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A successive approximation algorithm for the multiple knapsack problem

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Abstract It is well-known that the multiple knapsack problem is NP-hard, and does not admit an FPTAS even for the case of two identical knapsacks. Whereas the 0-1 knapsack problem with only one knapsack has been intensively studied, and some effective exact or approximation algorithms exist. A natural approach for the multiple knapsack problem is to pack the knapsacks successively by using an effective algorithm for the 0-1 knapsack problem. This paper considers such an approximation algorithm that packs the knapsacks in the nondecreasing order of their capacities. We analyze this algorithm for 2 and 3 knapsack problems by the worst-case analysis method and give all their error bounds.

Keywords Multiple knapsack problem · Approximation algorithm · Worst-case analysis

1 Introduction

The multiple knapsack problem (MKP for short) can be defined as follows: given a set of n items and m knapsacks such that each item j has a profit p_j and a weight w_j, and each knapsack i has a capacity c_i. The objective is to select a subset of items that can be packed into m knapsacks and the total profit of all items in the knapsacks is maximized. For $m = 1$, MKP reduces to the classical 0-1 knapsack problem (KP for short).

It is well-known that MKP is NP-hard, and is strongly NP-hard if m is a part of the input (Garey and Johnson 1979). Furthermore, MKP does not admit an FPTAS even for the case of two identical knapsacks (Chekuri and Khanna 2005). Caprara et al. (2000a, 2000b) consider the multiple subset sum problem (MSSP for short) that can

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be considered as a special case of MKP in which the profits and weights coincide. They present PTAS for the cases that the knapsacks are of the same or different sizes. Chekuri and Khanna (2005) derive a PTAS for the general MKP, but it is only of theoretical interest, since it requires huge computing time for any reasonably small value of the required accuracy.

For a multiple knapsack problem with the objective of maximizing total profit, an approximation algorithm is called of *ρ-approximation* for some $\rho < 1$, if it always delivers a solution with total profit at least ρz^*, where z^* denotes the optimal profit. We also called such ρ an error bound of the algorithm. When such ρ is taken as large as possible, it is called the *worst-case ratio* of the algorithm. The simple greedy algorithm, that packs the items one by one under the nonincreasing of the ratio p_j/w_j into the knapsacks, has an error bound zero (Caprara and Pferschy 2004). This is to say the solution yielded by the greedy algorithm may be arbitrarily bad compared to the optimal solution. Caprara et al. (2003) consider MSSP with identical knapsacks, and present a polynomial time 3/4-approximation algorithm.

A flagrant contrast to MKP's hardness, KP has some effective exact or approximation algorithms. KP has been intensively studied because of both its theoretical interest and its wide applicability. As a matter of fact, many instances of KP can be solved within acceptable running time although the problem is NP-hard (Garey and Johnson 1979). Exact algorithms for KP are mainly based on two approaches: branch-and-bound and dynamic programming. The best-known branch-and-bound algorithms for KP are those of Horowitz and Sahni (1974), Nauss (1976), Martello and Toth (1977). On the other hand, it is known that a dynamic programming can solve KP in $O(nc)$ running time, where c is the capacity of the knapsack. FPTAS also exists for KP (Ibarra and Kim 1975), and Lawler (1979) achieves an efficient $(1-\epsilon)$-approximation algorithm with a running time of $O(n\log 1/\epsilon + 1/\epsilon^4)$. The FPTAS with the best complexity currently known is due to Kellerer and Pferschy (1999). They present a FPTAS requiring a running time of $O(n\log 1/\epsilon + 1/\epsilon^3\log^2(1/\epsilon))$. More detailed results of KP and MPK can be found in Kellerer et al. (2004), Martello et al. (2000), and Pisinger and Toth (1998).

A natural idea is to apply an effective exact or approximation algorithm for KP one by one knapsack successively to get an approximate solution of MKP. Though this approach may not produce polynomial algorithms, it is expected to be fast and practical for small m as it inherits the characteristics of the KP algorithms. In some branch-and-bound algorithms for MKP, such as (Martello and Toth 1981) and (Pisinger 1999), this idea is applied to get a lower bound of MKP. Chekuri and Khanna (2005) consider such an approximation algorithm that packs knapsacks one at a time by applying the FPTAS for the single knapsack on the remaining items. In their algorithm, the knapsacks are sorted in any order, and they prove an error bound of $(1-\epsilon)1/2$. Interestingly, this natural idea also can be found in the literature of bin packing problem (Caprara and Pferschy 2004). This paper considers such an algorithm that pack the knapsacks in the nondecreasing order of their capacities for two and three knapsacks. We will call this algorithm successive approximation algorithm, and show that the error bound gotten by Chekuri and Khanna will be significantly improved for 2 or 3 knapsacks problem.

The remaining part of this paper is organized as follows. In Sect. 2, we present some notations for the successive approximation algorithm and give a summary of our results. The main results, the worst-case ratios of the algorithm using an exact KP algorithm successively, are proved in Sect. 3. If an approximation algorithm for KP is applied successively in the algorithm, the corresponding error bounds are presented in Sect. 4 as an extension of the main results.

2 Preliminaries

2.1 Notations

This paper considers two types of successive approximation algorithms. One is to pack the knapsacks one by one by using an exact algorithm of KP successively, where the exact algorithm is denoted as H. Another one is to use a $(1-\epsilon)$-approximation $(0 < \epsilon < 1)$ algorithm of KP, denoted as H_ϵ, successively for each knapsack. These two types of algorithms are denoted as H^m and H_ϵ^m respectively and are described formally as follows.

Successive approximation algorithm $H^m(H_\epsilon^m)$**:**

Step 1. Sort the knapsacks in order of nondecreasing capacities and set $i = 1$.

Step 2. Select the remaining items into the ith knapsack by using algorithm $H(H_\epsilon)$. If $i = m$, stop; otherwise, $i = i + 1$, repeat Step 2.

We assume that the knapsacks are sorted in an order such that $0 < c_1 \le c_2 \le \cdots \le c_m$ after Step 1. The following notations are used for any approximation algorithm of H^m and H_ϵ^m. Let z and z^* be the objective values of an approximation algorithm and the optimal algorithm for MKP respectively. Denote the set of items packed into the ith $(1 \le i \le m)$ knapsack in the approximation algorithm by S_i. If some items belonging to S_i $(1 \le i \le m-1)$ are packed into the jth knapsack in the optimal algorithm, denote the set of these items by S_{ij}. Let S_{mj} be the set of items which are packed into the jth knapsack in the optimal algorithm and do not belong to any of the sets $S_1, \ldots, S_{m-1}$.

Let $w(S)$ and $p(S)$ be the total weight and total profit of items in set S respectively. It is easily verified that

$$\begin{aligned} z &= \sum_{i=1}^{m} p(S_i), \\ z^* &= \sum_{i=1}^{m} \sum_{j=1}^{m} p(S_{ij}). \end{aligned} \tag{1}$$

2.2 Summary of results

This paper will show that the worst-case ratio of H^2 is 3/4 if $c_1 = c_2$, and 2/3 if $c_1 \ne c_2$. For the case of $m = 3$, the worst-case ratios of H^3 are sensitive to the change of knapsacks' capacities. The worst-case ratios of H^3 are presented in Table 1.

