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Delay-dependent stability analysis and synthesis of uncertain T–S fuzzy systems with time-varying delay

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Abstract

This paper considers the delay-dependent stability analysis and controller design for uncertain T–S fuzzy system with time-varying delay. A new method is provided by introducing some free-weighting matrices and employing the lower bound of time-varying delay. Based on the Lyapunov–Krasovskii functional method, sufficient condition for the asymptotical stability of the system is obtained. By constructing the Lyapunov–Krasovskii functional appropriately, we can avoid the supplementary requirement that the time-derivative of time-varying delay must be smaller than one. The fuzzy state feedback gain is derived through the numerical solution of a set of linear matrix inequalities (LMIs). The upper bound of time-delay can be obtained by using convex optimization such that the system can be stabilized for all time-delays. The efficiency of our method is demonstrated by two numerical examples.

Keywords: Nonlinear systems; Linear matrix inequality (LMI); Delay-dependent; T–S fuzzy systems

1. Introduction

Recently, Takagi–Sugeno (T–S) [15] fuzzy model has been paid considerable attention because it can combine the flexibility of fuzzy logic theory and rigorous mathematical theory of linear or nonlinear system into a unified framework.

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In [18], delay-independent stability and controller design were considered for a class of T–S fuzzy systems with constant state delay. For T–S fuzzy systems with time-varying delay, [1,2,4,9,11] investigated the stability and control problems based on Lyapunov–Krasovskii method under an assumption that the upper bound of time-derivative of time delay is less than one. Based on Razumikhin technique, the control problem for T–S fuzzy systems with time delays was considered in [3,16,26]. However, the obtained results were delay independent.

Generally speaking, delay-dependent results for time-delay systems are less conservative than those for the delay-independent case, especially for time-delay systems with actually small delay [12,21,22]. In recent few years, much attention has been paid to the study of delay-dependent stability and stabilization for time-delay systems [5,6,13,14,22–25]. However, only a few research works are concerned with the delay-dependent stability or/and stabilization for the T–S fuzzy systems with time delay. In [10], delay-dependent stabilization problem was investigated for a class of T–S fuzzy systems with time delay under an assumption that the upper bound of time-derivative of the time delay is less than one. In [7], authors considered the delay-dependent guaranteed cost controller design for T–S fuzzy systems based on both state feedback and generalized dynamic output feedback. However, only time-invariant delay case was considered in [7]. To the best of author's knowledge, up to now, the delay-dependent stability and stabilization problems of uncertain T–S fuzzy systems with time-varying delay have not been fully investigated. Therefore, it still remains challenging.

This paper deals with the delay-dependent stability and controller design problems of uncertain nonlinear time-varying delay systems via T–S fuzzy models. Sufficient conditions for stability analysis and controller design are derived based on Lyapunov–Krasovskii functional method. In this paper, there is no requirement for the information of derivative of the time delay, that is, our method allows fast time-varying delay. By solving a set of LMIs and using an optimal algorithm, the state feedback gain and the upper bound of the time delay can be obtained. The effectiveness and less conservativeness of the proposed method will be shown by two numerical examples.

2. System and problem description

Consider the Takagi–Sugeno fuzzy model with time-varying delay, the ith ruler is described by the following If–Then ruler:

$$\begin{aligned} R^i: \quad & \text{If} \quad z_1(t) \text{ is } W_1^i \text{ and } \cdots \text{ and } z_n(t) \text{ is } W_n^i, \\ & \text{Then} \quad \dot{x}(t)=(A_i+\Delta A_i(t))x(t)+(A_{di}+\Delta A_{di}(t))x(t-\tau(t))+(B_i+\Delta B_i(t))u(t), \end{aligned} \tag{1}$$

where $x(t)=\phi(t)$, $t \in [-\bar{\tau}, 0]$, A_i, A_{di} and $B_i (i=1,2,\ldots,n)$ are constant matrices with compatible dimensions, $x(t) \in R^r$ is the state vector and $u(t) \in R^m$ is the input vector, $\phi(t)$ is the initial condition of the state; W_j^i is the fuzzy set, $z_j(t)(j=1,2,\ldots,n)$ is the premise variable, $\bar{\tau}$ is the upper bound of time-delay $\tau(t)$. $\Delta A_i(t)$, $\Delta A_{di}(t)$ and $\Delta B_i(t)$ are time-varying matrices with appropriate dimensions, which are defined as

$$\Delta A_i(t) = D_{ai} F_{ai}(t) E_{ai}, \quad \Delta A_{di}(t) = D_{adi} F_{adi}(t) E_{adi}, \quad \Delta B_i(t) = D_{bi} F_{bi}(t) E_{bi}, \tag{2}$$

where $i=1,2,\ldots,n$, D_{ai}, D_{adi}, D_{bi} and E_{ai}, E_{adi}, E_{bi} are known constant real matrices with appropriate dimensions and $F_{ai}(t)$, $F_{adi}(t)$ and $F_{bi}(t)$ are unknown real time-varying matrices with Lebesgue measurable elements bounded by

$$F_{ai}^{\mathrm{T}}(t) F_{ai}(t) \leqslant I, \quad F_{adi}^{\mathrm{T}}(t) F_{adi}(t) \leqslant I, \quad F_{bi}^{\mathrm{T}}(t) F_{bi}(t) \leqslant I, \quad i=1,2,\ldots,n. \tag{3}$$

