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Dimensional entropy over sets and fibres

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Abstract
This paper is devoted to a study of dimensional entropy, especially entropy on the fibres of factor maps. We show that dimensional entropy and topological entropy of sets are not usually equal, while dimensional entropy over fibres always corresponds to entropy of the factor. Then we provide an estimate of the dimensional entropy of the image of a set under a factor map. Finally we solve a few questions stated recently by Dai and Jiang on distance entropy (which is a modification of dimensional entropy).

Mathematics Subject Classification: 37B40

1. Introduction

One of the main aims of the qualitative theory of dynamical systems is a description of the complexity of dynamics. A tool commonly used for this purpose is the so-called entropy theory, which is an important ingredient of topological dynamics, ergodic theory and many other fields related to modern theory of dynamical systems (e.g. see the survey paper by Katok [17], the book by Pesin [23] or forthcoming book by Downarowicz [10]).

In his fundamental paper of 1958, Kolmogorov introduced the measure-theoretic entropy in ergodic theory [18], and next in 1965 Adler, Konheim and McAndrew introduced the concept of topological entropy in the context of compact Hausdorff space equipped with a continuous self-map [1]. A classical result, connecting these two quantities is the variational principle [8, 12, 13, 21] which, generally speaking, says that for every continuous map on

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a compact metric space, the topological entropy equals the supremum of measure-theoretic entropies, where the supremum is taken over all invariant Borel probability measures for the considered system. Meanwhile, in [4] using separated and spanning sets, Bowen obtained another definition of entropy depending on the metric (called the metric entropy in that paper) and proved that in the context of compact metric space it coincides with the topological entropy of [1]. He also established a theorem which can be used to estimate entropy of factors in terms of entropy of fibres of the factor map [4, theorem 17]. Two years later, in 1973, motivated by the definition of the Hausdorff dimension, Bowen introduced the so-called dimensional entropy of a subset [5]. Even the single problem of relations between entropy and the Hausdorff dimension has received lots of attention in research papers (e.g. see [2, 22]) and while big progress has been made, still many questions remain open (e.g. see monograph [23] by Pesin).

The definition we will be particularly interested in for this paper is dimensional entropy, denoted h^B throughout (as we said before, introduced in [5]). The main reason is that dimensional entropy seems to be a more sensitive tool when calculating entropy capacity of subsets. For example, dimensional entropy of countable set is always zero, while topological entropy of such a set can be equal to the entropy of the whole system. In fact, it was proved in [26] that for each system there exists a countable closed subset with its topological entropy equal to that of the whole system.

In this paper we will mainly focus on dimensional entropy of subsets and their transformations by factor maps. It is well known that for continuous transformations on compact metric spaces, dimensional entropy of the whole space coincides with topological entropy of the system [5], while, as shown in section 3, it is not the case when we consider that of subsets. Strictly speaking, it may happen that dimensional entropy is strictly smaller than topological entropy for quite a large family of sets of the system (e.g. a dense subset of the hyperspace). For this reason, it seems rather hard in general to deduce information about the value of dimensional entropy when the value of topological entropy of this subset is known. Therefore, one of our main aims is to obtain estimates similar to [4, theorem 17], that is, we could estimate dimensional entropy of factors in terms of dimensional entropy of fibres of the factor map. Generally speaking, if we replace h_{top} (i.e. topological entropy) by h^B then still inequality from [4, theorem 17] is valid, that is

$$h^B(S, \pi(K)) \leqslant h^B(T, K) \leqslant h^B(S, \pi(K)) + h_{\mathrm{top}}(T|\pi). \tag{1.1}$$

We emphasize once again the fact that while these formulae look similar, values of h^B and h_{top} can be different, and so we also need quite a different approach to prove this new inequality. In the process of proving the above result, we also find that the supremum of dimensional entropy of fibres is the same as that of topological entropy of fibres.

The paper is organized as follows. In the next section we recall definitions of most of the concepts used in the latter parts of the paper. In section 3 we show that in general there is no relation between topological and dimensional entropy of subsets. Sections 4 and 5 aim to prove the theorem estimating dimensional entropy of factors in terms of dimensional entropy of fibres of the factor map.

While we are mostly interested in the context of (compact) metric spaces, similar to [5], we present our results using the weakest possible assumptions (e.g. consider also non-metrizable topological spaces). Most of our arguments, especially these leading to (1.1), are purely topological, i.e. we do not use properties of invariant measures and other tools from ergodic theory. Later, i.e. when (1.1) is established, we show how some tools from ergodic theory can be used to strengthen (1.1). This is done in section 6.

We finish the paper with the appendix, where we answer some questions on so-called distance entropy (an extension of dimensional entropy to non-compact setting) introduced in [7].

2. Preliminaries

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{R}$ the set of all positive integers, non-negative integers, integers and real numbers, respectively.

By a *dynamical system*, denoted by (X, T), we mean a topological space X equipped with a continuous self-map $T : X \to X$. If additionally T is a homeomorphism, we say that (X, T) is an *invertible dynamical system*. Given dynamical systems (X, T) and (Y, S), we say that $\pi : (X, T) \to (Y, S)$ is a *factor map between* (X, T) *and* (Y, S) if $\pi : X \to Y$ is a continuous surjection and $\pi \circ T = S \circ \pi$. In the above situation, we say that (Y, S) is a *factor* of (X, T), or that (X, T) is an *extension* of (Y, S).

Let (X, T) be a dynamical system, $K \subseteq X$ and let $\mathscr{W}$ be a family of subsets of X. We write $K \prec \mathscr{W}$ if there is $W \in \mathscr{W}$ such that $K \subseteq W$. If the condition $K \prec \mathscr{W}$ does not hold, we denote this fact by writing $K \not\prec \mathscr{W}$. If $\mathscr{U}, \mathscr{V}$ are two families of subsets of X, we say that $\mathscr{V}$ is *finer* than $\mathscr{U}$, denoted by $\mathscr{V} \prec \mathscr{U}$, if $V \prec \mathscr{U}$ for each $V \in \mathscr{V}$. We also simply write $\cup\mathscr{V}$ to denote the sum $\bigcup_{V\in\mathscr{V}} V \subset X$ and denote mesh $\mathscr{V} = \sup_{V\in\mathscr{V}}$ diam V.

By a *cover* of X we mean a finite family of Borel subsets with union X and a *partition* of X is any cover of X consisting of disjoint sets. If all elements of a cover $\mathscr{U}$ are open sets we say that $\mathscr{U}$ is an *open cover*. Denote by $\mathcal{C}_X$, $\mathcal{C}_X^o$ and $\mathcal{P}_X$, respectively, the set of covers, open covers and partitions of X. We emphasize here the fact that according to the introduced definition, *cover* always means a *finite cover*. While it is not that important on compact spaces (there is always a finite subcover), in a non-compact setting it becomes a necessity as otherwise the definition of topological entropy would have no sense in this setting.

Given two covers $\mathscr{U}, \mathscr{V} \in \mathcal{C}_X$ we define the cover $\mathscr{U} \vee \mathscr{V} \in \mathcal{C}_X$ by

$$\mathscr{U} \vee \mathscr{V} = \{U \cap V : U \in \mathscr{U}, V \in \mathscr{V}\}.$$

Note that if $\mathscr{U}, \mathscr{V} \in \mathcal{C}_X^o$ then $\mathscr{U} \vee \mathscr{V} \in \mathcal{C}_X^o$ and the same is true for partitions.

For each $\mathscr{U} \in \mathcal{C}_X$ and $n \in \mathbb{N}_0$ we denote $T^{-n}(\mathscr{U}) = \{T^{-n}(U) : U \in \mathscr{U}\}$. Similarly, for any $m, n \in \mathbb{N}_0$ with $m \leqslant n$ we denote

$$\mathscr{U}_m^n = T^{-m}(\mathscr{U}) \vee T^{-m-1}(\mathscr{U}) \vee \cdots \vee T^{-n}(\mathscr{U}).$$

2.1. Topological entropy

Given a dynamical system (X, T), $K \subseteq X$ and $\mathscr{U} \in \mathcal{C}_X$ we denote by $N(\mathscr{U}, K)$ the minimal cardinality among all sub-families $\mathscr{V} \subseteq \mathscr{U}$ covering K, that is $\cup\mathscr{V} \supseteq K$. For technical reasons we always put $N(\mathscr{U}, \emptyset) = 1$. Using the above notation, we define

$$h_{\mathscr{U}}(T, K) = \limsup_{n\to+\infty} \frac{1}{n} \log N(\mathscr{U}_0^{n-1}, K).$$

Note that if $\mathscr{U}, \mathscr{V} \in \mathcal{C}_X$ and $\mathscr{U} \prec \mathscr{V}$ then $h_{\mathscr{U}}(T, K) \geqslant h_{\mathscr{V}}(T, K)$. The *topological entropy of* K and *topological entropy of* (X, T) are defined, respectively, by

$$h_{\text{top}}(T, K) = \sup_{\mathscr{U}\in\mathcal{C}_X^o} h_{\mathscr{U}}(T, K) \qquad \text{and} \qquad h_{\text{top}}(T) = h_{\text{top}}(T, X).$$

Proposition 2.1. *Let (X, T) be a dynamical system, $\mathscr{U} \in \mathcal{C}_X$ and $K, K_1, K_2 \subseteq X$. Then*

(2.1.1) $h_{\mathscr{U}}(T, \emptyset) = 0$ *and so* $h_{\text{top}}(T, \emptyset) = 0$.
(2.1.2) $h_{\mathscr{U}}(T, K) = h_{\mathscr{U}}(T, T(K))$ *and so* $h_{\text{top}}(T, K) = h_{\text{top}}(T, T(K))$.
(2.1.3) $h_{\mathscr{U}}(T, K_1) \leqslant h_{\mathscr{U}}(T, K_1 \cup K_2)$ *and so* $h_{\text{top}}(T, K_1) \leqslant h_{\text{top}}(T, K_1 \cup K_2)$.
(2.1.4) $h_{\mathscr{U}}(T, K) \in [0, \log(\#\mathscr{U})]$, *here* $\#\mathscr{U}$ *denotes the cardinality of* $\mathscr{U}$.

Proof. The proof of (2.1.1) and (2.1.3) is straighforward. For the proof of (2.1.2) it is enough to observe that

$$K \subset \bigcup_{i=1}^{s}\bigcap_{j=0}^{n+1} T^{-j}(V_i^j) \Longrightarrow T(K) \subset \bigcup_{i=1}^{s} T\left(\bigcap_{j=0}^{n+1} T^{-j}(V_i^j)\right) \subset \bigcup_{i=1}^{s}\bigcap_{j=0}^{n} T^{-j}(V_i^{j+1}),$$

$$T(K) \subset \bigcup_{i=1}^{s}\bigcap_{j=0}^{n} T^{-j}(V_i^j) \Longrightarrow K \subset \bigcup_{i=1}^{s}\bigcap_{j=1}^{n+1} T^{-j}(V_i^{j-1}),$$

which immediately leads to the following inequality, finishing the proof:

$$\frac{1}{\#\mathscr{U}} N(\mathscr{U}_0^{n+1}, K) \leqslant N(\mathscr{U}_0^n, T(K)) \leqslant N(\mathscr{U}_0^{n+1}, K).$$

Finally, $N(\mathscr{U}_0^{n-1}, K) \leqslant (\#\mathscr{U})^n$ and so $h_{\mathscr{U}}(T, K) \leqslant \log(\#\mathscr{U})$. □

2.2. Dimensional entropy

Next we present another approach to entropy, introduced by Bowen in the spirit of the Hausdorff dimension [5].