For algorithm H_ϵ^m, if the worst-case ratio of $H^m (m = 2, 3)$ is ρ and H_ϵ is an $(1-\epsilon)$-approximation algorithm for KP, then we have $H_\epsilon^m (m = 2, 3)$ is a $\rho(1-\epsilon)$-approximation algorithm for MKP. Combining the results of the worst-case ratios of H^m, we can get the error bounds of H_ϵ^m for the cases of $m = 2$ and $m = 3$.

Table 1 The worst-case ratios of H^3

Case	Subcase	Ratio
$c_1 = c_2 = c_3$		3/4
$c_1 = c_2 < c_3$	$c_2 \le \frac{1}{2}c_3$	5/7
	$c_2 > \frac{1}{2}c_3$	2/3
$c_1 < c_2 = c_3$	$c_2 \ge 3c_1$	3/4
	$\frac{4}{3}c_1 < c_2 < 3c_1$	7/10
	$c_2 \le \frac{4}{3}c_1$	8/11
$c_1 < c_2 < c_3$	$c_1 + c_2 \le c_3$ or $2c_1 \le c_2$	2/3
	$c_1 + c_2 > c_3$ and $2c_1 > c_2$	3/5

3 Successive approximation algorithm H^m

In this section, we will present the worst-case ratios of H^2 and H^3. As algorithm H is an exact algorithms for KP, we can get the following Lemma 1.

Lemma 1 *Let S be a set of items. If $w(S) \le c_i$ and $S \cap S_j = \emptyset$ for each $1 \le j \le i-1$, then $p(S) \le p(S_i)$.*

As a matter of concision, if a set S satisfies the two conditions of Lemma 1, we will only mention the first one in the following part. For example, when it is said "As $w(S) \le c_i$, we have $p(S) \le p(S_i)$ according to Lemma 1", it implies that we can verify $S \cap S_j = \emptyset$ for each $1 \le j \le i-1$.

3.1 The worst-case ratios of H^2

For the case of $m = 2$, we have $z = p(S_1) + p(S_2)$ and $z^* = \sum_{i=1}^{2}\sum_{j=1}^{2} p(S_{ij})$ from (1). The optimal solution can be described like Fig. 1.

Theorem 1 *The worst-case ratio of H^2 is 3/4 if $c_1 = c_2$.*

Proof As $w(S_{11} \cup S_{21}) \le c_1$ and $w(S_{12} \cup S_{22}) \le c_2 = c_1$, according to Lemma 1 we have

$$z^* = p(S_{11} \cup S_{21}) + p(S_{12} \cup S_{22}) \le 2p(S_1). \tag{2}$$

At the same time, it is easy to see that $w(S_{11} \cup S_{12}) \le c_1$, $w(S_{21}) \le c_2$ and $w(S_{22}) \le c_2$. According to Lemma 1, we have

$$z^* = p(S_{11} \cup S_{12}) + p(S_{21}) + p(S_{22}) \le p(S_1) + 2p(S_2). \tag{3}$$

From (2) and (3), we have

$$z = p(S_1) + p(S_2) = \frac{1}{4}(2p(S_1)) + \frac{1}{2}(p(S_1) + 2p(S_2)) \ge \frac{3}{4}z^*. \tag{4}$$

The instance below will show that the bound is tight.

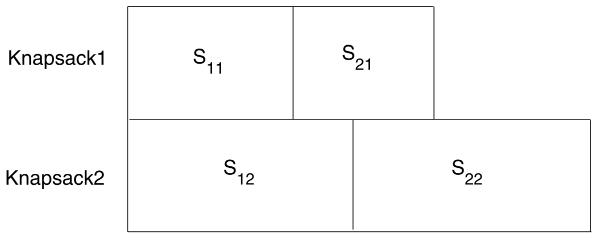

Fig. 1 The optimal solution of two knapsack problem

Table 2 Information of items in Instance 1

Item j	1	2	3	4
w_j	$\frac{1}{2}c_1 - \epsilon$	$\frac{1}{2}c_1 - \epsilon$	$\frac{1}{2}c_1 + \epsilon$	$\frac{1}{2}c_1 + \epsilon$
p_j	$1 + \epsilon$	$1 + \epsilon$	1	1

Table 3 Information of items in Instance 2

Item j	1	2	3
w_j	$c_1 - \epsilon$	$c_2 - c_1 + \epsilon$	c_1
p_j	$1 + \epsilon$	1	1

Instance 1 The information of the items is described in Table 2, in which $\epsilon > 0$ is small enough.

By algorithm H^2, items 1 and 2 are packed into knapsack 1, and one of items 3 and 4 is packed into knapsack 2, thus the objective value $z = 3 + 2\epsilon$. In an optimal solution, items 1 and 3 can be packed into knapsack 1, and items 2 and 4 can be packed into knapsack 2, thus the optimal value $z^* = 4 + 2\epsilon$. Therefore the bound is tight as $z/z^* \to 3/4$ when $\epsilon \to 0$ for the instance. □

Theorem 2 *The worst-case ratio of H^2 is $2/3$ if $c_1 < c_2$.*

Proof First, we observe that the result of (3) is also true for this case.

Next, notice that $w(S_{11} \cup S_{21}) \le c_1$, $w(S_{12}) \le c_1$, and $w(S_{22}) \le c_2$. According to Lemma 1 we have

$$z^* = p(S_{11} \cup S_{21}) + p(S_{12}) + p(S_{22}) \le 2p(S_1) + p(S_2). \tag{5}$$

From (3) and (5) , we have

$$z = p(S_1) + p(S_2) = \frac{1}{3}(p(S_1) + 2p(S_2)) + \frac{1}{3}(2p(S_1) + p(S_2)) \ge \frac{2}{3}z^*. \tag{6}$$

The instance below will show that the bound is tight.

Instance 2 The information of the items is described in Table 3, in which $\epsilon > 0$ is small enough.

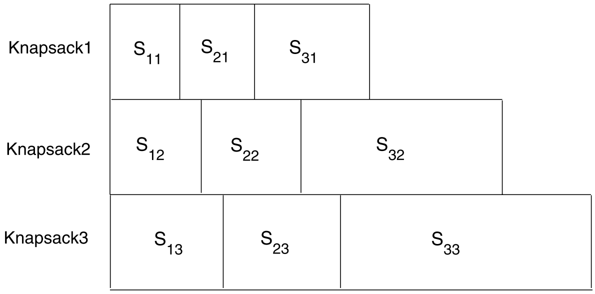

Fig. 2 The optimal solution of three knapsack problem

By algorithm H^2, item 1 is packed into knapsack 1, and one of item 2 and 3 is packed into knapsack 2, thus the objective value $z = 2 + \epsilon$. In an optimal solution, item 3 can be packed into knapsack 1, and items 1 and 2 can be packed into knapsack 2, thus the optimal value $z^* = 3 + \epsilon$. Therefore the bound is tight as $z/z^* \to 2/3$ when $\epsilon \to 0$ for the instance. □

3.2 The worst-case ratios of H^3

For the case of $m = 3$, we have $z = p(S_1) + p(S_2) + p(S_3)$ and $z^* = \sum_{i=1}^3 \sum_{j=1}^3 p(S_{ij})$ from (1). The optimal solution can be described like Fig. 2.

It is easily verified that

$$\sum_{j=1}^{3} p(S_{1j}) \leq p(S_1), \tag{7}$$

$$\sum_{j=1}^{3} p(S_{2j}) \leq p(S_2), \tag{8}$$

$$p(S_{3j}) \leq p(S_3), \quad j = 1, 2, 3. \tag{9}$$

In the following subsections, we will discuss the problem under three cases: three knapsacks with the same capacity, only two knapsacks with the same capacity, and knapsacks with different capacity.