Assumption 1. There exist two constants $\underline{\tau}$ and $\overline{\tau}$ such that

$$\underline{\tau} \leqslant \tau(t) \leqslant \overline{\tau}. \tag{4}$$

By using the center-average defuzzifier, product interference and singleton fuzzifier, the global dynamics of the T–S fuzzy system (1) can be inferred as

$$\dot{x}(t) = \sum_{i=1}^{n} \mu_i(z(t))[\bar{A}_i x(t) + \bar{A}_{di} x(t-\tau(t)) + \bar{B}_i u(t)], \tag{5}$$

where $\bar{A}_i = A_i + \Delta A_i(t)$, $\bar{A}_{di} = A_{di} + \Delta A_{di}(t)$ and $\bar{B}_i = B_i + \Delta B_i(t)$,

$$\mu_i(z(t)) = \omega_i(z(t)) \Big/ \sum_{i=1}^{n} \omega_i(z(t)), \quad \omega_i(z(t)) = \prod_{j=1}^{n} W_j^i(z_j(t)) \ ,$$

and $W_j^i(z_j(t))$ is the membership value of $z_j(t)$ in W_j^i, some basic properties of $\mu_i(z(t))$ are

$$\mu_i(z(t)) \geqslant 0, \quad \sum_{i=1}^{n} \mu_i(z(t)) = 1.$$

In this paper, a state feedback T–S fuzzy-model-based controller will be designed for the stabilization of the T–S fuzzy system (5). The ith controller rule is

$$\begin{aligned} R^i: \ \text{If} \quad & z_1(t) \text{ is } W_1^i \text{ and} \cdots \text{and } z_n(t) \text{ is } W_n^i, \\ \text{Then} \quad & u(t) = K_i x(t). \end{aligned} \tag{6}$$

The defuzzified output of the controller (6) rule is given by

$$u(t) = \sum_{i=1}^{n} \mu_i(z(t)) K_i x(t). \tag{7}$$

Combining (5) and (7), the closed-loop fuzzy system can be obtained

$$\begin{aligned} \dot{x}(t) = & \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i(z(t)) \mu_j(z(t)) [(\bar{A}_i + \bar{B}_i K_j) x(t) + \bar{A}_{di} x(t-\tau(t))], \\ & x(t) = \phi(t), \quad t \in [-\overline{\tau}, 0]. \end{aligned} \tag{8}$$

3. Main results

Define

$$\tau_0 = \tfrac{1}{2}(\overline{\tau} + \underline{\tau}), \quad \delta = \tfrac{1}{2}(\overline{\tau} - \underline{\tau}). \tag{9}$$

Then, it can be seen that $\tau(t) \in [\tau_0 - \delta, \ \tau_0 + \delta]$ and $\tau_0 \geqslant \delta$.

Remark 1. $\underline{\tau}$ and $\overline{\tau}$ are the lower and upper bound of $\tau(t)$. When $\delta = 0$, i.e., $\underline{\tau} = \overline{\tau}$, then $\tau(t)$ denotes a constant delay. The case when $\underline{\tau} = 0$, i.e., $\tau_0 = \delta = \overline{\tau}/2$, it implies that $0 \leqslant \tau(t) \leqslant \overline{\tau}$.

Using the Newton–Leibniz formula, we have

$$x(t) - x(t-\tau_0) - \int_{t-\tau_0}^{t} \dot{x}(s)\,\mathrm{d}s = 0, \tag{10}$$

$$x(t-\tau_0) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_0} \dot{x}(s)\,\mathrm{d}s = 0 \tag{11}$$

and from (8), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i(z(t))\mu_j(z(t))[(\bar{A}_i + \bar{B}_i K_j)x(t) + \bar{A}_{di}x(t-\tau(t)) - \dot{x}(t)] = 0. \tag{12}$$

Based on (10)–(12) and similar to the method used in [8,17], for arbitrary matrices N_{kij}, T_{kij} and $M_k (i, j = 1, 2, \ldots, n, k = 1, 2, 3, 4)$ with compatible dimensions, it can be seen that