Given a dynamical system (X, T) and a non-empty set $K \subseteq X$, $\mathfrak{C}(K)$ denotes the set of all countable families of subsets of X such that if $\mathscr{V} \in \mathfrak{C}(K)$ then $\cup\mathscr{V} \supseteq K$. Given $\mathscr{U} \in \mathcal{C}_X$, $\varepsilon > 0$ and $E \subset X$ define

$$n_{T,\mathscr{U}}(E) = \sup\left\{j \in \mathbb{N}_0 : T^i(E) \prec \mathscr{U} \text{ for each } 0 \leqslant i < j\right\}.$$

Note that $n_{T,\mathscr{U}}(E) \in [0, +\infty]$ and $n_{T,\mathscr{U}}(E) = 0$ exactly when $E \not\prec \mathscr{U}$. For any fixed $\varepsilon > 0$ let $\mathfrak{C}^{\varepsilon}(K)$ denote these $\mathscr{V} \in \mathfrak{C}(K)$ so that $n_{T,\mathscr{U}}(E) \geqslant \frac{1}{\varepsilon}$ for any $E \in \mathscr{V}$.

For each $\lambda \in \mathbb{R}$ and any $\mathscr{V} \in \mathfrak{C}(K)$ we define (applying the rule $0^0 = 0$)

$$m(T, \mathscr{U}, \mathscr{V}, \lambda) = \sum_{E\in\mathscr{V}} \left(\mathrm{e}^{-n_{T,\mathscr{U}}(E)}\right)^{\lambda},$$

and next

$$m_{T,\mathscr{U}}(K, \lambda, \varepsilon) = \inf_{\mathscr{V}\in\mathfrak{C}^{\varepsilon}(K)} m(T, \mathscr{U}, \mathscr{V}, \lambda).$$

For completeness, we also define $m_{T,\mathscr{U}}(\emptyset, \lambda, \varepsilon)$ by putting $m_{T,\mathscr{U}}(\emptyset, \lambda, \varepsilon) = +\infty$ if $\lambda < 0$, $m_{T,\mathscr{U}}(\emptyset, \lambda, \varepsilon) = 1$ if $\lambda = 0$ and $m_{T,\mathscr{U}}(\emptyset, \lambda, \varepsilon) = 0$ if $\lambda > 0$.

Observe that $m_{T,\mathscr{U}}(K, \lambda, \varepsilon)$ is decreasing as the function of ε so the following formulae are well defined

$$m_{T,\mathscr{U}}(K, \lambda) = \lim_{\varepsilon\to 0^+} m_{T,\mathscr{U}}(K, \lambda, \varepsilon) = \lim_{n\to+\infty} m_{T,\mathscr{U}}\left(K, \lambda, \frac{1}{n}\right).$$

It follows immediately that if $\lambda \geqslant \eta$ then $m_{T,\mathscr{U}}(K, \lambda) \leqslant m_{T,\mathscr{U}}(K, \eta)$. It is also not hard to verify that $m_{T,\mathscr{U}}(K, \lambda) \in (0, +\infty)$ for at most one λ.

Lemma 2.2. *Let $K \neq \emptyset$ and $\mathcal{M}(T, \mathscr{U}, K, n)$ denote the collection of all countable families $\mathcal{T} = \{(A_i, n_i) : i \in I\}$ with $\cup\{A_i : i \in I\} \supseteq K$ and such that for each $i \in I$, $n \leqslant n_i \in \mathbb{N}$ and $A_i \in \mathscr{U}_0^{n_i-1}$. Then, for each $\lambda \geqslant 0$,*

$$m_{T,\mathscr{U}}(K, \lambda) = \lim_{n\to+\infty} \inf\left\{\sum_{i\in I} \mathrm{e}^{-\lambda n_i} : \{(A_i, n_i) : i \in I\} \in \mathcal{M}(T, \mathscr{U}, K, n)\right\}. \tag{2.1}$$

Proof. First, note that if $n_{T,\mathscr{U}}(E) > k$ then we can find $A \in \mathscr{U}_0^{k-1}$ so that $E \subset A$ (and obviously $n_{T,\mathscr{U}}(E) \geqslant n_{T,\mathscr{U}}(A) \geqslant k$). In particular if $n_{T,\mathscr{U}}(E) = \infty$ then for any fixed $\lambda > 0$ there is $k \in \mathbb{N}_0$ such that $\mathrm{e}^{-k\lambda}$ is small enough. Since $\mathscr{V}$ is at most countable, this shows that for any $\delta, \varepsilon > 0$ and any $\mathscr{V} \in \mathfrak{C}^{\varepsilon}(K)$ we can find $\mathscr{W} \in \mathfrak{C}^{\varepsilon}(K)$ such that each element of $\mathscr{W}$ is an element from $\bigcup_{n\geqslant 0} \mathscr{U}_0^n$ and $\mathscr{V} \prec \mathscr{W}$, $m(T, \mathscr{U}, \mathscr{W}, \lambda) \leqslant m(T, \mathscr{U}, \mathscr{V}, \lambda) + \delta$. This proves that

$$m_{T,\mathscr{U}}\left(K, \lambda, \frac{1}{n}\right) = \inf\left\{\sum_{i\in I} \mathrm{e}^{-\lambda n_i} : \{(A_i, n_i) : i \in I\} \in \mathcal{M}(T, \mathscr{U}, K, n)\right\}$$

for any $n \in \mathbb{N}$ and so the result follows. □

Now that we have all of the ingredients introduced, we can define *the dimensional entropy of K relative to $\mathscr{U}$* as

$$\begin{aligned} h^B_{\mathscr{U}}(T, K) &= \inf\{\lambda \in \mathbb{R} : m_{T,\mathscr{U}}(K, \lambda) = 0\} \\ &= \sup\{\lambda \in \mathbb{R} : m_{T,\mathscr{U}}(K, \lambda) = +\infty\}. \end{aligned}$$

We also define the *dimensional entropy of K* by

$$h^B(T, K) = \sup_{\mathscr{U}\in\mathcal{C}^o_X} h^B_{\mathscr{U}}(T, K).$$

If $\mathscr{U}, \mathscr{V} \in \mathcal{C}^o_X$ satisfy $\mathscr{U} \prec \mathscr{V}$ then $h^B_{\mathscr{U}}(T, K) \geqslant h^B_{\mathscr{V}}(T, K)$. Thus if $\{\mathscr{U}_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{C}^o_X$ with $\lim_{n\to\infty} \operatorname{mesh}(\mathscr{U}_n) = 0$ and (X, T) is a dynamical system acting on a compact metric space, then

$$\lim_{n\to+\infty} h^B_{\mathscr{U}_n}(T, K) = h^B(T, K)$$

because in this setting the Lebesgue number exists for any $\mathscr{V} \in \mathcal{C}^o_X$.

The following facts are well known or elementary (e.g. see [5, propositions 1 and 2] and [15, proposition 2.3 and lemma 3.1]). While results in [15] are stated for X being a compact metric space, nothing more than the assumption that X is a compact topological space is used in the proofs.

Proposition 2.3. *Let (X, T) be a dynamical system on a compact topological space, let $K_1, K_2, \cdots \subseteq X$, $K \subseteq X$, $\mathscr{U} \in \mathcal{C}_X$, $\mathscr{V} \in \mathcal{C}^o_X$, $m \in \mathbb{N}$ and $i \in \mathbb{N}$. Then*

(2.3.1) $h_{\mathscr{V}}(T, X) = h^B_{\mathscr{V}}(T, X)$ *and so* $h_{\mathrm{top}}(T) = h^B(T, X)$.

(2.3.2) $h^B_{\mathscr{U}}(T, \emptyset) = 0$ *and so* $h^B(T, \emptyset) = 0$.

(2.3.3) $h^B_{\mathscr{U}}\left(T, \bigcup_{n\in\mathbb{N}} K_n\right) = \sup_{n\in\mathbb{N}} h^B_{\mathscr{U}}(T, K_n)$ *and so* $h^B\left(T, \bigcup_{n\in\mathbb{N}} K_n\right) = \sup_{n\in\mathbb{N}} h^B(T, K_n)$.

(2.3.4) $h^B_{T^{-i}(\mathscr{U})}(T^m, K) \geqslant h^B_{\mathscr{U}}(T^m, T^i(K))$ *and so* $h^B(T^m, K) \geqslant h^B(T^m, T^i(K))$.

(2.3.5) $h^B_{\mathscr{U}_0^{m-1}}(T^m, K) = m h^B_{\mathscr{U}}(T, K)$ *and so* $h^B(T^m, K) = m h^B(T, K)$.

(2.3.6) $h^B_{\mathscr{U}}(T, K) \leqslant \liminf_{n\to+\infty} \frac{1}{n} \log N(\mathscr{U}_0^{n-1}, K) \leqslant h_{\mathscr{U}}(T, K)$.

Corollary 2.4. *If (X, T) is a dynamical system on a compact topological space, $K \subseteq X$ and $\mathscr{U} \in \mathcal{C}_X$ then $h^B_{\mathscr{U}}(T, K) \leqslant \log(\#\mathscr{U})$.*

2.3. Measure-theoretic entropy

Let X be a compact metric space and let (X, T) be a dynamical system. $\mathscr{M}(X)$ denote the set of all Borel probability measures on X. By $\mathscr{M}(X, T)$ we denote the set of all T-invariant measures $\mu \in \mathscr{M}(X)$, i.e. $\mu(A) = \mu(T^{-1}(A))$ for any Borel set A. It is known that both $\mathscr{M}(X)$ and $\mathscr{M}(X, T)$ are convex, compact metric spaces when endowed with the weak*-topology. We denote by $\mathcal{B}_X$ the set of all Borel subsets of X. For any given partition $\alpha \in \mathcal{P}_X$, any measure $\mu \in \mathscr{M}(X)$ and any sub-σ-algebra $\mathscr{C} \subset \mathcal{B}_X$, we define the function

$$H_\mu(\alpha|\mathscr{C}) = \sum_{A\in\alpha} \int_X -\mathbb{E}_\mu(1_A|\mathscr{C})(x) \log \mathbb{E}_\mu(1_A|\mathscr{C})(x)\, \mathrm{d}\mu(x),$$

where $\mathbb{E}_\mu$ denotes the conditional μ-expectation. It is well known that $H_\mu(\alpha|\mathscr{C})$ increases with respect to α (i.e. its sub-partitions) and decreases with respect to $\mathscr{C}$ (i.e. is smaller for larger sub-σ-algebras). For any $\mathscr{U} \in \mathcal{C}_X$ we define

$$H_\mu(\mathscr{U}|\mathscr{C}) = \inf_{\alpha\in\mathcal{P}_X, \alpha\prec\mathscr{U}} H_\mu(\alpha|\mathscr{C}).$$