3.2.1 Three knapsacks with the same capacity

Theorem 3 *The worst-case ratio of H^3 is $3/4$ if $c_1 = c_2 = c_3$.*

Proof If there are $i, j \in \{1, 2, 3\}$ and $i \neq j$ such that $w(S_{3i} \cup S_{3j}) \leq c_3$, without loss of generality, assume $w(S_{31} \cup S_{32}) \leq c_3$, then we have $p(S_{31} \cup S_{32}) \leq p(S_3)$ according to Lemma 1. Together with (7–9), we have

$$\begin{aligned} z^* &= \sum_{j=1}^{3} p(S_{1j}) + \sum_{j=1}^{3} p(S_{2j}) + p(S_{31} \cup S_{32}) + p(S_{33}) \\ &\leq p(S_1) + p(S_2) + 2p(S_3). \end{aligned} \tag{10}$$

It is easily verified that $p(S_{21} \cup S_{31}) \le p(S_1)$, $p(S_{22} \cup S_{32}) \le p(S_2)$ and $p(S_{23} \cup S_{33}) \le p(S_2)$, thus we have

$$\begin{aligned} z^* &= \sum_{j=1}^{3} p(S_{1j}) + p(S_{21} \cup S_{31}) + p(S_{22} \cup S_{32}) + p(S_{23} \cup S_{33}) \\ &\le 2p(S_1) + 2p(S_2). \end{aligned} \tag{11}$$

From (10) and (11), we have

$$\begin{aligned} z &= p(S_1) + p(S_2) + p(S_3) \\ &= \frac{1}{2}(p(S_1) + p(S_2) + 2p(S_3)) + \frac{1}{4}(2p(S_1) + 2p(S_2)) \ge \frac{3}{4}z^*. \end{aligned} \tag{12}$$

Otherwise, i.e. $w(S_{3i} \cup S_{3j}) > c_3$ for each $i, j \in \{1, 2, 3\}$ and $i \ne j$, we have $\sum_{i=1}^{2} \sum_{j=2}^{3} w(S_{ij}) < c_1$ as $\sum_{i=1}^{3} \sum_{j=2}^{3} w(S_{ij}) \le c_2 + c_3$, $w(S_{32} \cup S_{33}) > c_3$ and $c_1 = c_2$. According to Lemma 1, we have $\sum_{i=1}^{2} \sum_{j=2}^{3} p(S_{ij}) \le p(S_1)$.

It is easily verified that $\sum_{i=1}^{3} p(S_{i1}) \le p(S_1)$. Noticing (9), we have

$$\begin{aligned} z^* &= \sum_{i=1}^{3} p(S_{i1}) + \sum_{i=1}^{2} \sum_{j=2}^{3} p(S_{ij}) + p(S_{32}) + p(S_{33}) \\ &\le 2p(S_1) + 2p(S_3). \end{aligned} \tag{13}$$

At the same time, we know $\sum_{i=1}^{3} \sum_{j=1}^{3} w(S_{ij}) \le c_1 + c_2 + c_3$ and $w(S_{31} \cup S_{33}) > c_3$, then we have $\sum_{i=1}^{2} \sum_{j=1}^{3} w(S_{ij}) + w(S_{32}) \le c_1 + c_2$. Therefore, at least one of the following two statements holds.

1. $\sum_{j=1}^{3} w(S_{1j}) + w(S_{21}) \le c_1$,
2. $w(S_{22} \cup S_{23} \cup S_{32}) \le c_2$.

If statement 1 is true, we have $\sum_{j=1}^{3} p(S_{1j}) + p(S_{21}) \le p(S_1)$ according to Lemma 1. We know that $p(S_{22} \cup S_{32}) \le p(S_2)$ and $p(S_{23} \cup S_{33}) \le p(S_2)$, together with (9), we obtain that

$$\begin{aligned} z^* &= \sum_{j=1}^{3} p(S_{1j}) + p(S_{21}) + p(S_{22} \cup S_{32}) + p(S_{23} \cup S_{33}) + p(S_{31}) \\ &\le p(S_1) + 2p(S_2) + p(S_3). \end{aligned} \tag{14}$$

If statement 2 is true, we have $p(S_{22} \cup S_{23} \cup S_{32}) \le p(S_2)$ according to Lemma 1. As $w(S_{21} \cup S_{31}) \le c_2$, we have $p(S_{21} \cup S_{31}) \le p(S_2)$. Together with (7) and (9), we obtain that

$$\begin{aligned} z^* &= \sum_{j=1}^{3} p(S_{1j}) + p(S_{22} \cup S_{23} \cup S_{32}) + p(S_{21} \cup S_{31}) + p(S_{33}) \\ &\le p(S_1) + 2p(S_2) + p(S_3), \end{aligned} \tag{15}$$

Table 4 Information of items in Instance 3

Item j	1,2,3,4	5,6,7	8,9,10
w_j	$\frac{1}{4}c_1 - \epsilon$	$\frac{1}{4}c_1$	$\frac{1}{2}c_1 + \epsilon$
p_j	$\frac{1}{4} + 2\epsilon$	$\frac{1}{4} + \epsilon$	$\frac{1}{2}$

with the same bound of (14).

From (13–15), we have

$$\begin{aligned} z &= p(S_1) + p(S_2) + p(S_3) \\ &= \frac{1}{4}(2p(S_1) + 2p(S_3)) + \frac{1}{2}(p(S_1) + 2p(S_2) + p(S_3)) \geq \frac{3}{4}z^*. \end{aligned} \tag{16}$$

From (12) and (16), we know $z \geq \frac{3}{4}z^*$ for all instances if $c_1 = c_2 = c_3$. The instance below will show that the bound is tight.

Instance 3 The information of the items is described in Table 4, in which $\epsilon > 0$ is small enough.

By algorithm H^3, items 1–4 are packed into knapsack 1, items 5–7 are packed into knapsack 2, and one of items 8–10 is packed into knapsack 3, thus the objective value is $z = 9/4 + 11\epsilon$. In an optimal solution, items 1, 5 and 8 can be packed into one knapsack, so do items 2, 6, 9, and items 3, 7, 10. Thus the optimal value is $z^* = 3 + 9\epsilon$. The bound is tight as $z/z^* \to 3/4$ when $\epsilon \to 0$ for the instance. □

3.2.2 Two knapsacks with the same capacity

In this subsection, Theorems 4 and 5 consider the case of $c_1 = c_2 < c_3$, and Theorems 6–8 deal with the case of $c_1 < c_2 = c_3$.

