MULTI-WAVE SOLUTIONS FOR A NON-ISOSPECTRAL KDV-TYPE EQUATION WITH VARIABLE COEFFICIENTS

by

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As a typical mathematical model in fluids and plasmas, Korteweg-de Vries equation is famous. In this paper, the Exp-function method is extended to a nonisospectral Korteweg-de Vries type equation with three variable coefficients, and multi-wave solutions are obtained. It is shown that the Exp-function method combined with appropriate ansatz may provide with a straightforward, effective and alternative method for constructing multi-wave solutions of variable-coefficient non-linear evolution equations.

Key words: Korteweg-de Vries type equation, exp-function method, multi-wave solution

Introduction

With the development of soliton theory, finding exact solutions of non-linear evolution equations (NLEE) has attached much attention and developed into a significant direction in non-linear science. Since proposed by He and Wu in 2006, the Exp-function method [1] has been applied to many equations, such as the double sine-Gordon equation [2], Maccari's system [3] and variable-coefficient Korteweg-de Vries (KdV) equation [4]. In addition, this method can be generalized for solving differential-difference equation [5], stochastic equation [6], and fractional differential equation [7].

Recently, Zhang [8] generalized the Exp-function method to obtain not only solitary wave solutions and periodic solutions but also rational solutions in a uniform way. Marinakis [9] generalized the Exp-function method and obtained multi-soliton solutions of the famous KdV equation. Based on Marinakis' work, Zhang *et al.* [10] obtained multi-soliton solutions and thus concluded with a uniform formula of the *N*-soliton solution of a KdV equation with two variable coefficients. In 2011, Zhang *et al.* [11] first generalized the Exp-function method for constructing multi-wave solutions of non-linear DDE by devising a rational ansatz of multiple exponential functions.

In this paper, we extend the Exp-function method for constructing multi-wave solutions of a non-isospectral KdV-type equation with variable coefficients $K_0(t)$, $K_1(t)$ and h(t) of time t:

$$u_t + K_0(t)(u_{xxx} + 6uu_x) + 4K_1(t)u_x - h(t)(xu_x + 2u) = 0$$
(1)

which was derived by Chan et al. from a non-isospectral Lax pair [12].

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Methodology

We describe the basic idea of the Exp-function method combined with a new and more general ansatz for multi-wave solutions of the given NLEE with variable coefficients, say, in three variables x, y, and t:

$$P(x, y, t, u, u_x, u_y, u_t, u_{xy}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, u_{tt}, \cdots) = 0$$
(2)

The Exp-function method for 1-wave solution is based on the assumption that eq. (2) has a solution:

$$u(x, y, t) = \frac{\sum_{i_1=0}^{p_1} a_{i_1}(x, y, t) e^{i_1 \xi_1(x, y, t)}}{\sum_{j_1=0}^{q_1} b_{j_1}(x, y, t) e^{j_1 \xi_1(x, y, t)}}$$
(3)

where $a_{i_1}(x, y, t)$, $b_{j_1}(x, y, t)$, and $\xi_1(x, y, t)$ are unknown functions of the indicated variables, the values of p_1 and q_1 can be determined by balancing the linear term of highest order in eq. (2) with the highest order non-linear term.

To seek *N*-soliton solutions for integer N > 1, we generalize eq. (3) to the form:

$$u(x, y, t) = \frac{\sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_N=0}^{p_N} a_{i_1 i_2 \cdots i_N}(x, y, t) e^{\sum_{s=1}^{i_s \xi_s} (x, y, t)}}{\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \cdots \sum_{j_N=0}^{q_N} b_{j_1 j_2 \cdots j_N}(x, y, t) e^{\sum_{s=1}^{N} i_s \xi_s(x, y, t)}}$$
(4)

Substituting eq. (4) with N = 2 into eq. (2) and equating to zero each coefficient of the same order power of the exponential functions yields a set of equations. Solving the set of equations, we can determine the 2-wave solution, and the following 3-wave solution by eq. (4) with N = 3, provided they exist. If possible, we may conclude with the uniform formula of *N*-wave solution for any N > 1.