Similar to $H_\mu(\alpha|\mathscr{C})$, $H_\mu(\mathscr{U}|\mathscr{C})$ increases with respect to $\mathscr{U}$ and decreases with respect to $\mathscr{C}$. When $\mu \in \mathscr{M}(X, T)$ and $T^{-1}(\mathscr{C}) \subset \mathscr{C}$ with respect to sets of μ-measure zero, then it is known (and not hard to check) that $a_n = H_\mu(\mathscr{U}_0^{n-1}|\mathscr{C})$ is a non-negative and sub-additive sequence (i.e. $a_{n+m} \leqslant a_n + a_m$) for any given $\mathscr{U} \in \mathcal{C}_X$. Then the following limit always exists:

$$h_\mu(T, \mathscr{U}|\mathscr{C}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathscr{U}_0^{n-1}|\mathscr{C}) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\mathscr{U}_0^{n-1}|\mathscr{C}).$$

The *relative measure-theoretical μ-entropy of (X, T) relevant to $\mathscr{C}$* is defined by

$$h_\mu(T|\mathscr{C}) = \sup_{\mathscr{U}\in\mathcal{P}_X} h_\mu(T, \mathscr{U}|\mathscr{C}).$$

Since X is a compact metric space, for every $\mathscr{U} \in \mathcal{C}_X^o$ there is $\delta > 0$ such that $\mathscr{V} \prec \mathscr{U}$ provided that $\mathscr{V} \in \mathcal{C}_X$ is a cover with $\mathrm{mesh}(\mathscr{V}) < \delta$. In particular, if we fix any sequence $\{\mathscr{U}_n\}_{n=1}^\infty \in \mathcal{C}_X^o$ with $\lim_{n\to\infty} \mathrm{mesh}(\mathscr{U}_n) = 0$ then $h_\mu(T|\mathscr{C}) = \lim_{n\to\infty} h_\mu(T, \mathscr{U}_n|\mathscr{C})$.

In the special case of $\mathscr{C} = \{\emptyset, X\}$ we simply write $H_\mu(\mathscr{U})$, $h_\mu(T, \mathscr{U})$ and $h_\mu(T)$ instead of $H_\mu(\mathscr{U}|\mathscr{C})$, $h_\mu(T, \mathscr{U}|\mathscr{C})$ and $h_\mu(T|\mathscr{C})$.

If $\pi: (X, T) \to (Y, S)$ is a factor map between dynamical systems on compact metric spaces then we define the *relative measure-theoretical μ-entropy of (X, T) with respect to π* by the formula

$$h_\mu(T|\pi) = h_\mu(T|\pi^{-1}(\mathcal{B}_Y)).$$

Similarly, we put $h_\mu(T, \mathscr{U}|\pi) = h_\mu(T, \mathscr{U}|\pi^{-1}(\mathcal{B}_Y))$.

Note that when $\mathscr{C} = \{\emptyset, X\}$ then simply $\mathbb{E}_\mu(1_A|\mathscr{C}) \equiv \mu(A)$ and so for any $\alpha \in \mathcal{P}_X$ we have

$$H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A) \log(\mu(A)).$$

Given partitions $\alpha, \beta \in \mathcal{P}_X$ and a sub-σ-algebra $\mathscr{C} \subset \mathcal{B}_X$, define

$$H_\mu(\alpha|\beta) = H_\mu(\alpha \vee \beta) - H_\mu(\beta),$$
$$H_\mu(\alpha|\mathscr{C} \vee \beta) = H_\mu(\alpha \vee \beta|\mathscr{C}) - H_\mu(\beta|\mathscr{C}).$$

Observe that

$$H_\mu(\alpha|\beta) = -\sum_{B\in\beta} \sum_{A\in\alpha\vee\beta, A\subset B} \mu(A) \log\left(\frac{\mu(A)}{\mu(B)}\right),$$

which shows that our definition is the same as the one introduced in [24] (and so also, the one used in [5]). We can also interpret $\mathscr{C} \vee \beta$ as $\mathscr{C} \vee \beta^*$ where β^* is the σ-algebra generated by β (and $\mathscr{C} \vee \mathscr{D}$ is the smallest σ-algebra containing union of σ-algebras $\mathscr{C}, \mathscr{D}$). Then it gives another way to interpret the above formulae. It can also be verified that $H_\mu(\alpha|\beta^*) = H_\mu(\alpha|\beta)$.

Let us summarize here a few basic properties of measure-theoretic entropy (see [14] and [25, theorem 4.12]).

Lemma 2.5. *Let X be a compact metric space and let (X, T) be an invertible dynamical system, $\mu \in \mathscr{M}(X, T)$, $\mathscr{U}, \mathscr{V} \in \mathcal{P}_X$ and let $\mathscr{C}$ be a sub-σ-algebra of $\mathcal{B}_X$ such that $T^{-1}(\mathscr{C}) \subset \mathscr{C}$ with respect to sets of μ-measure zero. Then*

(2.5.1) $h_\mu(T, \mathscr{U} \vee \mathscr{V}) \leqslant h_\mu(T, \mathscr{U}) + h_\mu(T, \mathscr{V})$,
(2.5.2) *if* $\mathscr{U} \prec \mathscr{V}$ *then* $h_\mu(T, \mathscr{U}) \geqslant h_\mu(T, \mathscr{V})$,
(2.5.3) $h_\mu(T, \mathscr{U}) \leqslant h_\mu(T, \mathscr{V}) + H_\mu(\mathscr{U}|\mathscr{V})$,
(2.5.4) $h_\mu(T|\mathscr{C}) = \sup_{\mathscr{W} \in \mathcal{C}_X^o} h_\mu(T, \mathscr{W}|\mathscr{C})$,
(2.5.5) $h_\mu(T, \mathscr{U}|\mathscr{C}) \leqslant h_\mu(T, \mathscr{V}|\mathscr{C}) + H_\mu(\mathscr{U}|\mathscr{V} \vee \mathscr{C})$,
(2.5.6) $H_\mu(\mathscr{U}|\mathscr{C} \vee \mathscr{V}) = H_\mu(\mathscr{U}|\mathscr{C} \vee \mathscr{V}^*)$.

3. Topological versus dimensional entropy of subsets

Let us assume for a moment that X is a compact metric space. In view of proposition 2.3 the reader may wonder what are the relations between topological and dimensional entropy when we evaluate it for a subset $K \subset X$ instead of X. While it is hard to provide a full characterization the following fact highlights possible problems that can arise. It also gives a very good motivation for further study of relations between values of h_{top} and h^B over subsets. If the class of subsets where these two quantities coincide could be characterized, it would be possible to use h_{top} and h^B interchangeably. Unfortunately, it is not clear in general, when these two quantities coincide. Before we start, let us recall [26, remark 5.13(4)].

Remark 3.1. Let (X, T) be a dynamical system on compact metric space. If $h_{\text{top}}(T) > 0$ then for every closed set K there exists a countable closed subset $K' \subset K$ such that $h_{\text{top}}(T, K') = h_{\text{top}}(T, K)$.

Let 2^X denote the hyperspace of non-empty closed subsets of a compact metric space (X, d). Endow 2^X with the Hausdorff metric H_d induced from X by d. It is well known that 2^X is also a compact metric space.

Corollary 3.2. *If (X, T) is a minimal dynamical system on compact metric space with $h_{\text{top}}(T) > 0$ then there exists a dense set $\mathfrak{Y} \subset 2^X$ so that if $A \in \mathfrak{Y}$ then $0 = h^B(T, A) < h_{\text{top}}(T, A) = h_{\text{top}}(T)$.*

Proof. Fix any non-empty open sets $U_1, \cdots, U_n \subset X$. Since T is minimal, it is easy to verify that for every i there is a closed set $K_i \subset U_i$ such that $h_{\text{top}}(T, K_i) = h_{\text{top}}(T)$. Then there are also countable closed subsets $K_i' \subset K_i$ with $h_{\text{top}}(T, K_i') = h_{\text{top}}(T, K_i)$. If we denote $A = \bigcup_i K_i'$ then A is a countable closed set, $A \subset \bigcup_i U_i$, $A \cap U_i \neq \emptyset$ for every i and additionally $h_{\text{top}}(T, A) = h_{\text{top}}(T)$. But A is countable, so we also have $h^B(T, A) = 0$. Denote by $\mathfrak{Y}$ the set of all possible subsets A that can be constructed as above starting from any finite sequence of open subsets of X. It is easy to verify that $\mathfrak{Y}$ is dense in 2^X (the easiest way to see it is by so-called Vietoris topology [16]), and so the proof is completed. □

For systems with finite entropy the situation is even more complex. Let us recall first an important fact about intermediate values of entropy over subsets (see [15]).

Theorem 3.3. *Let X be a compact metric space and let (X, T) be an invertible dynamical system with finite entropy. Then for each $0 \leqslant h \leqslant h_{\mathrm{top}}(T)$ there exists a non-empty compact subset K_h of X such that $h^B(T, K_h) = h = h_{\mathrm{top}}(T, K_h)$.*

Corollary 3.4. *Let X be a compact metric space and let (X, T) be an invertible minimal dynamical system with finite entropy. Then for any $0 \leqslant \alpha \leqslant \beta \leqslant h_{\mathrm{top}}(T)$ there is a dense subset $\mathcal{Y}_\alpha^\beta \subset 2^X$ such that*

$$h^B(T, K) = \alpha \qquad \text{and} \qquad h_{\mathrm{top}}(T, K) = \beta$$

for every $K \in \mathcal{Y}_\alpha^\beta$.

Proof. Let K_α be a set with $h^B(T, K_\alpha) = \alpha = h_{\mathrm{top}}(T, K_\alpha)$ (using theorem 3.3). If we fix any nonempty open set U then there is $N \in \mathbb{N}_0$ such that $\bigcup_{i=0}^N T^i(U) = X$ and so by proposition 2.3 there is $i \geqslant 0$ such that $h^B(T, T^i(U) \cap K_\alpha) = \alpha$.

Note that for any $x \in X$ we have $h^B(T, \{x\}) = 0$ and so by theorem 4.1 (which will be proved later) we find that $h^B(T, K) = h^B(T, T^{-i}(K))$. Then $K'_\alpha = T^{-i}(T^i(U) \cap K_\alpha) = U \cap T^{-i}(K_\alpha) \subset U$ has dimensional entropy $h^B(T, K'_\alpha) = \alpha$. Obviously, still $h_{\mathrm{top}}(T, K'_\alpha) = \alpha$ since it was the case for K_α and (X, T) is minimal.

If we fix any nonempty open set $U_1, \ldots, U_n$ then repeating arguments from corollary 3.2 we can find a countable closed set $K'_\beta \subset \bigcup_{i=1}^n U_i$ such that $0 = h^B(T, K'_\beta)$, $h_{\mathrm{top}}(T, K'_\beta) = \beta$ and $K'_\beta \cap U_i \neq \emptyset$ for all i. We can also assume that $K'_\alpha \subset U_1$.

