A Generalized Liouville Theorem for Entire Functions

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Abstract. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $f(z) \neq 0$ for any $z \in \mathbb{C}$. We show that if $g = |f(z)|^2 dz \otimes d\bar{z}$ is a complete Riemannian metric, then f must be a constant. As a corollary we give a new proof of the classical Liouville theorem.

1. INTRODUCTION. An entire function is a holomorphic function defined on $\mathbb{C}$. The classical Liouville theorem says that a bounded entire function must be a constant [**3**]. It was improved by the little Picard theorem [**1**] which says that an entire function f must be a constant if f omits two values. If f just omits one value, then f is not necessarily a constant as $f = e^z$ is an entire function with one value omitted. In this paper we show that under a certain geometric condition, an entire function with one value omitted is still a constant. More precisely, we are going to prove the following theorem.

Theorem 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $f(z) \neq 0$ for any $z \in \mathbb{C}$. If $g = |f(z)|^2 dz \otimes d\bar{z}$ is a complete Riemannian metric, then f must be a constant.*

A Riemannian metric is called complete if every bounded closed subset is relatively compact. The classical Hopf–Rinow theorem gives a different characterization of a complete Riemannian metric [**2**].

As a corollary of Theorem 1, we get the following classical Liouville theorem.

Theorem 2. *If $f : \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function, then f must be a constant.*

Proof. Suppose that $|f(z)| \leq A$ for some constant A and any $z \in \mathbb{C}$. Let γ be any smooth curve joining 0 and z. Then the length of γ with respect to $g = |e^f|^2 dz \otimes d\bar{z}$ is computed by

$$L(\gamma) = \int_\gamma |e^f||\gamma'(t)| = \int_\gamma e^{Ref}|\gamma'(t)| \geq e^{-A} \int_\gamma |\gamma'(t)|.$$

Let d_g, d_0 be the distance functions with respect to g, g_0, respectively, where $g_0 = dz \otimes d\bar{z}$. Then we have

$$d_g(0, z) \geq e^{-A} d_0(0, z).$$

Let K be any closed and bounded subset with respect to g. Then K is also bounded with respect to g_0. Since g_0 is a complete Riemannian metric, K must be relatively compact by Hopf–Rinow theorem. By the Hopf–Rinow theorem again, $g = |e^f|^2 dz \otimes d\bar{z}$ is a complete Riemannian metric. Then Theorem 1 implies that e^f must be a constant. Hence f must be a constant. ∎

http://dx.doi.org/10.4169/amer.math.monthly.122.10.1001
MSC: Primary 30D35, Secondary 30D15; 53C20

2. PROOF OF THEOREM 1. The proof is based on the following lemma.

Lemma 1. *Let $F : (M, g) \to (N, h)$ be a local diffeomorphism between Riemannian manifolds such that $F^*h = g$. If g is a complete Riemannain metric, then F is a covering map.*

Proof. See p. 144 in [**2**]. ■

Remark. In Lemma 1 we do not have to assume that F is onto in prior. In fact, by the assumption of Lemma 1, we can derive that F is onto.

Since f is an entire function, there is an entire function F such that $f(z) = \frac{\partial F}{\partial z}$. As $f(z) \neq 0$ for any $z \in \mathbb{C}$, F is a local biholomorphism between $\mathbb{C}$ by implicit function theorem. Let $g_0 = dz \otimes d\bar{z}$ be the standard Riemannian metric on $\mathbb{C}$. Then we have the following lemma.

Lemma 2. $F^*g_0 = |f(z)|^2 dz \otimes d\bar{z}$.

Proof.
$$\begin{aligned} F^*g_0 &= F^*(dz \otimes d\bar{z}) = F^*dz \otimes F^*d\bar{z} \\ &= \tfrac{\partial F}{\partial z} dz \otimes \tfrac{\partial \bar{F}}{\partial \bar{z}} d\bar{z} = |f(z)|^2 dz \otimes d\bar{z}. \end{aligned}$$
■

By assumption, $|f(z)|^2 dz \otimes d\bar{z}$ is a complete Riemannian metric. It follows that F^*g_0 is complete by Lemma 2. Combined with the fact that F is a local biholomorphism, F must be a covering map by Lemma 1. As $\mathbb{C}$ is simply connected, F must be a global biholomorphism. Since $Aut(\mathbb{C}) = \{az + b, a, b \in \mathbb{C}\}$ by Lemma 3 below, then $F = a_0 z + b_0$ for some constants a_0, b_0. Hence $f(z) = \frac{\partial F}{\partial z} = a_0$.

Lemma 3. $Aut(\mathbb{C}) = \{az + b \,|a, b \in \mathbb{C}\}$.

Proof. There are many known proofs so far. Here we provide a proof using Picard's great theorem. If $F \in Aut(\mathbb{C})$, then F can be expressed as a global convergence power series

$$F(z) = \Sigma_{n=0}^{\infty} a_n z^n. \tag{1}$$

Then $G(z) = F(\frac{1}{z}) = \Sigma_{n=0}^{\infty} \frac{a_n}{z^n}$ has an essential singularity at $z = 0$ unless the a_n eventually are all zero. If there were an essential singularity, then by Picard's great theorem, G would map multiple points to the same value which contradicts that F is injective. Hence G has no essential singularity and the a_n eventually are all zero. It follows that F is a polynomial. By the fundamental theorem of algebra, a polynomial $F(z)$ is injective if and only if $F(z) = az + b$ for some $a, b \in \mathbb{C}$. ■

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