Theorem 4 *The worst-case ratio of H^3 is $5/7$ if $c_1 = c_2 \leq \frac{1}{2}c_3$.*

Proof As $c_1 = c_2 \leq \frac{1}{2}c_3$, we have $w(S_{31} \cup S_{32}) \leq c_3$. With the same proof of getting (10) in Theorem 3, we have

$$z^* \leq p(S_1) + p(S_2) + 2p(S_3). \tag{17}$$

At the same time, noticing $w(S_{31}) \leq c_2$, we have $p(S_{31}) \leq p(S_2)$ according to Lemma 1. Similarly we have $p(S_{32}) \leq p(S_2)$. Together with (7–9), we have

$$\begin{aligned} z^* &= \sum_{j=1}^{3} p(S_{1j}) + \sum_{j=1}^{3} p(S_{2j}) + p(S_{31}) + p(S_{32}) + p(S_{33}) \\ &\leq p(S_1) + 3p(S_2) + p(S_3). \end{aligned} \tag{18}$$

If $w(S_{32} \cup S_{33}) \leq c_3$, we have $p(S_{32} \cup S_{33}) \leq p(S_3)$ according to Lemma 1. It is easy to see that $p(S_{31}) \leq p(S_1)$. Together with (7) and (8), we have

Table 5 Information of items in Instance 4

Item j	1,2	3	4	5,6
w_j	$\frac{1}{2}c_1 - 2\epsilon$	$\frac{1}{2}c_1 - \epsilon$	$c_3 - \frac{1}{2}c_1 + \epsilon$	$\frac{1}{2}c_1 + 2\epsilon$
p_j	$\frac{1}{2} + 2\epsilon$	$\frac{1}{2} + \epsilon$	$1 + \epsilon$	$\frac{1}{2}$

$$\begin{aligned} z^* &= \sum_{j=1}^{3} p(S_{1j}) + \sum_{j=1}^{3} p(S_{2j}) + p(S_{32} \cup S_{33}) + p(S_{31}) \\ &\le 2p(S_1) + p(S_2) + p(S_3). \end{aligned} \tag{19}$$

Otherwise, i.e. $w(S_{32} \cup S_{33}) > c_3$. We know $\sum_{i=1}^{3}\sum_{j=2}^{3} w(S_{ij}) \le c_2 + c_3$ and $c_1 = c_2$, then we have $\sum_{i=1}^{2}\sum_{j=2}^{3} w(S_{ij}) < c_1$. According to Lemma 1, we have $\sum_{i=1}^{2}\sum_{j=2}^{3} p(S_{ij}) < p(S_1)$. It is easy to see that $p(S_{11} \cup S_{21} \cup S_{31}) \le p(S_1)$ and $p(S_{32}) \le p(S_2)$. Together with (9), we also obtain that

$$\begin{aligned} z^* &= \sum_{i=1}^{2}\sum_{j=2}^{3} p(S_{ij}) + p(S_{11} \cup S_{21} \cup S_{31}) + p(S_{32}) + p(S_{33}) \\ &\le 2p(S_1) + p(S_2) + p(S_3). \end{aligned} \tag{20}$$

From (17–20), we have

$$\begin{aligned} z &= p(S_1) + p(S_2) + p(S_3) \\ &= \frac{2}{7}(p(S_1) + p(S_2) + 2p(S_3)) \\ &\quad + \frac{1}{7}(p(S_1) + 3p(S_2) + p(S_3)) + \frac{2}{7}(2p(S_1) + p(S_2) + p(S_3)) \ge \frac{5}{7}z^*. \end{aligned} \tag{21}$$

The instance below will show that the bound is tight.

Instance 4 The information of the items is described in Table 5, in which $\epsilon > 0$ is small enough.

By algorithm H^3, items 1 and 2 are packed into knapsack 1, item 3 is packed into knapsack 2, and item 4 is packed into knapsack 3, thus the objective value is $z = 5/2 + 6\epsilon$. In an optimal solution, items 1 and 5 can be packed into knapsack 1, items 2 and 6 can be packed into knapsack 2, and items 3 and 4 can be packed into knapsack 3. Thus the optimal value $z^* = 7/2 + 6\epsilon$. Therefore the bound is tight as $z/z^* \to 5/7$ when $\epsilon \to 0$ for the instance. □

Theorem 5 *The worst-case ratio of* H^3 *is* $2/3$ *if* $\frac{1}{2}c_3 < c_1 = c_2 < c_3$.

Proof If $w(S_{32} \cup S_{33}) \le c_3$, we have $p(S_{32} \cup S_{33}) \le p(S_3)$ according to Lemma 1. As $w(S_{31}) \le \min\{c_1, c_2, c_3\}$, we have $p(S_{31}) \le \min\{p(S_1), p(S_2), p(S_3)\}$ according

Table 6 Information of items in Instance 5

Item j	1,2	3	4	5,6
w_j	$c_1 - \frac{1}{2}c_3 - \epsilon$	$\frac{1}{2}c_3$	$\frac{1}{2}c_3$	$\frac{1}{2}c_3 + \epsilon$
p_j	$\frac{1}{2} + 2\epsilon$	$\frac{1}{2} + \epsilon$	$\frac{1}{2}$	$\frac{1}{2}$

to Lemma 1. Together with (7) and (8), we have

$$\begin{aligned} z^* &= \sum_{j=1}^{3} p(S_{1j}) + \sum_{j=1}^{3} p(S_{2j}) + p(S_{32} \cup S_{33}) + p(S_{31}) \\ &\le p(S_1) + p(S_2) + p(S_3) + \min\{p(S_1), p(S_2), p(S_3)\}. \end{aligned} \tag{22}$$

Therefore, we have

$$\begin{aligned} z &= p(S_1) + p(S_2) + p(S_3) \\ &= \frac{1}{4}(2p(S_1) + p(S_2) + p(S_3)) + \frac{1}{4}(p(S_1) + 2p(S_2) + p(S_3)) + \frac{1}{4}(p(S_1) \\ &\quad + p(S_2) + 2p(S_3)) \ge \frac{3}{4}z^* > \frac{2}{3}z^*. \end{aligned} \tag{23}$$

Otherwise, i.e. $w(S_{32} \cup S_{33}) > c_3$, with the same proof of getting (13) in Theorem 3, we have

$$z^* \le 2p(S_1) + 2p(S_3). \tag{24}$$

At the same time, we can get (18) in Theorem 4 based on a same proof, thus we have

$$z^* \le p(S_1) + 3p(S_2) + p(S_3). \tag{25}$$

From (24) and (25), we obtain that

$$\begin{aligned} z &= p(S_1) + p(S_2) + p(S_3) \\ &= \frac{1}{3}(2p(S_1) + 2p(S_3)) + \frac{1}{3}(p(S_1) + 3p(S_2) + p(S_3)) \ge \frac{2}{3}z^*. \end{aligned} \tag{26}$$

From (23) and (26), we know $z \ge \frac{2}{3}z^*$ for all instances if $\frac{1}{2}c_3 < c_1 = c_2 < c_3$. The instance below will show that the bound is tight.

Instance 5 The information of the items is described in Table 6, in which $\epsilon > 0$ is small enough.