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i(z(t))\mu_j(z(t))[x^{\mathrm{T}}(t)N_{1ij} + x^{\mathrm{T}}(t-\tau(t))N_{2ij} + x^{\mathrm{T}}(t-\tau_0)N_{3ij} + \dot{x}^{\mathrm{T}}(t)N_{4ij}]\\ &\quad\times\left[x(t) - x(t-\tau_0) - \int_{t-\tau_0}^{t} \dot{x}(s)\,\mathrm{d}s\right] = 0,\end{aligned} \tag{13}$$

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i(z(t))\mu_j(z(t))[x^{\mathrm{T}}(t)T_{1ij} + x^{\mathrm{T}}(t-\tau(t))T_{2ij} + x^{\mathrm{T}}(t-\tau_0)T_{3ij} + \dot{x}^{\mathrm{T}}(t)T_{4ij}]\\ &\quad\times\left[x(t-\tau_0) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_0} \dot{x}(s)\,\mathrm{d}s\right] = 0\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i(z(t))\mu_j(z(t))[x^{\mathrm{T}}(t)M_1 + x^{\mathrm{T}}(t-\tau(t))M_2 + x^{\mathrm{T}}(t-\tau_0)M_3 + \dot{x}^{\mathrm{T}}(t)M_4]\\ &\quad\times[(\bar{A}_i + \bar{B}_i K_j)x(t) + \bar{A}_{di}x(t-\tau(t)) - \dot{x}(t)] = 0.\end{aligned} \tag{15}$$

For given feedback gain K_j, combining (13)–(15), we can obtain the following stability condition based on Lyapunov–Krasovskii functional method.

Lemma 1. *For given scalars* $\underline{\tau} > 0$, $\overline{\tau} > 0$ *and matrix* K_j, *if there exist matrices* $P > 0$, $Q > 0$, $R_1 > 0$, $R_2 > 0$, N_{kij}, T_{kij} *and* M_k $(i, j = 1, 2, \ldots, n,\ k = 1, 2, 3, 4)$ *with compatible dimensions such that*

$$\begin{bmatrix} \Xi_{11}^{ii} & * \\ \Xi_{21}^{ii} & \Xi_{22} \end{bmatrix} < 0, \tag{16}$$

$$\begin{bmatrix} \Xi_{11}^{ij} + \Xi_{11}^{ji} & * & * \\ \Xi_{21}^{ij} & \Xi_{22} & * \\ \Xi_{21}^{ji} & 0 & \Xi_{22} \end{bmatrix} < 0, \quad 1 \leqslant i < j \leqslant n. \tag{17}$$

where $*$ denotes the transposed element in the symmetric position and

$$\Xi_{11}^{ij} = \begin{bmatrix} \Gamma_{11}^{ij} & * & * & * \\ \Gamma_{21}^{ij} & \Gamma_{22}^{ij} & * & * \\ \Gamma_{31}^{ij} & \Gamma_{32}^{ij} & \Gamma_{33}^{ij} & * \\ \Gamma_{41}^{ij} & \Gamma_{42}^{ij} & \Gamma_{43}^{ij} & \Gamma_{44}^{ij} \end{bmatrix}, \quad \Xi_{21}^{ij} = \begin{bmatrix} \tau_0 N_{1ij}^{\mathrm{T}} & \tau_0 N_{2ij}^{\mathrm{T}} & \tau_0 N_{3ij}^{\mathrm{T}} & \tau_0 N_{4ij}^{\mathrm{T}} \\ \delta T_{1ij}^{\mathrm{T}} & \delta T_{2ij}^{\mathrm{T}} & \delta T_{3ij}^{\mathrm{T}} & \delta T_{4ij}^{\mathrm{T}} \end{bmatrix},$$