Denote $K = K'_\alpha \cup K'_\beta$ and observe that K is a closed set such that $K \cap U_i \neq \emptyset$ for all i and $K \subset \bigcup_{i=1}^n U_i$. Additionally $h^B(T, K) = \max\{h^B(T, K'_\alpha), h^B(T, K'_\beta)\} = \alpha$ and similarly $h_{\mathrm{top}}(T, K) = \beta$. This shows that the set $\mathcal{Y}_\alpha^\beta$ is dense in 2^X. □

Remark 3.5. There is a large class of systems that fulfil assumptions of corollary 3.4. It includes some extensions of odometers [9] or Chacón flow [3].

4. Relative topological entropy

The main aim of this paper is to prove the following theorem:

Theorem 4.1. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems and $K \subseteq X$. If both X, Y are compact and additionally Y is a Hausdorff space then*

$$h^B(S, \pi(K)) \leqslant h^B(T, K) \leqslant h^B(S, \pi(K)) + \sup_{y \in Y} h^B(T, \pi^{-1}(y)). \tag{4.1}$$

We believe that in general in (4.1) one cannot replace $\sup_{y\in Y} h^B(T, \pi^{-1}(y))$ by $\sup_{y\in\pi(K)} h^B(T, \pi^{-1}(y))$, whereas, up to now we fail to prove or disprove it even in the setting of factor map of dynamical systems on compact metric spaces.

Intuitively, the reader may hope that (4.1) is true, since there is a well-known theorem by Bowen [4, theorem 17] which states (in the setting of compact metric spaces) that

$$h_{\mathrm{top}}(T, X) \leqslant h_{\mathrm{top}}(S, Y) + \sup_{y \in Y} h_{\mathrm{top}}(T, \pi^{-1}(y)). \tag{4.2}$$

Later this inequality was generalized to the following formula working for any $K \subseteq X$ (see [15, theorem 7.3]):

$$h_{\mathrm{top}}(S, \pi(K)) \leqslant h_{\mathrm{top}}(T, K) \leqslant h_{\mathrm{top}}(S, \pi(K)) + \sup_{y \in Y} h_{\mathrm{top}}(T, \pi^{-1}(y)). \tag{4.3}$$

We once again highlight the fact that it is not obvious that h_{top} can be replaced by h^B in the above formula (even when X is a compact metric space), since as we already demonstrated in section 3 for a given dynamical system on a compact metric space X values of functions $h_{\text{top}}(T, \cdot)$ and $h^B(T, \cdot)$ can differ on a dense subset of the hyperspace 2^X.

Before we go any further, let us recall some definitions which follow more modern terminology than that of [4], that is, let us express the above formulae in terms of relative entropy.

Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems and $\mathscr{U} \in \mathcal{C}_X^o$. Note that

$$a_n = \sup_{y\in Y} \log N(\mathscr{U}_0^{n-1}, \pi^{-1}(y))$$

is a sub-additive sequence and so the following limit always exists:

$$h_{\text{top}}(T, \mathscr{U}|\pi) = \lim_{n\to+\infty} \frac{1}{n} \sup_{y\in Y} \log N(\mathscr{U}_0^{n-1}, \pi^{-1}(y)). \tag{4.4}$$

We call the above defined quantity $h_{\text{top}}(T, \mathscr{U}|\pi) \in [0, \infty)$ the *topological entropy of $\mathscr{U}$ relative to π*. The *relative topological entropy of π* is defined by

$$h_{\text{top}}(T|\pi) = \sup_{\mathscr{U}\in\mathcal{C}_X^o} h_{\text{top}}(T, \mathscr{U}|\pi).$$

It is known that for any dynamical system on compact metric space (see [11, theorem 3]) the following condition holds (it is true in a little wider context; we will show it later for completeness)

$$h_{\text{top}}(T|\pi) = \sup_{y\in Y} h_{\text{top}}(T, \pi^{-1}(y)).$$

In this section, we are going to prove the following theorem.

Theorem 4.2. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems on compact spaces X, Y and $K \subseteq X$. If additionally Y is a Hausdorff space then*

$$h^B(S, \pi(K)) \leqslant h^B(T, K) \leqslant h^B(S, \pi(K)) + h_{\text{top}}(T|\pi).$$

However, before we can complete this task, we need some auxiliary lemmas.

Lemma 4.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems on compact spaces, let $K \subseteq X$ and $\mathscr{U} \in \mathcal{C}_Y^o$. Then $h^B_{\mathscr{U}}(S, \pi(K)) \leqslant h^B_{\pi^{-1}(\mathscr{U})}(T, K)$.*

Proof. Let $\lambda \in \mathbb{R}$ and $\mathscr{V} \in \mathfrak{C}(K)$, i.e. $\mathscr{V}$ is a countable family of non-empty subsets of X with $\cup\mathscr{V} \supseteq K$. We denote $\mathscr{C} = \pi^{-1}(\mathscr{U})$ and observe that $n_{T,\mathscr{C}}(E) = n_{S,\mathscr{U}}(\pi(E))$ for each $E \in \mathscr{V}$. Then $m(T, \mathscr{C}, \mathscr{V}, \lambda) = m(S, \mathscr{U}, \pi(\mathscr{V}), \lambda)$, where $\pi(\mathscr{V}) = \{\pi(E) : E \in \mathscr{V}\}$. Fix any $\delta > 0$ and let $\varepsilon > 0$ be so that if $\mathscr{V} \in \mathfrak{C}^\varepsilon(K)$ then $\pi(\mathscr{V}) \in \mathfrak{C}^\delta(\pi(K))$. Then we have the following:

$$\begin{aligned} m_{T,\mathscr{C}}(K, \lambda, \varepsilon) &= \inf_{\mathscr{V}\in\mathfrak{C}^\varepsilon(K)} m(T, \mathscr{C}, \mathscr{V}, \lambda) = \inf_{\mathscr{V}\in\mathfrak{C}^\varepsilon(K)} m(S, \mathscr{U}, \pi(\mathscr{V}), \lambda) \\ &\geqslant \inf_{\mathscr{V}\in\mathfrak{C}^\delta(\pi(K))} m(S, \mathscr{U}, \mathscr{V}, \lambda) = m_{S,\mathscr{U}}(\pi(K), \lambda, \delta). \end{aligned}$$

This immediately gives $m_{T,\mathscr{C}}(K, \lambda) \geqslant m_{S,\mathscr{U}}(\pi(K), \lambda)$, which implies $h^B_{\mathscr{U}}(S, \pi(K)) \leqslant h^B_{\mathscr{C}}(T, K)$ and so the proof is finished. □

Lemma 4.4. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems on compact spaces X, Y and $K \subseteq X$. If additionally Y is a Hausdorff space then for every $\mathscr{U} \in \mathcal{C}_X^o$ the following condition is satisfied:*

$$h^B_{\mathscr{U}}(T, K) \leqslant h^B(S, \pi(K)) + \sup_{y\in Y} \log N(\mathscr{U}, \pi^{-1}(y)).$$

Proof. Note that each continuous map from a compact space to a Hausdorff space must be a closed map (e.g. see [19]). Fix any $y \in Y$ and let $U = \cup U_i$ where the sum is taken over all sets $U_i \in \mathscr{U}$ such that $U_i \cap \pi^{-1}(y) \neq \emptyset$. Note that there is an open neighbourhood $y \in V$ such that $\pi^{-1}(V) \subset U$ as otherwise we can find a sequence $y_n \in Y$ such that $y \in \overline{\{y_n \ : \ n \in \mathbb{N}\}}$ and $\pi^{-1}(y_n) \setminus U \neq \emptyset$. Then if we denote $W = \overline{\{x_n \ : \ n \in \mathbb{N}\}}$ for some $x_n \in \pi^{-1}(y_n) \setminus U$ then $W \cap U = \emptyset$ while $\pi(W)$ is closed, in particular $y \in \pi(W)$. Such a situation is impossible by the definition of U. Thus using compactness of Y we can find points $y_1, \ldots, y_n \in Y$ their open neighbourhoods $\mathscr{V} = \{V_1, \ldots, V_n\} \in \mathcal{C}_Y^o$ (i.e. $y_i \in V_i$) so that for any fixed subcover $\mathscr{U}_i \subseteq \mathscr{U}$ with $\#\mathscr{U}_i = N(\mathscr{U}, \pi^{-1}(y_i))$ we have $\pi^{-1}(V_i) \subseteq \cup\mathscr{U}_i$.

Now we aim to prove the conclusion by showing that (when $K \neq \emptyset$)

$$h^B_{\mathscr{U}}(T, K) \leqslant h^B_{\mathscr{V}}(S, \pi(K)) + a, \text{ where } a = \sup_{y\in Y} \log N(\mathscr{U}, \pi^{-1}(y)). \tag{4.5}$$

Fix any $\lambda \geqslant 0$, $\varepsilon > 0$ and put $p = [\frac{1}{\varepsilon}] + 1$. By (2.1), it is enough to show that

$$m_{T,\mathscr{U}}(K, \lambda + a, \varepsilon) \leqslant \inf\left\{\sum_{j\in J} \mathrm{e}^{-\lambda n_j} : \{(A_j, n_j) : j \in J\} \in \mathcal{M}(S, \mathscr{V}, \pi(K), p)\right\}. \tag{4.6}$$

Fix any $\{(A_j, n_j) : j \in J\} \in \mathcal{M}(S, \mathscr{V}, \pi(K), p)$. If $j \in J$ then by the definition $A_j \in \mathscr{V}_0^{n_j-1}$ and so there are $s(j, 0), \cdots, s(j, n_j - 1) \in \{1, \cdots, n\}$ such that

$$A_j = \bigcap_{k=0}^{n_j-1} S^{-k}(V_{s(j,k)}).$$

For each $j \in J$ we define

$$\mathscr{C}_j = \left\{\bigcap_{k=0}^{n_j-1} T^{-k}(U_k) : U_k \in \mathscr{U}_{s(j,k)}, 0 \leqslant k \leqslant n_j - 1\right\}$$

and consider the following family of subsets of X:

$$\mathscr{C} = \bigcup_{j\in J} \mathscr{C}_j = \{E \ : \ E \in \mathscr{C}_j \text{ for some } j \in J\}.$$

Each set $\mathscr{C}_j$ is finite, so $\mathscr{C}$ is at most countable. Additionally, observe that

$$\begin{aligned}\pi^{-1}(S^{-k}(V_{s(j,k)})) &= (S^k \circ \pi)^{-1}(V_{s(j,k)}) = (\pi \circ T^k)^{-1}(V_{s(j,k)}) \\ &= T^{-k}(\pi^{-1}(V_{s(j,k)})) \subset T^{-k}(\cup\mathscr{U}_{s(j,k)})\end{aligned}$$

which in turn implies that $\mathscr{C} \in \mathfrak{C}(K)$, because

$$\begin{aligned}\bigcup_{j\in J} \cup\mathscr{C}_i &\supset \bigcup_{j\in J} \cup \bigcap_{k=0}^{n_j-1} T^{-k}(\pi^{-1}(V_{s(j,k)})) \\ &\supset \pi^{-1}\left(\bigcup_{j\in J} A_j\right) \supset \pi^{-1}(\pi(K)) \supseteq K.\end{aligned}$$

Simply by the definition, if $E \in \mathscr{C}_j$ for some $j \in J$, then $n_{T,\mathscr{U}}(E) \geqslant n_j \geqslant p$.