By algorithm H^3, items 1 and 2 are packed into knapsack 1, item 3 is packed into knapsack 2, and one of items 4, 5 and 6 is packed into knapsack 3, thus the objective value $z = 2 + 5\epsilon$. In an optimal solution, items 1 and 5 can be packed into knapsack 1, items 2 and 6 can be packed into knapsack 2, and items 3 and 4 can be packed into knapsack 3. Thus the optimal value $z^* = 3 + 5\epsilon$. Therefore the bound is tight as $z/z^* \to 2/3$ when $\epsilon \to 0$ for the instance. □

Theorem 6 *The worst-case ratio of H^3 is $3/4$ if $3c_1 \le c_2 = c_3$.*

Proof First, with the same proof of getting (11) in Theorem 3, we have

$$z^* \le 2p(S_1) + 2p(S_2). \tag{27}$$

Next, we will prove that

$$z^* \le p(S_1) + p(S_2) + 2p(S_3). \tag{28}$$

We can verify that (28) will be true if one of the following statements (29–32) does not hold.

$$\sum_{j=1}^{3} w(S_{1j}) + w(S_{31}) > c_1, \tag{29}$$

$$\sum_{j=1}^{3} w(S_{2j}) + w(S_{31}) > c_2, \tag{30}$$

$$w(S_{31}) + w(S_{32}) > c_2, \tag{31}$$

$$w(S_{31}) + w(S_{33}) > c_2. \tag{32}$$

For example, if (29) does not hold, we have $\sum_{j=1}^{3} w(S_{1j}) + w(S_{31}) \le c_1$, and then $\sum_{j=1}^{3} p(S_{1j}) + p(S_{31}) \le p(S_1)$ according to Lemma 1. Together with (8) and (9), we have

$$\begin{aligned} z^* &= \sum_{j=1}^{3} p(S_{1j}) + p(S_{31}) + \sum_{j=1}^{3} p(S_{2j}) + p(S_{32}) + p(S_{33}) \\ &\le p(S_1) + p(S_2) + 2p(S_3). \end{aligned} \tag{33}$$

Similarly, we can get the same result for the other cases. Assume that (29–32) all hold. As $w(S_{31}) \le c_1$, then from (29) and (30), we have

$$\sum_{j=1}^{3} w(S_{1j}) + \sum_{j=1}^{3} w(S_{2j}) + w(S_{31}) > c_2, \tag{34}$$

and from (31) and (32), we have

$$w(S_{32}) + w(S_{33}) > 2c_2 - 2c_1. \tag{35}$$

We know that

$$\sum_{i=1}^{3} \sum_{j=1}^{3} w(S_{ij}) \le c_1 + 2c_2. \tag{36}$$

Table 7 Information of items in Instance 6

Item j	1	2	3	4
w_j	$\frac{1}{2}c_1$	$c_2 - \frac{1}{2}c_1$	c_1	c_2
p_j	$1+2\epsilon$	$1+\epsilon$	1	1

From (34), (35) and (36), we have $3c_1 > c_2$, which is conflict with the condition of $3c_1 \le c_2$, thus we know (28) must hold.

From (27) and (28), we obtain that

$$\begin{aligned} z &= p(S_1) + p(S_2) + p(S_3) \\ &= \frac{1}{4}(2p(S_1) + 2p(S_2)) + \frac{1}{2}(p(S_1) + p(S_2) + 2p(S_3)) \ge \frac{3}{4}z^*. \end{aligned} \tag{37}$$

The instance below will show that the bound is tight.

Instance 6 The information of the items is described in Table 7, in which $\epsilon > 0$ is small enough.

By algorithm H^3, item 1 is packed into knapsack 1, item 2 is packed into knapsack 2, and one of items 3 and 4 is packed into knapsack 3, thus the objective value $z = 3 + 3\epsilon$. In an optimal solution, item 3 can be packed into knapsack 1, items 1 and 2 can be packed into knapsack 2, and item 4 can be packed into knapsack 3. Thus the optimal value $z^* = 4 + 3\epsilon$. Therefore the bound is tight as $z/z^* \to 3/4$ when $\epsilon \to 0$ for the instance. □

Theorem 7 *The worst-case ratio of H^3 is $7/10$ if $\frac{4}{3}c_1 < c_2 = c_3 < 3c_1$.*

Proof If there are $i, j \in \{1, 2, 3\}$ and $i \neq j$ such that $w(S_{3i} \cup S_{3j}) \le c_3$, with the same proof of getting (12) in Theorem 3, we have

$$z \ge \frac{3}{4}z^* > \frac{7}{10}z^*. \tag{38}$$

Otherwise, we have $w(S_{3i} \cup S_{3j}) > c_3$ for each $i, j \in \{1, 2, 3\}$ and $i \neq j$. We know that at least one of $p(S_{12})$ and $p(S_{13})$ is no more than $\frac{1}{2}p(S_1)$ as $\sum_{j=1}^{3} p(S_{1j}) \le p(S_1)$. Without loss of generality, we assume that $p(S_{13}) \le \frac{1}{2}p(S_1)$. We can verify that $\sum_{i=1}^{2}\sum_{j=1}^{2} w(S_{ij}) \le c_1$ as $w(S_{31} \cup S_{32}) > c_3$, $\sum_{i=1}^{3}\sum_{j=1}^{2} w(S_{ij}) \le c_1 + c_2$ and $c_2 = c_3$. Therefore, we have $\sum_{i=1}^{2}\sum_{j=1}^{2} p(S_{ij}) \le p(S_1)$ according to Lemma 1. Together with (8) and (9), we have

$$\begin{aligned} z^* &= \sum_{i=1}^{2}\sum_{j=1}^{2} p(S_{ij}) + p(S_{13}) + p(S_{23} \cup S_{33}) + p(S_{31}) + p(S_{32}) \\ &\le \frac{3}{2}p(S_1) + p(S_2) + 2p(S_3). \end{aligned} \tag{39}$$

Table 8 Information of items in Instance 7

Item j	1,2	3	4,5	6	7,8
w_j	$\frac{1}{2}(3c_1 - c_2) - \epsilon$	$c_1 - \epsilon$	$\frac{1}{2}(c_2 - c_1) + \frac{1}{2}\epsilon$	c_1	$c_2 - c_1 + \frac{1}{2}\epsilon$
p_j	$\frac{1}{2} + 2\epsilon$	$\frac{1}{2}$	$\frac{1}{2} + \epsilon$	1	1

Table 9 Information of items in Instance 8

Item j	1,2	3,4,5	6	7,8
w_j	$\frac{1}{4}c_2 - 2\epsilon$	$\frac{1}{4}c_2 + \epsilon$	c_1	$\frac{1}{2}c_2 + \epsilon$
p_j	$\frac{1}{2} + 2\epsilon$	$\frac{1}{2} + \epsilon$	1	1

With the same proof of getting (11), (14) and (15) in Theorem 3, we have

$$z^* \leq 2p(S_1) + 2p(S_2). \tag{40}$$

$$z^* \leq p(S_1) + 2p(S_2) + p(S_3). \tag{41}$$

From (39), (40) and (41), we have

$$\begin{aligned} z &= p(S_1) + p(S_2) + p(S_3) \\ &= \frac{4}{10}\left(\frac{3}{2}p(S_1) + p(S_2) + 2p(S_3)\right) \\ &\quad + \frac{1}{10}(2p(S_1) + 2p(S_2)) + \frac{2}{10}(p(S_1) + 2p(S_2) + p(S_3)) \geq \frac{7}{10}z^*. \end{aligned} \tag{42}$$

From (38) and (42), we know $z \geq \frac{7}{10}z^*$ for all instances if $\frac{4}{3}c_1 < c_2 = c_3 < 3c_1$. The two instances below will show that the bound is tight.

Instance 7 This instance considers the case of $2c_1 \leq c_2 = c_3 < 3c_1$. The information of the items is described in Table 8, in which $\epsilon > 0$ is small enough.

By algorithm H^3, items 1 and 2 are packed into knapsack 1, items 3, 4 and 5 are packed into knapsack 2, and one of items 6, 7 and 8 is packed into knapsack 3, thus the objective value $z = 7/2 + 6\epsilon$. In an optimal solution, item 6 can be packed into knapsack 1, items 1, 4 and 7 can be packed into knapsack 2, and items 2, 5 and 8 can be packed into knapsack 3. Thus the optimal value $z^* = 5 + 6\epsilon$. Therefore the bound is tight as $z/z^* \to 7/10$ when $\epsilon \to 0$ for the instance.