Multi-wave solutions

Let us apply the method described in the section *Methodology* to solve the non-isospectral KdV-type equation. To begin with, we suppose that eq. (1) admits 1-wave solution in the form:

$$u(x,t) = \frac{a_1(t)\exp(\xi_1)}{\left[1 + b_1\exp(\xi_1)\right]^2}$$
(5)

where $\xi_1 = k_1(t)x + s_1(t) + w_1$, $k_1(t)$, $s_1(t)$, and $a_1(t)$ are undetermined functions of t, w_1 , and b_1 are constants to be determined.

Substituting eq. (5) into eq. (1), and using Mathematica, then equating to zero each coefficient of the same order power of $x^{\theta} e^{g_{\xi_1}} (\theta = 0, 1; \theta = 1, 2, 3, 4)$ yields a set of equations for $k_1(t)$, $s_1(t)$, $a_1(t)$ and b_1 . Solving the set of equations, we have:

$$a_1(t) = 2b_1 k_{10}^2 e^{2\int h(t)dt}, \ k_1(t) = k_{10} e^{\int h(t)dt}, \ s_1(t) = -\int [k_{10}^3 K_0(t) e^{3\int h(t)dt} + 4k_{10} K_1(t) e^{\int h(t)dt}] dt$$

from which we obtain the 1-wave solution of eq. (1):

$$u(x,t) = \frac{2b_1 k_{10}^2 e^{2\int h(t)dt} e^{\xi_1}}{(1+b_1 e^{\xi_1})^2} = 2[\ln(1+b_1 e^{\xi_1})]_{xx}$$
(6)

where $\xi_1 = xk_{10}e^{\int h(t)dt} - \int [k_{10}^3 K_0(t)e^{3\int h(t)dt} + 4k_{10}K_1(t)e^{\int h(t)dt}]dt + w_1$, b_1 , k_{10} , and w_1 are arbitrary constants.

We next suppose that eq. (1) has 2-wave solution in the form:

$$u(x,t) = \frac{a_{10}(t)e^{\xi_1} + a_{01}(t)e^{\xi_2} + a_{11}(t)e^{\xi_1 + \xi_2} + a_{21}(t)e^{2\xi_1 + \xi_2} + a_{12}(t)e^{\xi_1 + 2\xi_2}}{(1 + b_1e^{\xi_1} + b_2e^{\xi_2} + b_3e^{\xi_1 + \xi_2})^2}$$
(7)

where $\xi_1 = k_1(t)x + s_1(t) + w_1$, $\xi_2 = k_2(t)x + s_2(t) + w_2$, $k_1(t)$, $k_2(t)$, $s_1(t)$, $s_2(t)$, $a_{10}(t)$, $a_{01}(t)$, $a_{11}(t)$, $a_{21}(t)$, $a_{12}(t)$ are undetermined functions of *t*, and w_1 , w_2 , b_1 , b_2 , and b_3 are constants to be determined.

Substituting eq. (7) into eq. (1) and using Mathematica yields:

$$\begin{aligned} a_{10}(t) &= 2b_1k_{10}^2 e^{2\int h(t)dt}, \quad a_{01}(t) = 2b_2k_{20}^2 e^{2\int h(t)dt}, \quad a_{11}(t) = 2b_1b_2e^{2\int h(t)dt}(k_{10} - k_{20})^2, \\ a_{21}(t) &= \frac{2b_1^2b_2k_{20}^2 e^{2\int h(t)dt}(k_{10} - k_{20})^2}{(k_{10} + k_{20})^2}, \quad a_{12}(t) = \frac{2b_1b_2^2k_{10}^2 e^{2\int h(t)dt}(k_{10} - k_{20})^2}{(k_{10} + k_{20})^2}, \quad b_3 = \frac{b_1b_2(k_{10} - k_{20})^2}{(k_{10} + k_{20})^2}, \\ k_1(t) &= k_{10}e^{\int h(t)dt}, \quad s_1(t) = -\int [k_{10}^3K_0(t)e^{3\int h(t)dt} + 4k_{10}K_1(t)e^{\int h(t)dt}]dt, \\ k_2(t) &= k_{20}e^{\int h(t)dt}, \quad s_2(t) = -\int [k_{20}^3K_0(t)e^{3\int h(t)dt} + 4k_{20}K_1(t)e^{\int h(t)dt}]dt, \end{aligned}$$