Additionally, note that

$$\max_{1\leqslant i\leqslant n} \#\mathscr{U}_i = \max_{1\leqslant i\leqslant n} N(\mathscr{U}, \pi^{-1}(y_i)) \leqslant \sup_{y\in Y} N(\mathscr{U}, \pi^{-1}(y)),$$

which immediately leads to the following inequalities:

$$\begin{aligned} m(T, \mathscr{U}, \mathscr{C}, \lambda + a) &\leqslant \sum_{j\in J} \mathrm{e}^{-n_j(\lambda+a)} \#\mathscr{C}_j \\ &\leqslant \sum_{j\in J} \mathrm{e}^{-n_j(\lambda+a)} \left(\max_{1\leqslant i\leqslant n} \#\mathscr{U}_i\right)^{n_j} \\ &\leqslant \sum_{j\in J} \mathrm{e}^{-n_j(\lambda+a)} (\mathrm{e}^{a})^{n_j} = \sum_{j\in J} \mathrm{e}^{-n_j\lambda}. \end{aligned}$$

Since $\{(A_j, n_j) : j \in J\}$ was arbitrary, we obtain (4.6), which ends the proof. □

Proof of theorem 4.2. By lemma 4.3 we obtain $h^B(S, \pi(K)) \leqslant h^B(T, K)$.

For the proof of the second inequality, fix any $n \in \mathbb{N}$ and observe that by the application of lemma 4.4 to $\mathscr{U}_0^{n-1}$ and $\pi : (X, T^n) \to (Y, S^n)$ we obtain

$$\frac{1}{n} h^B_{\mathscr{U}_0^{n-1}}(T^n, K) \leqslant \frac{1}{n} h^B(S^n, \pi(K)) + \frac{1}{n} \sup_{y\in Y} \log N(\mathscr{U}_0^{n-1}, \pi^{-1}(y)).$$

This by proposition 2.3 gives

$$h^B_{\mathscr{U}}(T, K) \leqslant h^B(S, \pi(K)) + \frac{1}{n} \sup_{y\in Y} \log N(\mathscr{U}_0^{n-1}, \pi^{-1}(y)).$$

The proof is finished by letting $n \to +\infty$. □

5. Relative dimensional entropy

As we mentioned earlier, we always have $h^B(T, K) \leqslant h_{\mathrm{top}}(T, K)$ and sometimes this inequality is strict. In particular, we have $\sup_{y\in Y} h^B(T, \pi^{-1}(y)) \leqslant h_{\mathrm{top}}(T|\pi)$; however, it is not immediately clear if these two quantities are equal. If it is the case, then we can express the statement of theorem 4.2 using only h^B obtaining a condition analogous to that mentioned earlier [15]. This is the main aim of this section.

Strictly speaking we are going to prove the following result, and then combined with theorem 4.2 from it we obtain directly theorem 4.1.

Theorem 5.1. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems acting on compact topological spaces and let $\mathfrak{U} \in \mathfrak{C}_X^o$. If Y is a Hausdorff space then*

$$h_{\mathrm{top}}(T, \mathscr{U}|\pi) = \sup_{y\in Y} h_{\mathscr{U}}(T, \pi^{-1}(y)) = \sup_{y\in Y} h^B_{\mathscr{U}}(T, \pi^{-1}(y)).$$

Before proving theorem 5.1, we need the following.

Lemma 5.2. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems acting on compact topological spaces and let $\mathfrak{U} \in \mathfrak{C}_X^o$. If Y is a Hausdorff space then*

$$h_{\mathrm{top}}(T, \mathscr{U}|\pi) \leqslant \sup_{y\in Y} h^B_{\mathscr{U}}(T, \pi^{-1}(y)).$$

Proof. It is sufficient to show that when $\lambda > 0$ is such that $m_{T,\mathscr{U}}(\pi^{-1}(y), \lambda) = 0$ for every $y \in Y$ then $h_{\mathrm{top}}(T, \mathscr{U}|\pi) \leqslant \lambda$.

To see that indeed it is the case, fix any $\lambda > 0$ such that each $y \in Y$ satisfies the condition $m_{T,\mathscr{U}}(\pi^{-1}(y), \lambda) = 0$.

Now, if we fix any $y \in Y$ then, since $m_{T,\mathscr{U}}(\pi^{-1}(y), \lambda) = 0$, by (2.1) there exists a family $\{(E_i(y), n(E_i(y)))\}_{i=1}^{\infty} \in \mathcal{M}(T, \mathscr{U}, \pi^{-1}(y), 1)$ such that

$$\sum_{i=1}^{\infty} \mathrm{e}^{-\lambda n(E_i(y))} < 1.$$

Note that for every $i \in \mathbb{N}$ we have $n(E_i(y)) \geqslant 1$ and $E_i(y) \in \mathscr{U}_0^{n(E_i(y))-1}$ (in particular, $E_i(y)$ is an open subset of X).

Observe that by compactness of $\pi^{-1}(y)$ there is $N(y) \in \mathbb{N}$ such that $\pi^{-1}(y) \subset \cup\{E_i(y) : i = 1, \cdots, N(y)\}$. Using the assumption that Y is a Hausdorff space, we can repeat arguments from the proof of lemma 4.4 obtaining an open neighbourhood $U(y) \ni y$ such that $\cup\{E_i(y) : i = 1, \cdots, N(y)\} \supseteq \pi^{-1}(U(y))$.

By compactness of Y there are points $z_1, \cdots, z_p \in Y$ with $\{U(z_j) : j = 1, \cdots, p\} \in \mathcal{C}_Y^o$. For every $y \in Y$ fix a number $\phi(y) \in \{1, \cdots, p\}$ such that $y \in U(z_{\phi(y)})$. Then

$$\bigcup_{i=1}^{N(z_{\phi(y)})} E_i(z_{\phi(y)}) \supseteq \pi^{-1}(U(z_{\phi(y)})) \supseteq \pi^{-1}(y).$$

Our aim is to find an estimate of $N(\mathscr{U}_0^{n-1}, \pi^{-1}(y))$ for every $y \in Y$.

To do so, we need some more notation. Fix any $y \in Y$, $t \in \mathbb{N}$ and a sequence $(j_1, \cdots, j_t) \subseteq \mathbb{N}$ and next, for $q = 1, 2, \cdots, t+1$ put $a(1) = 0$ and $a(q) = a(q-1) + n(E_{j_{q-1}}(z_{\phi(S^{a(q-1)}(y))}))$ when $q > 1$. Obviously $a(q)$ depends on the choice of y as well as on the choice of corresponding tuple.

If $1 \leqslant j_q \leqslant N(z_{\phi(S^{a(q)}y)})$ for every $q \in \{1, \cdots, t\}$ then we say that the tuple $(j_1, \cdots, j_t)$ is *admissible (for y)*. For an admissible tuple we define

$$C(y, j_1, \cdots, j_t) = \pi^{-1}(y) \cap \bigcap_{q=1}^{t} T^{-a(q)} E_{j_q}(z_{\phi(S^{a(q)}y)}).$$

Observe that if $(j_1, \cdots, j_t)$ is admissible then

$$C(y, j_1, \cdots, j_t) \prec \mathscr{U}_0^{a(t+1)-1}. \tag{5.1}$$

For each $n \in \mathbb{N}$ and y denote by $I_n(y)$ the collection of all tuples $(j_1, \cdots, j_t)$ such that $(j_1, \cdots, j_t)$ is admissible and $a(t) < n \leqslant a(t+1)$.

We claim that the following inclusion holds

$$\cup\{C(y, j_1, \cdots, j_t) : (j_1, \cdots, j_t) \in I_n(y)\} \supseteq \pi^{-1}(y). \tag{5.2}$$

For the proof of the claim fix any $x \in \pi^{-1}(y)$. Then there exists $j_1 \in \{1, \cdots, N(z_{\phi(y)})\}$ with $x \in E_{j_1}(z_{\phi(y)})$. Obviously $x \in C(y, j_1)$. Next, if $x \in C(y, j_1, \cdots, j_t)$ for some admissible tuple $(j_1, \cdots, j_t)$. Observe that there exists $j_{t+1} \in \{1, \cdots, N(z_{\phi(S^{a(t+1)}(y))})\}$ such that $T^{a(t+1)}(x) \in E_{j_{t+1}}(z_{\phi(S^{a(t+1)}(y))})$, because

$$\begin{aligned} T^{a(t+1)}(x) \in T^{a(t+1)}(\pi^{-1}(y)) &\subseteq \pi^{-1}(S^{a(t+1)}(y)) \\ &\subseteq \bigcup_{i=1}^{N(z_{\phi(S^{a(t+1)}(y))})} E_i(z_{\phi(S^{a(t+1)}(y))}). \end{aligned}$$

Note that $(j_1, \cdots, j_t, j_{t+1})$ is still an admissible tuple. By the above inductive procedure we can define an admissible tuple of arbitrary length. But since n is fixed, after finitely many steps we will obtain $a(t+1) \geqslant n$. This means that eventually we find that $(j_1, \cdots, j_t) \in I_n(y)$ and by the method of construction also $x \in C(y, j_1, \cdots, j_t)$. This ends the proof of the claim.

Let us continue with the proof of the theorem. Note that for any admissible tuple $(j_1, \cdots, j_t)$ we have $a(t) = a(t+1) - n(E_{j_t}(z_{\phi(S^{a(t)}(y))}))$ and $a(t) < n \leqslant a(t+1)$, which gives

$$n - a(t) = n - a(t+1) + n(E_{j_t}(z_{\phi(S^{a(t)}(y))})). \tag{5.3}$$

Put

$$\begin{aligned}
\alpha &= \max\{n(E_k(z_j)) : 1 \leqslant j \leqslant p, 1 \leqslant k \leqslant N(z_j)\}, \\
\beta &= \max_{1 \leqslant j \leqslant p} \sum_{i=1}^{\infty} \mathrm{e}^{-\lambda n(E_i(z_j))},
\end{aligned}$$

and observe that $1 \leqslant \alpha < +\infty$ and $0 \leqslant \beta < 1$. By (5.1), (5.2) and (5.3) we obtain

$$\begin{aligned}
N(\mathscr{U}_0^{n-1}, \pi^{-1}(y)) &\leqslant \# I_n(y) \\
&\leqslant \sum_{t \geqslant 1} \sum_{(j_1,\cdots,j_t) \in I_n(y)} \mathrm{e}^{\lambda(n-a(t))} \\
&= \sum_{t \geqslant 1} \sum_{(j_1,\cdots,j_t) \in I_n(y)} \mathrm{e}^{\lambda(n-a(t+1)+n(E_{j_t}(z_{\phi(S^{a(t)}(y))})))} \\
&\leqslant \mathrm{e}^{\lambda(n+\alpha)} \sum_{t \geqslant 1} \sum_{(j_1,\cdots,j_t) \in I_n(y)} \mathrm{e}^{-\lambda a(t+1)} \\
&= \mathrm{e}^{\lambda(n+\alpha)} \sum_{t \geqslant 1} \sum_{(j_1,\cdots,j_t) \in I_n(y)} \prod_{1 \leqslant q \leqslant t} \mathrm{e}^{-\lambda n(E_{j_q}(z_{\phi(S^{a(q)}(y))}))}.
\end{aligned} \tag{5.4}$$