Instance 8 This instance considers the case of $\frac{4}{3}c_1 < c_2 = c_3 < 2c_1$. The information of the items is described in Table 9, in which $\epsilon > 0$ is small enough.

By algorithm H^3, items 1 and 2 are packed into knapsack 1, items 3, 4 and 5 are packed into knapsack 2, and one of items 6, 7 and 8 is packed into knapsack 3, thus the objective value $z = 7/2 + 7\epsilon$. In an optimal solution, item 6 can be packed into knapsack 1, items 1, 4 and 7 can be packed into knapsack 2, and items 2, 5 and 8 can be packed into knapsack 3. Thus the optimal value $z^* = 5 + 6\epsilon$. Therefore the bound is tight as $z/z^* \to 7/10$ when $\epsilon \to 0$ for the instance. □

Theorem 8 *The worst-case ratio of* H^3 *is* $8/11$ *if* $c_1 < c_2 = c_3 \leq \frac{4}{3}c_1$.

Proof If there are $i, j \in \{1, 2, 3\}$ and $i \neq j$ such that $w(S_{3i} \cup S_{3j}) \leq c_3$, with the same proof of getting (12) in Theorem 3, we have

$$z \geq \frac{3}{4}z^* > \frac{8}{11}z^*. \tag{43}$$

Otherwise, we have $w(S_{3i} \cup S_{3j}) > c_3$ for each $i, j \in \{1, 2, 3\}$ and $i \neq j$. We can get (14) and (15) in Theorem 3 based on a same proof, thus we have

$$z^* \leq p(S_1) + 2p(S_2) + p(S_3). \tag{44}$$

If $w(S_{12} \cup S_{13} \cup S_{22}) \leq c_1$ or $w(S_{12} \cup S_{13} \cup S_{23}) \leq c_1$, without loss of generality, we assume that $w(S_{12} \cup S_{13} \cup S_{22}) \leq c_1$, then $p(S_{12} \cup S_{13} \cup S_{22}) \leq p(S_1)$ according to Lemma 1. It is easily verified that $\sum_{i=1}^{3} p(S_{i1}) \leq p(S_1)$. Noticing $c_2 = c_3$, together with (8) and (9), we have

$$\begin{aligned} z^* &= \sum_{i=1}^{3} p(S_{i1}) + p(S_{12} \cup S_{13} \cup S_{22}) + p(S_{23} \cup S_{33}) + p(S_{32}) \\ &\leq 2p(S_1) + p(S_2) + p(S_3). \end{aligned} \tag{45}$$

Let $p(S_{11} \cup S_{21}) = \lambda p(S_1)$, then we know $0 \leq \lambda \leq 1$, and obtain that

$$\begin{aligned} z^* &= p(S_{11} \cup S_{21}) + p(S_{12} \cup S_{13} \cup S_{22}) + p(S_{23} \cup S_{33}) + p(S_{31}) + p(S_{32}) \\ &\leq (1 + \lambda)p(S_1) + p(S_2) + 2p(S_3). \end{aligned} \tag{46}$$

As $w(S_{31} \cup S_{32}) > c_3$, $\sum_{i=1}^{3} \sum_{j=1}^{2} w(S_{ij}) \leq c_1 + c_2$ and $c_2 = c_3$, we have $\sum_{i=1}^{2} \sum_{j=1}^{2} w(S_{ij}) < c_1$. According to Lemma 1, we obtain that $\sum_{i=1}^{2} \sum_{j=1}^{2} p(S_{ij}) \leq p(S_1)$, then $p(S_{12} \cup S_{22}) \leq (1 - \lambda)p(S_1)$. Similarly, we have $p(S_{13} \cup S_{23}) \leq (1 - \lambda)p(S_1)$. Together with (8) and (9), we have

$$\begin{aligned} z^* &= p(S_{11} \cup S_{21}) + p(S_{12} \cup S_{22}) + p(S_{13} \cup S_{23}) + \sum_{j=1}^{3} p(S_{3j}) \\ &\leq (2 - \lambda)p(S_1) + 3p(S_3). \end{aligned} \tag{47}$$

Combining (46) and (47) to eliminate λ, we have

$$z^* \leq \frac{3}{2}p(S_1) + \frac{1}{2}p(S_2) + \frac{5}{2}p(S_3). \tag{48}$$

From (44), (45) and (48), we have

$$
\begin{aligned}
z &= p(S_1) + p(S_2) + p(S_3) \\
&= \frac{4}{11}(p(S_1) + 2p(S_2) + p(S_3)) \\
&\quad + \frac{2}{11}(2p(S_1) + p(S_2) + p(S_3)) + \frac{2}{11}\left(\frac{3}{2}p(S_1) + \frac{1}{2}p(S_2) + \frac{5}{2}p(S_3)\right) \\
&\geq \frac{8}{11}z^*.
\end{aligned}
\tag{49}
$$

Otherwise, i.e. $w(S_{12} \cup S_{13} \cup S_{22}) > c_1$ and $w(S_{12} \cup S_{13} \cup S_{23}) > c_1$, we can get that $w(S_{22}) < c_2 - c_1$, $w(S_{23}) < c_2 - c_1$ as $\sum_{i=1}^{3}\sum_{j=2}^{3} w(S_{ij}) \leq c_2 + c_3$ and $w(S_{32} \cup S_{33}) > c_3$. Thus we have $w(S_{12} \cup S_{13}) > 2c_1 - c_2$, then $\sum_{i=2}^{3}\sum_{j=2}^{3} w(S_{ij}) < 3c_2 - 2c_1 \leq 2c_1$ as $c_1 < c_2 = c_3 \leq \frac{4}{3}c_1$. Therefore, we can claim that at least one of the two cases of $w(S_{22} \cup S_{32}) \leq c_1$ and $w(S_{23} \cup S_{33}) \leq c_1$ holds. Without loss of generality, we assume $w(S_{22} \cup S_{32}) \leq c_1$. According to Lemma 1, we have $p(S_{22} \cup S_{32}) \leq p(S_1)$.

With a similar proof of getting (14) and (15) in Theorem 3, and noticing that $p(S_{22} \cup S_{32}) \leq p(S_1)$ in (14) and $p(S_{21} \cup S_{31}) \leq p(S_1)$ in (15), we have

$$
z^* \leq 2p(S_1) + p(S_2) + p(S_3). \tag{50}
$$

At the same time, we can verify that $w(S_{12}) + w(S_{13}) \geq w(S_{22}) + w(S_{23})$ as $c_1 < c_2 = c_3 \leq \frac{4}{3}c_1$ and $w(S_{32} \cup S_{33}) > c_3$, thus at least one of the two cases of $w(S_{12}) \geq w(S_{23})$ and $w(S_{13}) \geq w(S_{22})$ holds. Without loss of generality, we assume $w(S_{12}) \geq w(S_{23})$, then $w(S_{22} \cup S_{23} \cup S_{32}) \leq c_2$. According to Lemma 1, we have $p(S_{22} \cup S_{23} \cup S_{32}) \leq p(S_2)$.

