \tilde{M} + D^2P_2 + D^2v + D^2h \quad \texttt{in} \quad B_1.$$

Set $M_2 = \tilde{M} + D^2P_2 + D^2h(0)$. It follows that for any $x \in B_{\delta_0}$,

$$|D^2u(x) - M_2| = |D^2v(x) + D^2h(x) - D^2h(0)| \leqslant |D^2v(x)| + |D^2h(x) - D^2h(0)|.$$

By (23) and (26), we have (16) holds.

Corollary 3.1. *Assume that all the hypotheses of Lemma 3.2 hold, and ϵ_0 and δ_0 are given by it. If there exists $M_1 \in \mathcal{B}_2$ satisfying*

$$\Big(\frac{1}{|B_1|}\int_{B_1}|D^2u-M_1|^2\Big)^{\frac{1}{2}} \leqslant \sqrt{\frac{|B_{\delta_0}|}{|B_1|}}\epsilon_0\delta_0^{\frac{\ln(1/2)}{\ln\delta_0}}, \tag{27}$$

then $u\in C^{2,\alpha}(\overline{B_{1/2}})$ with $\alpha=\frac{\ln(1/2)}{\ln\delta_0}$, moreover,

$$[D^2u]_{\alpha,\overline{B}_{1/2}} \leqslant 2C_1\delta_0^{-\frac{n}{2}}\epsilon_0, \tag{28}$$

where C_1 is the constant given by Corollary 2.1.

Proof. We only need to show that for any $x_0\in\overline{B_{1/2}}$ and any $k=1,2\cdots$, there exists $M_{x_0,k}\in\mathcal{B}_2$ such that

$$\Big(\frac{1}{|B_{\delta_0^k}(x_0)|}\int_{B_{\delta_0^k}(x_0)}|D^2u-M_{x_0,k}|^2\Big)^{\frac{1}{2}} \leqslant \epsilon_0\delta_0^{k\frac{\ln(1/2)}{\ln\delta_0}}. \tag{29}$$

Then we infer (28) by Corollary 2.1 since $\delta_0^{\frac{\ln(1/2)}{\ln\delta_0}}=1/2$.

In order to prove (29), we use the mathematical induction method. First, for $k=1$, since $B_{\delta_0}(x_0)\subset B_1$, (27) implies (29) clearly for $M_{x_0,1}=M_1$. Suppose that there exists a matrix $M_{x_0,k}\in\mathcal{B}_2$ such that (29) holds for $k=m$. Then for $k=m+1$, set $y=\frac{x-x_0}{\delta_0^m}$ and $v(y)=\frac{u(x_0+\delta_0^m y)}{\delta_0^{2m}}$ for any $x\in B_{\delta_0^m}(x_0)$ and $y\in B_1$. Since $D_x^2u(x)=D_y^2v(y)$ as $y=\frac{x-x_0}{\delta_0^m}$, we have (14) still holds with u replaced by v and

$$F(D^2v(y))=0 \quad \texttt{in} \quad B_1.$$

By the induction hypothesis, there exists $M_{x_0,m}\in\mathcal{B}_2$ such that

$$\begin{aligned}\Big(\tfrac{1}{|B_1|}\textstyle\int_{B_1}|D^2v-M_{x_0,m}|^2\Big)^{\frac{1}{2}} &= \Big(\tfrac{1}{|B_{\delta_0^m}(x_0)|}\textstyle\int_{B_{\delta_0^m}(x_0)}|D^2u-M_{x_0,m}|^2\Big)^{\frac{1}{2}}\\ &\leqslant \epsilon_0\delta_0^{m\frac{\ln(1/2)}{\ln\delta_0}} \leqslant \epsilon_0.\end{aligned}$$

According to Lemma 3.2, there exists $M_{x_0,m+1}\in\mathcal{B}_2$ such that

$$\Big(\frac{1}{|B_{\delta_0}|}\int_{B_{\delta_0}}|D^2v-M_{x_0,m+1}|^2\Big)^{\frac{1}{2}} \leqslant \frac{1}{2}\epsilon_0\delta_0^{m\frac{\ln(1/2)}{\ln\delta_0}},$$

that is,

$$\Big(\frac{1}{|B_{\delta_0^{m+1}}(x_0)|}\int_{B_{\delta_0^{m+1}}(x_0)}|D^2u-M_{x_0,m+1}|^2\Big)^{\frac{1}{2}} \leqslant \epsilon_0\delta_0^{(m+1)\frac{\ln(1/2)}{\ln\delta_0}}.$$

Then, (29) holds for $k=m+1$.

proof of Theorem 1.1

Let $\mathcal{D}=D^2u(B_{3/4})$, then $\mathcal{D}$ is bounded in $\mathcal{S}_n$ since D^2u is continuous. By F is locally $C^{1,\beta}$ almost everywhere, there exists K such that (3) holds for any $M\in\mathcal{D}$ and any $N\in\mathcal{D}$ where F is differentiable. Let ϵ_0 and δ_0 be given by Lemma 3.2. (Recall ϵ_0 and δ_0 depend only on λ,Λ,n,β and K.) By (4), there exists a uniform constant $0<r_0<1/4$ depending only on ρ such that, for any $x_0\in B_{1/2}$,

$$\sup_{x\in B_{r_0}(x_0)}|D^2u(x)-D^2u(x_0)| \leqslant \frac{1}{2}\epsilon_0\delta_0^{\frac{n}{2}} \leqslant \epsilon_0,$$

then we have

$$\Big(\frac{1}{|B_{r_0}(x_0)|}\int_{B_{r_0}(x_0)}|D^2u-D^2u(x_0)|^2\Big)^{\frac{1}{2}} \leqslant \sqrt{\frac{|B_{\delta_0}|}{|B_1|}}\epsilon_0\delta_0^{\frac{\ln(1/2)}{\ln\delta_0}}.$$

Set $y = \frac{x-x_0}{r_0}$ and $v(y) = \frac{u(x_0+r_0 y)}{r_0^2}$ for any $x \in B_{r_0}(x_0)$, then $D^2v(y) = D^2u(x)$ for $y \in B_1$. It follows that all the hypotheses of Lemma 3.2 hold for $v \in C^2(B_1)$. According to corollary 3.1, for $\alpha = \frac{\ln \delta_0}{\ln(1/2)}$, we have

$$[D^2 v]_{\alpha,\overline{B_{1/2}}} \leqslant C_2 \delta_0^{-\frac{n}{2}} \|v\|_{C^2(\overline{B_1})},$$

where C_2 is a constant depending only on n, α since $\|D^2 v\|_{L^2(B_1)} \leqslant \epsilon_0$. Then we derive

$$[D^2 u]_{\alpha,\overline{B}_{r_0/2}(x_0)} \leqslant C_2 \delta_0^{-\frac{n}{2}} r_0^{-\alpha} \|u\|_{C^2(\overline{B_{3/4}})}.$$

By the standard covering method, we have (5) holds.

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