$$\Xi_{22} = \begin{bmatrix} -\tau_0 R_1 & * \\ 0 & -\delta R_2 \end{bmatrix},$$

$$\begin{aligned}
\Gamma_{11}^{ij} &= Q + N_{1ij} + N_{1ij}^{\mathrm{T}} + M_1 \bar{A}_i + \bar{A}_i^{\mathrm{T}} M_1^{\mathrm{T}} + M_1 \bar{B}_i K_j + K_j^{\mathrm{T}} \bar{B}_i^{\mathrm{T}} M_1^{\mathrm{T}}, \\
\Gamma_{21}^{ij} &= N_{2ij} - T_{1ij}^{\mathrm{T}} + \bar{A}_{di}^{\mathrm{T}} M_1^{\mathrm{T}} + M_2 \bar{A}_i + M_2 \bar{B}_i K_j, \\
\Gamma_{22}^{ij} &= -T_{2ij} - T_{2ij}^{\mathrm{T}} + M_2 \bar{A}_{di} + \bar{A}_{di}^{\mathrm{T}} M_2^{\mathrm{T}}, \\
\Gamma_{31}^{ij} &= -N_{1ij}^{\mathrm{T}} + N_{3ij} + T_{1ij}^{\mathrm{T}} + M_3 \bar{A}_i + M_3 \bar{B}_i K_j, \\
\Gamma_{32}^{ij} &= -N_{2ij}^{\mathrm{T}} + T_{2ij}^{\mathrm{T}} - T_{3ij} + M_3 \bar{A}_{di}, \\
\Gamma_{33}^{ij} &= -Q - N_{3ij} - N_{3ij}^{\mathrm{T}} + T_{3ij} + T_{3ij}^{\mathrm{T}}, \\
\Gamma_{41}^{ij} &= P + N_{4ij} - M_1^{\mathrm{T}} + M_4 \bar{A}_i + M_4 \bar{B}_i K_j, \\
\Gamma_{42}^{ij} &= -T_{4ij} - M_2^{\mathrm{T}} + M_4 \bar{A}_{di}, \\
\Gamma_{43}^{ij} &= -N_{4ij} + T_{4ij} - M_3^{\mathrm{T}}, \\
\Gamma_{44}^{ij} &= \tau_0 R_1 + 2\delta R_2 - M_4 - M_4^{\mathrm{T}},
\end{aligned}$$

$1 \leqslant i < j \leqslant n$, *then system* (8) *is asymptotically stable when* $\tau(t)$ *satisfies Assumption* 1.

Proof. By using the similar method in [20], we can construct a Lyapunov–Krasovskii functional as

$$V(x_t) = V_1(x_t) + V_2(x_t), \tag{18}$$

where

$$\begin{aligned}
V_1(x_t) &= x^{\mathrm{T}}(t) P x(t) + \int_{t-\tau_0}^{t} x^{\mathrm{T}}(s) Q x(s)\,\mathrm{d}s + \int_{t-\tau_0}^{t} \int_{s}^{t} \dot{x}^{\mathrm{T}}(v) R_1 \dot{x}(v)\,\mathrm{d}v\,\mathrm{d}s, \\
V_2(x_t) &= 2\delta \int_{t-\tau_0+\delta}^{t} \dot{x}^{\mathrm{T}}(s) R_2 \dot{x}(s)\,\mathrm{d}s + \int_{t-\tau_0-\delta}^{t-\tau_0+\delta} \int_{s}^{t-\tau_0+\delta} \dot{x}^{\mathrm{T}}(v) R_2 \dot{x}(v)\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned}$$

where $P > 0$, $Q > 0$, $R_1 > 0$ and $R_2 > 0$.

Taking the derivative of $V_1(x_t)$ and $V_2(x_t)$ yields

$$\dot{V}_1(x_t) = 2x^{\mathrm{T}}(t)P\dot{x}(t) + x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau_0)Qx(t-\tau_0) + \tau_0\dot{x}^{\mathrm{T}}(t)R_1\dot{x}(t) - \int_{t-\tau_0}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)\,\mathrm{d}s, \tag{19}$$

$$\dot{V}_2(x_t) = 2\delta\dot{x}^{\mathrm{T}}(t)R_2\dot{x}(t) - \int_{t-\tau_0-\delta}^{t-\tau_0+\delta}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\,\mathrm{d}s. \tag{20}$$

With (13)–(15) and (18)–(20) we can get

$$\begin{aligned}\dot{V}(x_t) = {} & 2x^{\mathrm{T}}(t)P\dot{x}(t) + x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau_0)Qx(t-\tau_0) + \tau_0\dot{x}^{\mathrm{T}}(t)R_1\dot{x}(t) \\ & - \int_{t-\tau_0}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)\,\mathrm{d}s + 2\delta\dot{x}^{\mathrm{T}}(t)R_2\dot{x}(t) - \int_{t-\tau_0-\delta}^{t-\tau_0+\delta}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\,\mathrm{d}s \\ & + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i(z(t))\mu_j(z(t))\zeta^{\mathrm{T}}(t)N_{ij}\left[x(t) - x(t-\tau_0) - \int_{t-\tau_0}^{t}\dot{x}(s)\,\mathrm{d}s\right] \\ & + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i(z(t))\mu_j(z(t))\zeta^{\mathrm{T}}(t)T_{ij}\left[x(t-\tau_0) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_0}\dot{x}(s)\,\mathrm{d}s\right] \\ & + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i(z(t))\mu_j(z(t))\zeta^{\mathrm{T}}(t)M_{ij}[(\bar{A}_i + \bar{B}_iK_j)x(t) + \bar{A}_{di}x(t-\tau(t)) - \dot{x}(t)],\end{aligned} \tag{21}$$

where

$$\zeta^{\mathrm{T}}(t) = [x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau(t)) \quad x^{\mathrm{T}}(t-\tau_0) \quad \dot{x}^{\mathrm{T}}(t)], \tag{22}$$