We consider $1/3$ as a threshold of λ, which is defined before. If $\lambda \leq 1/3$, together with (7) and (9), we can get

$$
\begin{aligned}
z^* &= p(S_{11} \cup S_{21}) + p(S_{12} \cup S_{13}) + p(S_{22} \cup S_{23} \cup S_{32}) + p(S_{31}) + p(S_{33}) \\
&\leq \frac{4}{3}p(S_1) + p(S_2) + 2p(S_3).
\end{aligned}
\tag{51}
$$

From (44), (50) and (51), we have

$$
\begin{aligned}
z &= p(S_1) + p(S_2) + p(S_3) \\
&= \frac{3}{11}(p(S_1) + 2p(S_2) + p(S_3)) + \frac{2}{11}(2p(S_1) + p(S_2) + p(S_3)) \\
&\quad + \frac{3}{11}\left(\frac{4}{3}p(S_1) + p(S_2) + 2p(S_3)\right) \geq \frac{8}{11}z^*.
\end{aligned}
\tag{52}
$$

If $\lambda > 1/3$, we find that (47) is true as well, then we have

$$
z^* \leq \frac{5}{3}p(S_1) + 3p(S_3). \tag{53}
$$

Table 10 Information of items in Instance 9

Item j	1,2,3	4	5,6	7	8,9
w_j	$\frac{1}{4}c_2 - 2\epsilon$	$\frac{1}{4}c_2$	$\frac{1}{4}c_2 + \epsilon$	$\frac{1}{2}c_2$	$\frac{1}{2}c_2 + \epsilon$
p_j	$\frac{1}{3} + 2\epsilon$	$\frac{1}{3} + \epsilon$	$\frac{1}{3} + \epsilon$	$\frac{2}{3}$	$\frac{2}{3}$

From (44), (50) and (53), we have

$$\begin{aligned} z &= p(S_1) + p(S_2) + p(S_3) \\ &= \frac{9}{22}(p(S_1) + 2p(S_2) + p(S_3)) \\ &\quad + \frac{4}{22}(2p(S_1) + p(S_2) + p(S_3)) + \frac{3}{22}\left(\frac{5}{3}p(S_1) + 3p(S_3)\right) \geq \frac{8}{11}z^*. \end{aligned} \tag{54}$$

From (43), (49), (52) and (54), we know $z \geq \frac{8}{11}z^*$ for all instances if $c_1 < c_2 = c_3 \leq \frac{4}{3}c_1$. The instance below will show that the bound is tight.

Instance 9 The information of the items is described in Table 10, in which $\epsilon > 0$ is small enough.

By algorithm H^3, items 1, 2 and 3 are packed into knapsack 1, items 4, 5 and 6 are packed into knapsack 2, and one of items 7, 8 and 9 is packed into knapsack 3. The objective value is $z = 8/3 + 9\epsilon$. In an optimal solution, items 4 and 7 can be packed into knapsack 1, items 1, 5 and 8 can be packed into knapsack 2, and items 2, 6 and 9 can be packed into knapsack 3. Thus the optimal value is $z^* = 11/3 + 7\epsilon$. Therefore the bound is tight as $z/z^* \to 8/11$ when $\epsilon \to 0$ for the instance. □

3.2.3 *Knapsacks with different capacity*

In this subsection, we will consider the case that all knapsacks with different capacity, i.e. $c_1 < c_2 < c_3$, and present three theorems covered all possible cases.

Theorem 9 *The worst-case ratio of H^3 is $2/3$ if $c_1 + c_2 \leq c_3$ and $c_1 < c_2$.*

Proof As $c_1 + c_2 \leq c_3$, we have $w(S_{31} \cup S_{32}) \leq c_3$. With the same proof of getting (10) in Theorem 3, we have

$$z^* \leq p(S_1) + p(S_2) + 2p(S_3). \tag{55}$$

At the same time, according to (7–9), we have

$$\begin{aligned} z^* &= \sum_{i=1}^{3} p(S_{i1}) + p(S_{12} \cup S_{13}) + p(S_{22} \cup S_{32}) + p(S_{23}) + p(S_{33}) \\ &\leq 2p(S_1) + 2p(S_2) + p(S_3). \end{aligned} \tag{56}$$

Table 11 Information of items in Instance 10

Item j	1	2	3
w_j	$c_1+\epsilon$	c_2	$c_3-c_1-\epsilon$
p_j	$1+\epsilon$	1	1

From (55) and (56), we have

$$z = p(S_1)+p(S_2)+p(S_3)$$
$$= \frac{1}{3}(p(S_1)+p(S_2)+2p(S_3)) + \frac{1}{3}(2p(S_1)+2p(S_2)+p(S_3)) \geq \frac{2}{3}z^*. \quad (57)$$

The instance below will show that the bound is tight.

Instance 10 The information of the items is described in Table 11, in which $\epsilon > 0$ is small enough.

We notice that none of these items can be packed into knapsack 1. By algorithm H^3, item 1 is packed into knapsack 2, and one of items 2 and 3 is packed into knapsack 3, thus the objective value $z = 2+\epsilon$. In an optimal solution, item 2 can be packed into knapsack 2, and items 1 and 3 can be packed into knapsack 3. Thus the optimal value $z^* = 3+\epsilon$. Therefore the bound is tight as $z/z^* \to 2/3$ when $\epsilon \to 0$ for the instance. □

Theorem 10 *The worst-case ratio of H^3 is $2/3$ if $2c_1 \leq c_2 < c_3$.*

Proof If $w(S_{31} \cup S_{33}) \leq c_3$, with a similar proof of getting (10) in Theorem 3, we have

$$z^* \leq p(S_1)+p(S_2)+2p(S_3). \quad (58)$$

Notice that (56) also holds, then we have

$$z = p(S_1)+p(S_2)+p(S_3)$$
$$= \frac{1}{3}(p(S_1)+p(S_2)+2p(S_3)) + \frac{1}{3}(2p(S_1)+2p(S_2)+p(S_3)) \geq \frac{2}{3}z^*. \quad (59)$$

Otherwise, i.e. $w(S_{31} \cup S_{33}) > c_3$, we have $w(S_{11} \cup S_{21} \cup S_{13} \cup S_{23}) < c_1$ as $\sum_{i=1}^{3} w(S_{i1}) + \sum_{i=1}^{3} w(S_{i3}) \leq c_1 + c_3$. Therefore, $w(S_{21} \cup S_{23} \cup S_{31}) < 2c_1 \leq c_2$ as $w(S_{31}) \leq c_1$ and $2c_1 \leq c_2$. According to Lemma 1, we have $p(S_{21} \cup S_{23} \cup S_{31}) \leq p(S_2)$. Together with (7–9), we have

$$z^* = \sum_{j=1}^{3} p(S_{1j}) + p(S_{21} \cup S_{23} \cup S_{31}) + p(S_{22} \cup S_{32}) + p(S_{33})$$
$$\leq p(S_1)+2p(S_2)+p(S_3). \quad (60)$$

At the same time, considering (7–9), we also have

$$z^* = \sum_{j=1}^{3} p(S_{1j}) + p(S_{21} \cup S_{31}) + p(S_{22} \cup S_{23}) + p(S_{32}) + p(S_{33}) \\ \leq 2p(S_1) + p(S_2) + 2p(S_3). \quad (61)$$