from which we obtain 2-wave solution of eq. (1):

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$$u(x,t) = 2[\ln(1+b_1e^{\xi_1}+b_2e^{\xi_2}+b_1b_2e^{\xi_1+\xi_2+B_{12}})]_{xx}$$
(8)

where $\xi_i = xk_{i0}e^{\int h(t)dt} - \int [k_{i0}^3 K_0(t)e^{3\int h(t)dt} + 4k_{i0}K_1(t)e^{\int h(t)dt}]dt + w_i$, b_i , k_{i0} , and w_i are arbitrary constants, i = 1, 2, and $e^{B_{12}} = (k_{10} - k_{20})^2 / (k_{10} + k_{20})^2$.

Similarly, we can also determine the 3-wave solution of eq. (1):

$$u(x,t) = 2[\ln(1+b_1e^{\xi_1}+b_2e^{\xi_2}+b_3e^{\xi_3}+b_1b_2e^{\xi_1+\xi_2+B_{12}} + b_1b_3e^{\xi_1+\xi_3+B_{13}} + b_2b_3e^{\xi_2+\xi_3+B_{23}} + b_1b_2b_3e^{\xi_1+\xi_2+\xi_3+B_{13}+B_{23}})]_{xx}$$
(9)

where $\xi_i = xk_{i0}e^{\int h(t)dt} - \int [k_{i0}^3 K_0(t)e^{3\int h(t)dt} + 4k_{i0}K_1(t)e^{\int h(t)dt}]dt + w_i$, b_i , k_{i0} , and w_i are arbitrary constants, i = 1, 2, 3, and $e^{B_{ij}} = (k_{i0} - k_{j0})^2 / (k_{i0} + k_{j0})^2 (1 \le i < j \le 3)$.

By analyzing the obtained solutions (6), (8) and (9), we can conclude with a uniform formula of *N*-wave solution for any N > 1 of eq. (1) as:

$$u(x,t) = 2\left[\ln\left(\sum_{\mu=0,1}\prod_{i=1}^{N}b_{i}^{\mu_{i}}e^{\sum_{i=1}^{N}\mu_{i}\xi_{i}+\sum_{1\leq i< j\leq N}\mu_{i}\mu_{j}B_{ij}}\right)\right]_{xx}$$
(10)

where $\xi_i = xk_{i0}e^{\int h(t)dt} - \int [k_{i0}^3K_0(t)e^{3\int h(t)dt} + 4k_{i0}K_1(t)e^{\int h(t)dt}]dt + w_i$, b_i , k_{i0} , w_i are arbitrary constants, the sum $\Sigma_{\mu=0.1}$ refers to all combinations of each $\mu = 0.1$ for i = 1, 2, ..., N, and $e^{B_{ij}} = (k_{i0} - k_{i0})^2 / (k_{i0} + k_{i0})^2 (1 \le i < j \le N)$.

Conclusions

In this paper, multi-wave solutions of the nonisospectral KdV-type equation with variable coefficients have successfully been obtained, from which the uniform formula of *N*wave solution is derived. It is due to the devised new and more general ansatz (4). The paper shows that the Exp-function method combined with appropriate anstaz may provide us with a straightforward, effective and alternative method for constructing multi-wave solutions or testing their existence and can be extended to other NLEE with variable coefficients.

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