$$N_{ij}^{\mathrm{T}} = [N_{1ij}^{\mathrm{T}} \;\; N_{2ij}^{\mathrm{T}} \;\; N_{3ij}^{\mathrm{T}} \;\; N_{4ij}^{\mathrm{T}}], \tag{23}$$

$$T_{ij}^{\mathrm{T}} = [T_{1ij}^{\mathrm{T}} \;\; T_{2ij}^{\mathrm{T}} \;\; T_{3ij}^{\mathrm{T}} \;\; T_{4ij}^{\mathrm{T}}], \tag{24}$$

$$M^{\mathrm{T}} = [M_1^{\mathrm{T}} \;\; M_2^{\mathrm{T}} \;\; M_3^{\mathrm{T}} \;\; M_4^{\mathrm{T}}]. \tag{25}$$

In (21), by using Lemma 1 in [21], we can easily get the following inequalities:

$$-\zeta^{\mathrm{T}}(t)N_{ij}\int_{t-\tau_0}^{t}\dot{x}(s)\,\mathrm{d}s \leqslant \tau_0\zeta^{\mathrm{T}}(t)N_{ij}R_1^{-1}N_{ij}^{\mathrm{T}}\zeta(t) + \int_{t-\tau_0}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)\,\mathrm{d}s, \tag{26}$$

$$\begin{aligned}-\zeta^{\mathrm{T}}(t)T_{ij}\int_{t-\tau(t)}^{t-\tau_0}\dot{x}(s)\,\mathrm{d}s &= \zeta^{\mathrm{T}}(t)T_{ij}\int_{t-\tau_0}^{t-\tau(t)}\dot{x}(s)\,\mathrm{d}s \\ &\leqslant \delta\zeta^{\mathrm{T}}(t)T_{ij}R_2^{-1}T_{ij}^{\mathrm{T}}\zeta(t) + \int_{t-\tau_0}^{t-\tau(t)}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\,\mathrm{d}s \\ &\leqslant \delta\zeta^{\mathrm{T}}(t)T_{ij}R_2^{-1}T_{ij}^{\mathrm{T}}\zeta(t) + \int_{t-\tau_0-\delta}^{t-\tau_0+\delta}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\,\mathrm{d}s \quad \text{as } \tau(t)\leqslant\tau_0,\end{aligned} \tag{27}$$

$$-\zeta^{\mathrm{T}}(t)T_{ij}\int_{t-\tau(t)}^{t-\tau_0}\dot{x}(s)\,\mathrm{d}s \leqslant \delta\zeta^{\mathrm{T}}(t)T_{ij}R_2^{-1}T_{ij}^{\mathrm{T}}\zeta(t)+\int_{t-\tau(t)}^{t-\tau_0}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\,\mathrm{d}s$$
$$\leqslant \delta\zeta^{\mathrm{T}}(t)T_{ij}R_2^{-1}T_{ij}^{\mathrm{T}}\zeta(t)+\int_{t-\tau_0-\delta}^{t-\tau_0+\delta}\dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)\,\mathrm{d}s \quad \text{as } \tau(t)>\tau_0. \tag{28}$$

Combining (21)–(28), we can get

$$\begin{aligned}\dot{V}(t) &\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i(z(t))\mu_j(z(t))[\zeta^{\mathrm{T}}(t)\Xi_{11}^{ij}\zeta(t)+\tau_0\zeta^{\mathrm{T}}(t)N_{ij}R_1^{-1}N_{ij}^{\mathrm{T}}\zeta(t)+\delta\zeta^{\mathrm{T}}(t)T_{ij}R_2^{-1}T_{ij}^{\mathrm{T}}\zeta(t)]\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i(z(t))\mu_j(z(t))\zeta^{\mathrm{T}}(t)[\Xi_{11}^{ij}+\tau_0N_{ij}R_1^{-1}N_{ij}^{\mathrm{T}}+\delta T_{ij}R_2^{-1}T_{ij}^{\mathrm{T}}]\zeta(t)\\
&=\sum_{i=1}^{n}\mu_i^2(z(t))\zeta^{\mathrm{T}}(t)(\Xi_{11}^{ii}+\tau_0N_{ii}R_1^{-1}N_{ii}^{\mathrm{T}}+\delta T_{ii}R_2^{-1}T_{ii}^{\mathrm{T}})\zeta(t)\\
&\quad+\sum_{i=1}^{n-1}\sum_{j>i}^{n}\mu_i(z(t))\mu_j(z(t))\zeta^{\mathrm{T}}(t)(\Xi_{11}^{ij}+\Xi_{11}^{ji}+\tau_0N_{ij}R_1^{-1}N_{ij}^{\mathrm{T}}+\tau_0N_{ji}R_1^{-1}N_{ji}^{\mathrm{T}}\\
&\quad+\delta T_{ij}R_2^{-1}T_{ij}^{\mathrm{T}}+\delta T_{ji}R_2^{-1}T_{ji}^{\mathrm{T}})\zeta(t).\end{aligned} \tag{29}$$