From (60) and (61), we have

$$z = p(S_1) + p(S_2) + p(S_3) \\ = \frac{1}{3}(p(S_1) + 2p(S_2) + p(S_3)) + \frac{1}{3}(2p(S_1) + p(S_2) + 2p(S_3)) \geq \frac{2}{3}z^*. \quad (62)$$

From (59) and (62), we know $z \geq \frac{2}{3}z^*$ for all instances if $2c_1 \leq c_2 < c_3$. Considering Instance 10, we can find that the objective value $z = 2 + \epsilon$ and the optimal value $z^* = 3 + \epsilon$. Therefore the bound is tight as $z/z^* \to 2/3$ when $\epsilon \to 0$ for the instance. □

Theorem 11 *The worst-case ratio of H^3 is $3/5$ if $c_1 + c_2 > c_3$, $c_1 < c_2 < 2c_1$ and $c_2 < c_3$.*

Proof This theorem covers all possible cases which are not considered before. With the same proof of getting (18) in Theorem 4, we have

$$z^* \leq p(S_1) + 3p(S_2) + p(S_3). \quad (63)$$

At the same time, we can get (61) in Theorem 10 based on a same proof, thus we have

$$z^* \leq 2p(S_1) + p(S_2) + 2p(S_3). \quad (64)$$

From (63) and (64), we have

$$z = p(S_1) + p(S_2) + p(S_3) \\ = \frac{1}{5}(p(S_1) + 3p(S_2) + p(S_3)) + \frac{2}{5}(2p(S_1) + p(S_2) + 2p(S_3)) \geq \frac{3}{5}z^*. \quad (65)$$

The two instances below will show that the bound is tight.

Instance 11 This instance considers the case of $c_2 \geq \frac{2}{3}c_3$ under the conditions of this theorem. The information of the items are described in Table 12, in which $\epsilon > 0$ is small enough.

By algorithm H^3, item 1 is packed into knapsack 1, item 2 is packed into knapsack 2, and one of items 3, 4 and 5 is packed into knapsack 3, thus the objective value $z = 3 + 3\epsilon$. In an optimal solution, item 3 can be packed into knapsack 1, items 1 and 4 can be packed into knapsack 2, and items 2 and 5 can be packed into knapsack 3. Thus the optimal value $z^* = 5 + 3\epsilon$. Therefore the bound is tight as $z/z^* \to 3/5$ when $\epsilon \to 0$ for the instance.

Table 12 Information of items in Instance 11

Item j	1	2	3	4	5
w_j	$c_1 - \frac{1}{2}c_2 - \epsilon$	$\frac{1}{2}c_2 + 2\epsilon$	c_1	$\frac{3}{2}c_2 - c_1 + \epsilon$	$c_3 - \frac{1}{2}c_2 - 2\epsilon$
p_j	$1 + 2\epsilon$	$1 + \epsilon$	1	1	1

Table 13 Information of items in Instance 12

Item j	1	2	3	4	5
w_j	$c_1 + c_2 - c_3 - \epsilon$	$c_3 - c_2 + 2\epsilon$	c_1	$c_3 - c_1 + \epsilon$	$c_2 - 2\epsilon$
p_j	$1 + 2\epsilon$	$1 + \epsilon$	1	1	1

Instance 12 This instance considers the case of $c_2 < \frac{2}{3}c_3$ under the conditions of this theorem. The information of the items are described in Table 13, in which $\epsilon > 0$ is small enough.

By algorithm H^3, item 1 is packed into knapsack 1, item 2 is packed into knapsack 2, and one of items 3, 4 and 5 is packed into knapsack 3, thus the objective value $z = 3 + 3\epsilon$. In an optimal solution, item 3 can be packed into knapsack 1, items 1 and 4 can be packed into knapsack 2, and items 2 and 5 can be packed into knapsack 3. Thus the optimal value $z^* = 5 + 3\epsilon$. Therefore the bound is tight as $z/z^* \to 3/5$ when $\epsilon \to 0$ for the instance. □

4 Conclusion and extension

This paper investigates a natural approximation algorithm for the multiple knapsack problem with two or three knapsacks. This algorithm packs the knapsacks one by one in the nondecreasing order of their capacities successively. In Sect. 3, we get the worst-case ratios of this algorithm if an exact algorithm for KP is applied in the procedure.

If a $(1-\epsilon)$-approximation algorithm for KP is applied for each knapsack, we have the following lemma.

Lemma 2 *Let S be a set of items. If $w(S) \le c_i$ and $S \cap S_j = \emptyset$ for each $1 \le j \le i-1$, then $(1-\epsilon)p(S) \le p(S_i)$.*

Consequently, following the same framework and the similar proofs in Sect. 3, we can get the theorem below. For simplicity, we omit the detailed proof.

Theorem 12 *If the worst-case ratio of $H^m (m = 2, 3)$ is ρ and H_ϵ is a $(1-\epsilon)$-approximation algorithm for KP, then $H_\epsilon^m (m = 2, 3)$ is a $\rho(1-\epsilon)$-approximation algorithm for MKP.*

Combining this theorem and the worst-case ratios of H^m gotten in Sect. 3, we can get the error bounds of H_ϵ^m for the cases of $m = 2$ and $m = 3$.

An interesting problem is what are the error bounds of the successive approximation algorithm when there are four or more knapsacks. It needs further research.

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References

Caprara A, Pferschy U (2004) Worst-case analysis of the subset sum algorithm for bin packing. Oper Res Lett 32:159–166

Caprara A, Kellerer H, Pferschy U (2000a) The multiple subset sum problem. SIAM J Optim 6:308–319

Caprara A, Kellerer H, Pferschy U (2000b) A PTAS for the multiple subset sum problem with different knapsack capacities. Inf Process Lett 73:111–118

Caprara A, Kellerer H, Pferschy U (2003) A 3/4-approximation algorithm for multiple subset sum. J Heuristics 9:99–111

Chekuri C, Khanna S (2005) A polynomial time approximation scheme for the multiple knapsack problem. SIAM J Comput 35:713–728

Garey MR, Johnson DS (1979) Computers and intractability: a guide to the theory of NP-completeness. Freeman, San Francisco

Horowitz E, Sahni S (1974) Computing partitions with applications to the knapsack problem. J ACM 21:277–292

Ibarra OH, Kim CE (1975) Fast approximation algorithms for the knapsack and sum of subset problem. J ACM 22:463–468

Kellerer H, Pferschy U (1999) A new fully polynomial time approximation scheme for the knapsack problem. J Comb Optim 3:59–71

Kellerer H, Pferschy U, Pisinger D (2004) Knapsack problems. Springer, Berlin

Lawler EL (1979) Fast approximation algorithms for knapsack problems. Math Oper Res 4:339–356

Martello S, Toth P (1977) An upper bound for the zero-one knapsack problem and a branch and bound algorithm. Eur J Oper Res 1:169–175

Martello S, Toth P (1981) A branch and bound algorithm for the zero-one multiple knapsack problem. Discrete Appl Math 3:275–288

Martello S, Pisinger D, Toth P (2000) New trends in exact algorithms for the 0-1 knapsack problem. Eur J Oper Res 123:325–332

Nauss RM (1976) An efficient algorithm for the 0-1 knapsack problem. Manag Sci 23:27–31

Pisinger D (1999) An exact algorithm for the large multiple knapsack problems. Eur J Oper Res 114:528–541

Pisinger D, Toth P (1998) Knapsack problems. In: Handbook of combinatorial optimization, vol 1. Kluwer Academic, Dordrecht, pp 299–428