Now observe that for any given $t \in \mathbb{N}$, if $(j_1, \cdots, j_t)$ is admissible then consider the set Γ consisting of all admissible tuples $(j_1, \cdots, j_t, j_{t+1})$, then all these tuples from Γ generate the same values of $a(1), \cdots, a(t+1)$, and so

$$\begin{aligned}
&\sum_{(j_1,\cdots,j_t,j_{t+1}) \in \Gamma} \prod_{1 \leqslant q \leqslant t+1} \mathrm{e}^{-\lambda n(E_{j_q}(z_{\phi(S^{a(q)}(y))}))} \\
&\quad \leqslant \left(\sum_{i=1}^{N(z_{\phi(S^{a(t+1)}(y))})} \mathrm{e}^{-\lambda n(E_i(z_{\phi(S^{a(t+1)}(y))}))} \right) \prod_{1 \leqslant q \leqslant t} \mathrm{e}^{-\lambda n(E_{j_q}(z_{\phi(S^{a(q)}(y))}))} \\
&\quad \leqslant \beta \prod_{1 \leqslant q \leqslant t} \mathrm{e}^{-\lambda n(E_{j_q}(z_{\phi(S^{a(q)}(y))}))},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{(j_1,\cdots,j_t,j_{t+1}) \text{ is admissible}} \prod_{1 \leqslant q \leqslant t+1} \mathrm{e}^{-\lambda n(E_{j_q}(z_{\phi(S^{a(q)}(y))}))} \\
&\quad \leqslant \beta \sum_{(j_1,\cdots,j_t) \text{ is admissible}} \prod_{1 \leqslant q \leqslant t} \mathrm{e}^{-\lambda n(E_{j_q}(z_{\phi(S^{a(q)}(y))}))} \\
&\quad \leqslant \cdots \leqslant \beta^t \sum_{(j_1) \text{ is admissible}} \mathrm{e}^{-\lambda n(E_{j_1}(z_{\phi(S^{a(1)}(y))}))} \\
&\quad = \beta^t \sum_{(j_1) \text{ is admissible}} \mathrm{e}^{-\lambda n(E_{j_1}(z_{\phi(y)}))} \leqslant \beta^{t+1}.
\end{aligned} \tag{5.5}$$

Combining (5.4) with (5.5) we finally obtain that

$$
\begin{aligned}
N(\mathscr{U}_0^{n-1}, \pi^{-1}(y)) &\leqslant \mathrm{e}^{\lambda(n+\alpha)} \sum_{t\geqslant 1} \sum_{(j_1,\cdots,j_t) \text{ is admissible}} \prod_{1\leqslant q\leqslant t} \mathrm{e}^{-\lambda n(E_{j_q}(z_{\phi(S^{a(q)}(y))}))} \\
&\leqslant \mathrm{e}^{\lambda(n+\alpha)} \sum_{t\geqslant 1} \beta^t \\
&= \mathrm{e}^{\lambda(n+\alpha)} \frac{\beta}{1-\beta}.
\end{aligned} \tag{5.6}
$$

Note that $0 < \beta < 1$ is a constant and the expression in (5.6) is independent of the selection of $y \in Y$, so directly from the definition we find that $h_{\mathrm{top}}(T, \mathscr{U}|\pi) \leqslant \lambda$, which completes the proof. □

Proof of theorem 5.1. Combining proposition 2.3 and lemma 5.2 we have

$$
\begin{aligned}
\sup_{y\in Y} \liminf_{n\to\infty} \frac{1}{n} \log N(\mathscr{U}_0^{n-1}, \pi^{-1}(y)) &\leqslant \sup_{y\in Y} h_{\mathscr{U}}(T, \pi^{-1}(y)) \\
&\leqslant \limsup_{n\to\infty} \frac{1}{n} \sup_{y\in Y} \log N(\mathscr{U}_0^{n-1}, \pi^{-1}(y)) \\
&= h_{\mathrm{top}}(T, \mathscr{U}|\pi) \\
&\leqslant \sup_{y\in Y} h^B_{\mathscr{U}}(T, \pi^{-1}(y)) \\
&\leqslant \sup_{y\in Y} \liminf_{n\to\infty} \frac{1}{n} \log N(\mathscr{U}_0^{n-1}, \pi^{-1}(y)).
\end{aligned}
$$

The proof is finished. □

As a direct consequence of theorem 4.1, we also obtain the following theorem (note that as shown in [26] while topological entropy of a countable set can be strictly positive, its dimensional entropy h^B is always zero).

Theorem 5.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems on compact topological spaces. If Y is a Hausdorff space and each fibre $\pi^{-1}(y)$, $y \in Y$ is at most countable then $h_{\mathrm{top}}(T|\pi) = 0$ (and so π preserves topological entropy).*

With the additional assumption that $\#\pi^{-1}(y) < +\infty$ and that (X, T), (Y, S) are dynamical systems on compact metric spaces, the above fact follows directly from the well-known Bowen formula (see also (4.3)) proved first in [4] (see also [6, theorem 7.1]), estimating topological entropy of factors (the proof strongly uses the fact that Lebesgue number is well defined).

We should emphasize here one important fact (which seems to be well known for many years, but hardly found in the literature until [10]). In the case of a factor map between dynamical systems on compact metric spaces, it is possible to prove even more than stated in theorem 5.3. First, if (X, T) is μ-ergodic (not necessarily invertible) and $h_\mu(T|\pi) > 0$ then for almost every y the fibre $\pi^{-1}(y)$ is uncountable [10, theorem 4.1.15]. In particular, using the variational principle concerning relative entropy [11, theorem 5] we find that at least one fibre must be uncountable (it can also be deduced from some other results, e.g. [27, theorem 4.2]).

6. More on entropies over fibres

Throughout this whole section all spaces will be *compact metric spaces*. We will use this assumption without any further reference.

As a direct corollary of theorem 5.1, we have that in quite a general setting

$$\sup_{y\in Y} h^B(T,\pi^{-1}(y)) = \sup_{y\in Y} h_{\mathrm{top}}(T,\pi^{-1}(y)).$$

In the context of invertible dynamical systems on compact metric spaces we can say even more. Namely, we could prove the following result.

Theorem 6.1. *Let $\pi : (X,T)\to(Y,S)$ be a factor map between invertible dynamical systems and $\nu\in\mathscr{M}(Y,S)$. Then $h^B(T,\pi^{-1}(y)) = h_{\mathrm{top}}(T,\pi^{-1}(y))$ for ν-a.e. $y\in Y$.*

In fact, it is just a direct corollary of the following theorem.

Theorem 6.2. *Let $\pi : (X,T)\to(Y,S)$ be a factor map and let $\nu\in\mathscr{M}(Y,S)$. Then*

$$\sup_{\mu\in\mathscr{M}(X,T),\pi\mu=\nu} h_\mu(T|\pi) = \int_Y h_{\mathrm{top}}(T,\pi^{-1}(y))\,\mathrm{d}\nu(y) = \int_Y h^B(T,\pi^{-1}(y))\,\mathrm{d}\nu(y).$$

Before proceeding, first we prove:

Theorem 6.3. *Let $\pi : (X,T)\to(Y,S)$ be a factor map between dynamical systems and let $\mathscr{U}\in\mathfrak{C}^o_X$. Then the following functions are Borel measurable:*

$$\hat{h}^B_{\mathscr{U}}\colon Y\ni y\mapsto h^B_{\mathscr{U}}(T,\pi^{-1}(y))\in[0,+\infty),$$
$$\hat{h}^B\colon Y\ni y\mapsto h^B(T,\pi^{-1}(y))\in[0,+\infty].$$

Proof. First of all, by corollary 2.4 image of $\hat{h}^B_{\mathscr{U}}$ is indeed bounded. We will prove its Borel measurability.

Let $\lambda>0, n\in\mathbb{N}, y_0\in Y$ and $\delta>0$ with $m_{T,\mathscr{U}}(\pi^{-1}(y_0),\lambda,\frac{1}{n})<\delta$. There exists a countable family $\mathscr{V}\in\mathfrak{C}^{1/n}(\pi^{-1}(y_0))$ such that $m(T,\mathscr{U},\mathscr{V},\lambda)<\delta$. But let $K\subset X$ and $\eta>0$ given any cover in $\mathscr{W}\in\mathfrak{C}^\eta(K)$ and any $\xi>0$ we can replace its elements by open sets in such a way that new family $\mathscr{W}'\in\mathfrak{C}^\eta(K)$ has the following properties: $\mathscr{W}\prec\mathscr{W}'$ and $|m(T,\mathscr{U},\mathscr{W},\lambda)-m(T,\mathscr{U},\mathscr{W}',\lambda)|<\xi$. Thus, without loss of generality we may assume that $\mathscr{V}$ consists of open sets.

There is an open neighbourhood $V\ni y_0$ such that $\cup\mathscr{V}\supseteq\pi^{-1}(V)$, thus $\mathscr{V}\in\mathfrak{C}^{1/n}(\pi^{-1}(y))$ for each $y\in V$. In particular

$$m_{T,\mathscr{U}}\left(\pi^{-1}(y),\lambda,\frac{1}{n}\right)\leqslant m(T,\mathscr{U},\mathscr{V},\lambda)<\delta.$$

This shows that $\{y\in Y : m_{T,\mathscr{U}}(\pi^{-1}(y),\lambda,\frac{1}{n})<\delta\}$ is an open subset of Y. Next note that if $m_{T,\mathscr{U}}(\pi^{-1}(y),\lambda)>t$ then there is n such that $m_{T,\mathscr{U}}(\pi^{-1}(y),\lambda,\frac{1}{n})>t$ and vice-versa. Thus, for each $t\geqslant 0$ we have

$$\{y\in Y \;:\; m_{T,\mathscr{U}}(\pi^{-1}(y),\lambda)>t\} = \bigcup_{n=1}^{\infty}\left\{y\in Y \;:\; m_{T,\mathscr{U}}(\pi^{-1}(y),\lambda,\frac{1}{n})>t\right\}$$

and so the function $y\mapsto m_{T,\mathscr{U}}(\pi^{-1}(y),\lambda)$ is Borel measurable. Similarly,

$$\{y\in Y : h^B_{\mathscr{U}}(T,\pi^{-1}(y))>t\} = \bigcup_{m\in\mathbb{N}}\left\{y\in Y : m_{T,\mathscr{U}}\left(\pi^{-1}(y),t+\frac{1}{m}\right)>0\right\}$$

for each $t\geqslant 0$, which shows that $\hat{h}^B_{\mathscr{U}}$ is a Borel measurable function.