By using Schur complements, we can show that $\Xi_{11}^{ii}+\tau_0N_{ii}R_1^{-1}N_{ii}^{\mathrm{T}}+\delta T_{ii}R_2^{-1}T_{ii}^{\mathrm{T}}<0$ is equivalent to (16) and $\Xi_{11}^{ij}+\Xi_{11}^{ji}+\tau_0N_{ij}R_1^{-1}N_{ij}^{\mathrm{T}}+\tau_0N_{ji}R_1^{-1}N_{ji}^{\mathrm{T}}+\delta T_{ij}R_2^{-1}T_{ij}^{\mathrm{T}}+\delta T_{ji}R_2^{-1}T_{ji}^{\mathrm{T}}<0$ is equivalent to (17). Thus, (16) and (17) imply $\dot{V}(x_t)<0$, which can further imply the asymptotical stability of system (8). This completes the proof. □

The parameter uncertainties $\Delta A_i(t)$, $\Delta A_{di}(t)$ and $\Delta B_i(t)$ are contained in (16) and (17). So Lemma 1 cannot be directly used to determine the stability of closed-loop system (8). Then the following result is given to provide a sufficient condition for the asymptotical stability of system (8).

Theorem 1. *For given scalars* $\underline{\tau}>0, \bar{\tau}>0$ *and matrix* K_j, *if there exist matrices* $P>0$, $Q>0$, $R_1>0$, $R_2>0$, N_{kij}, T_{kij} *and* M_k $(i,j=1,2,\ldots,n,\ k=1,2,3,4)$ *with compatible dimensions and scalars* $\varepsilon_1,\varepsilon_2$ *and* ε_3 *such that*

$$\begin{bmatrix}\hat{\Xi}_{11}^{ii} & * & *\\ \Xi_{21}^{ii} & \Xi_{22} & *\\ \Xi_{31}^{i} & 0 & \Xi_{33}\end{bmatrix}<0, \tag{30}$$

$$\begin{bmatrix} \hat{\Xi}_{11}^{ij} + \hat{\Xi}_{11}^{ji} & * & * & * & * \\ \Xi_{21}^{ij} & \Xi_{22} & * & * & * \\ \Xi_{21}^{ji} & 0 & \Xi_{22} & * & * \\ \Xi_{31}^{i} & 0 & 0 & \Xi_{33} & * \\ \Xi_{31}^{j} & 0 & 0 & 0 & \Xi_{33} \end{bmatrix} < 0, \quad 1 \leqslant i < j \leqslant n, \tag{31}$$

where

$$\hat{\Xi}_{11}^{ij} = \begin{bmatrix} \hat{\Gamma}_{11}^{ij} & * & * & * \\ \hat{\Gamma}_{21}^{ij} & \hat{\Gamma}_{22}^{ij} & * & * \\ \hat{\Gamma}_{31}^{ij} & \hat{\Gamma}_{32}^{ij} & \hat{\Gamma}_{32}^{ij} & * \\ \hat{\Gamma}_{41}^{ij} & \hat{\Gamma}_{42}^{ij} & \hat{\Gamma}_{43}^{ij} & \hat{\Gamma}_{44}^{ij} \end{bmatrix}, \quad \Xi_{31}^{i} = \begin{bmatrix} D_{ai}^{\mathrm{T}} M_1^{\mathrm{T}} & D_{ai}^{\mathrm{T}} M_2^{\mathrm{T}} & D_{ai}^{\mathrm{T}} M_3^{\mathrm{T}} & D_{ai}^{\mathrm{T}} M_4^{\mathrm{T}} \\ D_{adi}^{\mathrm{T}} M_1^{\mathrm{T}} & D_{adi}^{\mathrm{T}} M_2^{\mathrm{T}} & D_{adi}^{\mathrm{T}} M_3^{\mathrm{T}} & D_{adi}^{\mathrm{T}} M_4^{\mathrm{T}} \\ D_{bi}^{\mathrm{T}} M_1^{\mathrm{T}} & D_{bi}^{\mathrm{T}} M_2^{\mathrm{T}} & D_{bi}^{\mathrm{T}} M_3^{\mathrm{T}} & D_{bi}^{\mathrm{T}} M_4^{\mathrm{T}} \end{bmatrix},$$

$$\Xi_{33} = \begin{bmatrix} -\varepsilon_1 I & * & * \\ 0 & -\varepsilon_2 I & * \\ 0 & 0 & -\varepsilon_3 I \end{bmatrix},$$

Ξ_{22} *is as given in Lemma* 1 *and*

$$\begin{aligned}
\hat{\Gamma}_{11}^{ij} &= Q + N_{1ij} + N_{1ij}^{\mathrm{T}} + M_1 A_i + A_i^{\mathrm{T}} M_1^{\mathrm{T}} + M_1 B_i K_j + K_j^{\mathrm{T}} B_i^{\mathrm{T}} M_1^{\mathrm{T}} \\
&\quad + \varepsilon_1 E_{ai}^{\mathrm{T}} E_{ai} + \varepsilon_3 K_j^{\mathrm{T}} E_{bi}^{\mathrm{T}} E_{bi} K_j, \\
\hat{\Gamma}_{21}^{ij} &= N_{2ij} - T_{1ij}^{\mathrm{T}} + A_{di}^{\mathrm{T}} M_1^{\mathrm{T}} + M_2 A_i + M_2 B_i K_j, \\
\hat{\Gamma}_{22}^{ij} &= -T_{2ij} - T_{2ij}^{\mathrm{T}} + M_2 A_{di} + A_{di}^{\mathrm{T}} M_2^{\mathrm{T}} + \varepsilon_2 E_{adi}^{\mathrm{T}} E_{adi}, \\
\hat{\Gamma}_{31}^{ij} &= -N_{1ij}^{\mathrm{T}} + N_{3ij} + T_{1ij}^{\mathrm{T}} + M_3 A_i + M_3 B_i K_j, \\
\hat{\Gamma}_{32}^{ij} &= -N_{2ij}^{\mathrm{T}} + T_{2ij}^{\mathrm{T}} - T_{3ij} + M_3 A_{di}, \\
\hat{\Gamma}_{33}^{ij} &= -Q - N_{3ij} - N_{3ij}^{\mathrm{T}} + T_{3ij} + T_{3ij}^{\mathrm{T}}, \\
\hat{\Gamma}_{41}^{ij} &= P + N_{4ij} - M_1^{\mathrm{T}} + M_4 A_i + M_4 B_i K_j, \\
\hat{\Gamma}_{42}^{ij} &= -T_{4ij} - M_2^{\mathrm{T}} + M_4 A_{di}, \\
\hat{\Gamma}_{43}^{ij} &= -N_{4ij} + T_{4ij} - M_3^{\mathrm{T}}, \\
\hat{\Gamma}_{44}^{ij} &= \tau_0 R_1 + 2\delta R_2 - M_4 - M_4^{\mathrm{T}},
\end{aligned}$$

$1 \leqslant i < j \leqslant n$, *then system* (8) *is asymptotically stable when* $\tau(t)$ *satisfies Assumption* 1.

Proof. Replace $A_i + \Delta A_i(t)$ with $\bar{A}_i$, $A_{di} + \Delta A_{di}(t)$ with $\bar{A}_{di}$ and $B_i + \Delta B_i(t)$ with $\bar{B}_i$ in (16) and (17), combine (2) and (3) and use Schur complements, we can obtain (30) and (31). □

For the case of time-invariant delay, we can get the following corollary based on Theorem 1.

Corollary 1. *For given scalar* $\tau > 0$ *and matrix* K_j, *if there exist matrices* $P > 0, Q > 0, R > 0, N_{kij}, T_{kij}$ *and* M_k $(i, j = 1, 2, \dots, n, k = 1, 2, 3)$ *with compatible dimensions and scalars* $\varepsilon_1, \varepsilon_2$ *and* ε_3 *such that*

$$\begin{bmatrix} \Omega_{11}^{ii} & * & * \\ \Omega_{21}^{ii} & -\tau R & * \\ \Xi_{31}^{i} & 0 & \Xi_{33} \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} \Omega_{11}^{ij} + \Omega_{11}^{ji} & * & * & * & * \\ \Omega_{21}^{ij} & -\tau R & * & * & * \\ \Omega_{21}^{ji} & 0 & -\tau R & * & * \\ \Xi_{31}^{i} & 0 & 0 & \Xi_{33} & * \\ \Xi_{31}^{j} & 0 & 0 & 0 & \Xi_{33} \end{bmatrix} < 0, \quad 1 \leqslant i < j \leqslant n, \tag{33}$$

where

$$\Omega_{11}^{ij} = \begin{bmatrix} \Pi_{11}^{ij} & * & * \\ \Pi_{21}^{ij} & \Pi_{22}^{ij} & * \\ \Pi_{31}^{ij} & \Pi_{32}^{ij} & \Pi_{33}^{ij} \end{bmatrix}, \quad \Omega_{21}^{ij} = [\tau N_{1ij}^{\mathrm{T}} \;\; \tau N_{2ij}^{\mathrm{T}} \;\; \tau N_{3ij}^{\mathrm{T}}], \quad \Xi_{31}^{i}, \Xi_{31}^{j}, \Xi_{33}$$