Fix any $\{\mathcal{U}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}_X^o$ with $\lim_{n\to\infty} \mathrm{mesh}(\mathcal{U}_n) = 0$. Then obviously $\hat{h}^B(y) = \lim_{n\to+\infty} \hat{h}^B_{\mathcal{U}_n}(y)$ and so given any $y \in Y$ and $t \in \mathbb{R}$ we see that $\hat{h}^B(y) > t$ if and only if $\hat{h}^B_{\mathcal{U}_n}(y) > t$ for some n. This shows that $\hat{h}^B$ is also Borel measurable. □

We also need the following result (see [15, theorem 4.1 and proposition 4.2]).

Theorem 6.4. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between invertible dynamical systems, $\alpha \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X, T)$ and $\nu = \pi\mu \in \mathcal{M}(Y, S)$ with $\mu = \int_Y \mu_y \,\mathrm{d}\nu(y)$ the disintegration of μ over ν. Then there exists $f \in L^1(\nu)$ such that $\int_Y f(y)\,\mathrm{d}\nu(y) = h_\mu(T, \alpha|\pi)$ and if $\mathcal{U} \in \mathcal{C}_X$ satisfies that each element of $\mathcal{U}$ has a non-empty intersection with at most $M \in \mathbb{N}$ elements of α, then $h^B_{\mathcal{U}}(T, Z_y) \geqslant f(y) - \log M$ for ν-a.e. $y \in Y$ and for any Borel subset $Z_y \subseteq X$ with $\mu_y(Z_y) > 0$.*

Then we have

Theorem 6.5. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between invertible dynamical systems, $\alpha \in \mathcal{P}_X$, $\mathcal{U} \in \mathcal{C}_X^o$ and $\mu \in \mathcal{M}(X, T)$. Assume that each element of $\mathcal{U}$ has a non-empty intersection with at most $M \in \mathbb{N}$ elements of α. Then*

$$h_\mu(T, \alpha|\pi) \leqslant \int_Y h^B_{\mathcal{U}}(T, \pi^{-1}(y))d(\pi\mu)(y) + \log M$$
$$\leqslant \int_Y h^B(T, \pi^{-1}(y))d(\pi\mu)(y) + \log M.$$

Proof. It is a direct corollary of theorems 6.3 and 6.4. Simply,

$$h^B(T, \pi^{-1}(y)) + \log M \geqslant h^B_{\mathcal{U}}(T, \pi^{-1}(y)) + \log M \geqslant f(y)$$

for ν-a.e. $y \in Y$ (as $\mu_y(\pi^{-1}y) = 1$ for ν-a.e. $y \in Y$) and both functions $\hat{h}^B_{\mathcal{U}} \doteq h^B_{\mathcal{U}}(T, \pi^{-1}(\cdot))$ and $\hat{h}^B \doteq h^B(T, \pi^{-1}(\cdot))$ are Borel measurable. □

Let us also recall three important lemmas proved first by Bowen (see [5]).

Lemma 6.6. *Let X be a topological space and $\alpha \in \mathcal{P}_X$, $p \in \mathbb{N}$. Assume that at most $p \in \mathbb{N}$ elements of α can have a point in the intersection of their closures. Then there exists $\mathcal{U} \in \mathcal{C}_X^o$ such that each element of $\mathcal{U}$ has a non-empty intersection with at most p elements of α.*

Proof. Let α be the partition from the assumptions and say $\alpha = \{A_1, \cdots, A_n\}$ for some $n \in \mathbb{N}$. If $n \leqslant p$ then any $\mathcal{U} \in \mathcal{C}_X^o$ satisfies the conclusion. Hence, now assume that $n > p$. Denote by S the set of all subsets $\{l_1 < \cdots < l_{n-p}\} \subseteq \{1, \cdots, n\}$. Additionally, if $\{l_1 < \cdots < l_{n-p}\}$ is such a subset then denote by U the set of all points which are not contained in $\overline{A_j}$, $j = l_1, \cdots, l_{n-p}$. Now let $\mathcal{U}$ to be the family of all such U as above (i.e. defined by different sequences l_j). We claim that $\mathcal{U}$ has the required property. It suffices to prove that the family $\mathcal{U}$ covers X. But if $x \in X$ then by assumption there exists $\{l_1 < \cdots < l_{n-p}\} \in S$ such that $x \notin \overline{A_j}$ for each $j = l_1, \cdots, l_{n-p}$, which implies that x must be contained in some element of the constructed $\mathcal{U}$. This finishes our proof. □

Lemma 6.7 ([5, lemma 2]). *Let (X, T) be a dynamical system, $\mu \in \mathcal{M}(X, T)$ and $\mathcal{U} \in \mathcal{C}_X^o$. Then for every $n > 0$ there is $\alpha_n \in \mathcal{P}_X$ such that $\alpha_n \prec \mathcal{U}_0^{n-1}$ and any $x \in X$ is in the closure of at most $n\#\mathcal{U} > 0$ elements of α.*

Lemma 6.8 ([5, lemma 3]). *Let (X, T) be a dynamical system and $\mu \in \mathcal{M}(X, T)$. Given $\beta \in \mathcal{P}_X$ and $\varepsilon > 0$ there is $\mathcal{U} \in \mathcal{C}_X^o$ such that $H_\mu(\beta|\alpha) < \varepsilon$ for every $\alpha \in \mathcal{P}_X$ such that $\alpha \prec \mathcal{U}$.*

Now we are to prove theorem 6.2.

Proof of theorem 6.2. First, let us recall that similar to $\hat{h}^B$ in theorem 6.3, the function $Y \ni y \mapsto h_{\mathrm{top}}(T, \pi^{-1}(y)) \in [0, +\infty]$ is also Borel measurable (e.g. see [20, lemma 3.3]) so both integrals above are well defined. The first equality was established in [20, theorem 2.1] so we only need to prove the second one.

By proposition 2.3, $h^B_{\mathscr{U}}(T, \pi^{-1}(y)) \leqslant h_{\mathscr{U}}(T, \pi^{-1}(y))$ for any $y \in Y$ and so $\int_Y h(T, \pi^{-1}(y))\,\mathrm{d}\nu(y) \geqslant \int_Y h^B(T, \pi^{-1}(y))\,\mathrm{d}\nu(y)$. Then it is sufficient to prove

$$\int_Y h^B(T, \pi^{-1}(y))\,\mathrm{d}\nu(y) \geqslant \sup_{\mu \in \mathscr{M}(X,T), \pi\mu = \nu} h_\mu(T|\pi).$$

Let $\mu \in \mathscr{M}(X, T)$, $\beta \in \mathcal{P}_X$ and $\varepsilon > 0$ be fixed. By lemma 6.8 there exists $\mathscr{U} \in \mathcal{C}^o_X$ such that if $\alpha \in \mathcal{P}_X$ satisfies $\alpha \prec \mathscr{U}$ then $H_\mu(\beta|\alpha) < \varepsilon/2$. Now by lemma 6.7 for each $n \in \mathbb{N}$ there exists $\alpha_n \in \mathcal{P}_X$ such that $\alpha_n \prec \mathscr{U}_0^{n-1}$ and at most $n\#\mathscr{U}$ elements of α_n can have a point in the intersection of their closures. Moreover, by lemma 6.6 there exists $\mathscr{U}_n \in \mathcal{C}^o_X$ such that each element of $\mathscr{U}_n$ has a non-empty intersection with at most $n\#\mathscr{U}$ elements of α_n.

Using basic facts collected in lemma 2.5 we obtain that for any $\mathscr{W}, \mathscr{V} \in \mathcal{P}_X$,

$$\begin{aligned} H_\mu(\mathscr{W}_0^{n-1}|\pi) &\leqslant H_\mu(\mathscr{V}_0^{n-1}|\pi) + H_\mu(\mathscr{W}_0^{n-1}|\mathscr{V}_0^{n-1}) \\ &\leqslant H_\mu(\mathscr{V}_0^{n-1}|\pi) + \sum_{i=0}^{n-1} H_\mu(T^{-i}(\mathscr{W})|T^{-i}(\mathscr{V})) \\ &= H_\mu(\mathscr{V}_0^{n-1}|\pi) + nH_\mu(\mathscr{W}|\mathscr{V}). \end{aligned}$$

Then directly from the definition of measure-theoretic entropy we obtain estimate

$$h_\mu(T, \mathscr{U}|\pi) \leqslant h_\mu(T, \mathscr{W}|\pi) + H_\mu(\mathscr{U}|\mathscr{W}).$$

In our setting, the above fact together with theorem 6.5 gives

$$\begin{aligned} h_\mu(T, \beta|\pi) &= \frac{1}{n} h_\mu(T^n, \beta_0^{n-1}|\pi) \\ &\leqslant \frac{1}{n} h_\mu(T^n, \alpha_n|\pi) + \frac{1}{n} H_\mu(\beta_0^{n-1}|\alpha_n) \\ &\leqslant \frac{1}{n}\left(\int_Y h^B_{\mathscr{U}_n}(T^n, \pi^{-1}(y))\,\mathrm{d}(\pi\mu)(y) + \log(n\#\mathscr{U})\right) + \frac{1}{n} H_\mu(\beta_0^{n-1}|\alpha_n) \\ &\leqslant \frac{1}{n}\left(\int_Y h^B(T^n, \pi^{-1}(y))\,\mathrm{d}(\pi\mu)(y) + \log(n\#\mathscr{U})\right) + \frac{1}{n}\sum_{i=0}^{n-1} H_\mu(T^{-i}\beta|\alpha_n). \end{aligned}$$

Now, note that $\alpha_n \prec \mathscr{U}_0^{n-1}$, $T^i\alpha_n \prec \mathscr{U}$ for any $i = 0, 1, \cdots, n-1$, thus

$$\frac{1}{n}\sum_{i=0}^{n-1} H_\mu(T^{-i}\beta|\alpha_n) = \frac{1}{n}\sum_{i=0}^{n-1} H_\mu(\beta|T^i\alpha_n) < \varepsilon/2$$

and so for n large enough we finally obtain

$$h_\mu(T, \beta|\pi) \leqslant \int_Y h^B(T, \pi^{-1}(y))\,\mathrm{d}(\pi\mu)(y) + \varepsilon.$$

But β and ε were arbitrary, thus directly from the above inequality we obtain

$$h_\mu(T|\pi) \leqslant \int_Y h^B(T, \pi^{-1}(y))\,\mathrm{d}(\pi\mu)(y)$$

which ends the proof. $\square$

Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems, $\mathscr{U} \in \mathcal{C}_X^o$ and $\nu \in \mathscr{M}(Y, S)$. It can be proved that (see [28, section 4.4.2])

$$\sup_{\mu \in \mathscr{M}(X,T), \pi\mu=\nu} h_\mu(T, \mathscr{U}|\pi) = \int_Y h_{\mathscr{U}}(T, \pi^{-1}(y))\, \mathrm{d}\nu(y). \tag{6.1}$$

Then theorem 6.1 leads to the following natural question:

Question 6.9. *Does the following condition hold for any $\mathscr{U} \in \mathcal{C}_X^o$:*

$$\sup_{\mu \in \mathscr{M}(X,T), \pi\mu=\nu} h_\mu(T, \mathscr{U}|\pi) = \int_Y h_{\mathscr{U}}^B(T, \pi^{-1}(y))\, \mathrm{d}\nu(y)? \tag{6.2}$$

Note that if the above condition holds then we could obtain a local version of theorem 6.1: $h_{\mathscr{U}}^B(T, \pi^{-1}(y)) = h_{\mathscr{U}}(T, \pi^{-1}(y))$ for ν-a.e. $y \in Y$.

Appendix

A.1. Distance entropy

Recently motivated by the definition of dimensional entropy Dai and Jiang introduced the distance entropy of a subset in the setting of separable metric spaces [7].

Let (X, T) be a dynamical system on a separable metric space (X, d). For any $E \subset X$ denote

$$l_{T,\varepsilon}(E) = \sup\left\{n \in \mathbb{N}_0 \ :\ \mathrm{diam}(T^i(E)) < \varepsilon \text{ for every } 0 \leqslant i < n\right\}.$$

Note that if $l_{T,\varepsilon}(E) = 0$ then $\mathrm{diam}(E) \geqslant \varepsilon$ and if $l_{T,\varepsilon}(E) = +\infty$ then $\mathrm{diam}(T^i(E)) < \varepsilon$ for every $i \in \mathbb{N}_0$. Denote by $\widetilde{\mathfrak{C}}^\varepsilon(K)$ these $\mathscr{V} \in \mathfrak{C}(K)$ so that $l_{T,\mathscr{U}}(E) > -\log\varepsilon$ for every $E \in \mathscr{V}$ (recall the definition of $\mathfrak{C}(K)$ introduced in previous subsection).

In a way similar to dimensional entropy, for each $\lambda \in \mathbb{R}$, $\varepsilon > 0$ and any $\mathscr{V} \in \mathfrak{C}(K)$ we define (again, applying the rule $0^0 = 0$)

$$D(T, \varepsilon, \mathscr{V}, \lambda) = \sum_{E \in \mathscr{V}} \left(e^{-l_{T,\varepsilon}(E)}\right)^\lambda,$$

and next

$$\mathfrak{M}_T^\lambda(K, \varepsilon) = \inf_{\mathscr{V} \in \widetilde{\mathfrak{C}}^\varepsilon(K)} D(T, \varepsilon, \mathscr{V}, \lambda).$$

Finally, we define λ-*measure (with respect to T)* by

$$\mathfrak{M}_T^\lambda(K) = \lim_{\varepsilon \to 0^+} \mathfrak{M}_T^\lambda(K, \varepsilon).$$

Note that for any $s < t$ we have $\mathfrak{M}_T^s(K, \varepsilon) \geqslant \varepsilon^{s-t}\mathfrak{M}_T^t(K, \varepsilon)$ and so there is at most one λ such that $\mathfrak{M}_T^\lambda(K) \in (0, \infty)$. Similar to dimensional entropy, we define

$$\mathrm{ent}_H(T, K) = \inf\left\{\lambda \in \mathbb{R} : \mathfrak{M}_T^\lambda(K) = 0\right\}.$$

It was proved in [7] that the value of ent_H does not change if we replace metric d by a uniformly equivalent metric d'. In particular, if X is a compact metric space then ent_H depends only on the topology (induced by d). It can also be proved that if X is a compact metric space then $\mathrm{ent}_H(T, K) = h^B(T, K)$ for every $K \subset X$ (see [7, proposition 1]).

A.2. Solutions of two questions on distance entropy

In [7, question 5.1] the authors asked the following question:

Question A.1. *Let (X, d) be a metric space. If $T: X \to X$ is a uniformly continuous and pointwise periodic map, is $h_d(T) = 0$?*

We recall that a map is *pointwise periodic* if each $x \in X$ is a periodic point, that is, $T^n x = x$ for some $n \in \mathbb{N}$. The motivation for this question was [7, propositions 8 and 9] which can be summarized as follows.

Remark A.2. If X is a topological space and T is pointwise periodic then $h^B(T) = 0$. If additionally X is a separable metric space then also $\mathrm{ent}_H(T) = 0$.

Even if we add the assumption that X is separable to the above question, the answer is still negative as shown by the following example.

Example A.3. *Consider any transitive dynamical system on compact metric spaces with dense periodic points, let say, full shift on two symbols (X, σ) (i.e. $X = \{0, 1\}^{\mathbb{Z}}$ and $\sigma(x)_i = x_{i+1}$) equipped with a metric ρ compatible with the Tikhonov topology. Let h_ρ denote the entropy calculated using (n, ε)-separated sets (in metric ρ) as introduced by Bowen [4] (since this definition is quite standard and we use it only in this example, we do not recall this definition). It is well known that $h_\rho(\sigma, X) = \log 2$ and that $h_\rho(\sigma, \overline{K}) = h_\rho(\sigma, K)$ for any subset K, in particular $h_\rho(\sigma, X) = h_\rho(\sigma, \mathrm{Per}(\sigma))$, here $\mathrm{Per}(\sigma)$ denotes the set of all periodic points of (X, σ). But $\sigma(\mathrm{Per}(\sigma)) = \mathrm{Per}(\sigma)$ so $Y = \mathrm{Per}(\sigma)$ with metric induced by ρ together with transformation $S = \sigma|_{\mathrm{Per}(\sigma)}$ induces a dynamical system (Y, S) acting on the separable metric space Y. Obviously S is uniformly continuous, pointwise periodic and $h_\rho(S) = h_\rho(\sigma, \mathrm{Per}(\sigma)) > 0$.*

Now, let us turn to [7, question 2.1] which was left as another problem for further research. The question (rewritten in suitable terminology) is as follows.

Question A.4. *Let (Y, S) be a dynamical system acting on a compact metric space and let (X, T) be a dynamical system acting on a separable metric space X. If (Y, S) is a factor of (X, T) via a factor map π then does the following condition hold:*

$$\mathrm{ent}_H(T) \leqslant \mathrm{ent}_H(S) + \sup_{y \in Y} \mathrm{ent}_H(T, \pi^{-1}(y))?$$

As shown by theorem 4.1 the answer to the above question is positive when considering a factor map between dynamical systems on compact metric spaces. Whereas, in general the answer is negative as shown below.

Example A.5. *There exists a dynamical system (Y, S), where Y is a compact metric space, a uniformly continuous map $T: X \to X$ acting on a separable metric space and a factor map $\pi: (X, T) \to (Y, S)$, such that*

$$\mathrm{ent}_H(T) = \log 4 > \log 2 = \mathrm{ent}_H(S) + \sup_{y \in Y} \mathrm{ent}_H(T, \pi^{-1}(y)). \tag{A.1}$$

Proof. Let $X = \mathbb{R}^2 \setminus \{0\}$ and let $T : X \ni (x_1, x_2) \mapsto (2x_1, 2x_2) \in X$. We also denote $Y = \{(x_1, x_2) \in \mathbb{R}^2: x_1^2 + x_2^2 = 1\}$, and put $S: Y \ni (x_1, x_2) \mapsto (x_1, x_2) \in Y$. We endow both X, Y with a metric induced by the Euclidean metric in $\mathbb{R}^2$.

We will now calculate the value of all terms appearing in (A.1). First note that $\mathrm{ent}_H(S, Y) = h^B(S, Y) = 0$.

Now, if we consider the map $\widetilde{T} : \mathbb{R}^2 \ni (x_1, x_2) \mapsto (2x_1, 2x_2) \in \mathbb{R}^2$ with a Euclidean metric, then by [7, corollary 8] we find that $\mathrm{ent}_H(\widetilde{T}) = 2\log 2$. Additionally, by [7, theorem 2.1.(2)] distance entropy is stable under countable sums, in particular

$$\begin{aligned}\mathrm{ent}_H(\widetilde{T}) &= \max\left\{\mathrm{ent}_H(\widetilde{T}, \{0\}), \mathrm{ent}_H(\widetilde{T}, \mathbb{R}^2 \setminus \{0\})\right\}\\ &= \mathrm{ent}_H(\widetilde{T}, \mathbb{R}^2 \setminus \{0\}).\end{aligned}$$

But if $E \subset \mathbb{R}^2 \setminus \{0\}$ then $l_{T,\varepsilon}(E) = l_{\widetilde{T},\varepsilon}(E)$ and so $\mathrm{ent}_H(T) = \mathrm{ent}_H(\widetilde{T}) = \log 4$.

Finally, it was demonstrated in [7, example 2.3] that if we endow $Z = (0, \infty)$ with the Euclidean metric and consider the map $F : Z \ni x \mapsto 2x \in Z$ then $\mathrm{ent}_H(F) = \log 2$. But for any $y \in Y$ the fibre $\pi^{-1}(y)$ together with restriction of T to that fibre can be isometrically identified with the dynamical system (Z, F). Using the fact that every uniformly continuous conjugacy preserves distance entropy [7, theorem 2.3], we finally obtain that $\mathrm{ent}_H(T, \pi^{-1}(y)) = \log 2$ for every $y \in Y$. This ends the proof. $\square$

The factor map π in example A.5 is not uniformly continuous (arbitrarily close to 0 we can find points which project onto antipodal points on Y), but even this additional assumption does not make the situation better. It will be demonstrated by next example. While it is a simple modification of example A.5, it is less transparent and so we decided to present both of them.

Example A.6. There exists a dynamical system (Y, S) acting on a compact metric space, a dynamical system $(\widehat{X}, \widehat{T})$, where $\widehat{T}$ is a uniformly continuous map acting on a separable metric space, and a uniformly continuous factor map $\hat{\pi} : (\widehat{X}, \widehat{T}) \to (Y, S)$, such that

$$\mathrm{ent}_H(\widehat{T}) = \log 4 > \log 2 = \mathrm{ent}_H(S) + \sup_{y\in Y} \mathrm{ent}_H(\widehat{T}, \hat{\pi}^{-1}(y)). \tag{A.2}$$

Proof. Let (Y, S) be the map constructed in example A.5. Denote $\widehat{X} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 > 1\}$ and endow it with metric induced by the Euclidean metric on $\mathbb{R}^2$. We define $\widehat{T} : \widehat{X} \to \widehat{X}$, by the formula

$$(x_1, x_2) \mapsto \left(2x_1 - \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, 2x_2 - \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right)$$

together with the natural projection $\hat{\pi} : \widehat{X} \to Y$ given by

$$(x_1, x_2) \mapsto \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right).$$

We define a map $\eta : \widehat{X} \to X$ as follows

$$(x_1, x_2) \mapsto \left(x_1 - \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, x_2 - \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right),$$

where (X, T) was constructed in example A.5. Note that η is uniformly continuous and $\widehat{X}$ is uniformly continuous as well. Simple calculations show that $\eta \circ \widehat{T} = T \circ \eta$, so using [7, theorem 2.3] we obtain that $\mathrm{ent}_H(\widehat{T}, \widehat{X}) \geqslant \mathrm{ent}_H(T, X) = 2\log 2$.

Finally, observe that for any $y \in Y$ we have $(\widehat{T}, \hat{\pi}^{-1}(y))$ can be isometrically transformed to the map $(1, +\infty) \ni x \mapsto 2x - 1 \in (1, +\infty)$ which in fact is isometric with (Z, F)

from example A.5. This shows that $\mathrm{ent}_H(\widehat{T}, \hat{\pi}^{-1}(y)) = \mathrm{ent}_H(F) = \log 2$ which ends the proof. □

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