are as given before and

$$\begin{aligned}
\Pi_{11}^{ij} &= Q + N_{1ij} + N_{1ij}^{\mathrm{T}} + M_1 A_i + A_i^{\mathrm{T}} M_1^{\mathrm{T}} + M_1 B_i K_j + K_j^{\mathrm{T}} B_i^{\mathrm{T}} M_1^{\mathrm{T}} + \varepsilon_1 E_{ai}^{\mathrm{T}} E_{ai} \\
&\quad + \varepsilon_3 K_j^{\mathrm{T}} E_{bi}^{\mathrm{T}} E_{bi} K_j, \\
\Pi_{21}^{ij} &= N_{2ij} - N_{1ij}^{\mathrm{T}} + A_{di}^{\mathrm{T}} M_1^{\mathrm{T}} + M_2 A_i + M_2 B_i K_j, \\
\Pi_{22}^{ij} &= -Q - N_{2ij} - N_{2ij}^{\mathrm{T}} + M_2 A_{di} + A_{di}^{\mathrm{T}} M_2^{\mathrm{T}} + \varepsilon_1 E_{adi}^{\mathrm{T}} E_{adi}, \\
\Pi_{31}^{ij} &= P + N_{3ij} + M_3 A_i + M_3 B_i K_j - M_1^{\mathrm{T}}, \\
\Pi_{32}^{ij} &= -N_{3ij} + M_3 A_{di} - M_2^{\mathrm{T}}, \\
\Pi_{33}^{ij} &= -M_3 - M_3^{\mathrm{T}} + \tau R,
\end{aligned}$$

$1 \leqslant i < j \leqslant n$, *then system* (8) *with time-invariant delay is asymptotically stable*.

In terms of Theorem 1, we are now in a position to design the feedback gain K_j, which can guarantee the asymptotical stability of the closed-loop system (8).

Theorem 2. *For given scalars* $\underline{\tau} > 0, \bar{\tau} > 0$ *and* $\rho_i (i = 2, 3, 4)$, $\rho_4 \neq 0$, *if there exist matrices* $\tilde{P} > 0, \tilde{Q} > 0, \tilde{R}_1 > 0, \tilde{R}_2 > 0, \tilde{N}_{kij}, \tilde{T}_{kij}, X$ *and* $Y_j (i, j = 1, 2, \dots, n, k = 1, 2, 3, 4)$ *with appropriate* *dimensions and scalars* μ_1, μ_2 *and* μ_3 *such that*

$$\begin{bmatrix} \tilde{\Xi}_{11}^{ii} & * & * \\ \tilde{\Xi}_{21}^{ii} & \tilde{\Xi}_{22} & * \\ \Xi_{41}^{ii} & 0 & \Xi_{44} \end{bmatrix} < 0, \tag{34}$$

$$\begin{bmatrix} \tilde{\Xi}_{11}^{ij} + \tilde{\Xi}_{11}^{ji} & * & * & * & * \\ \tilde{\Xi}_{21}^{ij} & \tilde{\Xi}_{22} & * & * & * \\ \tilde{\Xi}_{21}^{ji} & 0 & \tilde{\Xi}_{22} & * & * \\ \Xi_{41}^{ij} & 0 & 0 & \Xi_{44} & * \\ \Xi_{41}^{ji} & 0 & 0 & 0 & \Xi_{44} \end{bmatrix} < 0, \quad 1 \leqslant i < j \leqslant n, \tag{35}$$

where

$$\tilde{\Xi}_{11}^{ij} = \begin{bmatrix} \tilde{\Gamma}_{11}^{ij} & * & * & * \\ \tilde{\Gamma}_{21}^{ij} & \tilde{\Gamma}_{22}^{ij} & * & * \\ \tilde{\Gamma}_{31}^{ij} & \tilde{\Gamma}_{32}^{ij} & \tilde{\Gamma}_{32}^{ij} & * \\ \tilde{\Gamma}_{41}^{ij} & \tilde{\Gamma}_{42}^{ij} & \tilde{\Gamma}_{43}^{ij} & \tilde{\Gamma}_{44}^{ij} \end{bmatrix}, \quad \tilde{\Xi}_{21}^{ij} = \begin{bmatrix} \tau_0 \tilde{N}_{1ii}^{\mathrm{T}} & \tau_0 \tilde{N}_{2ii}^{\mathrm{T}} & \tau_0 \tilde{N}_{3ii}^{\mathrm{T}} & \tau_0 \tilde{N}_{4ii}^{\mathrm{T}} \\ \delta \tilde{T}_{1ii}^{\mathrm{T}} & \delta \tilde{T}_{2ii}^{\mathrm{T}} & \delta \tilde{T}_{3ii}^{\mathrm{T}} & \delta \tilde{T}_{4ii}^{\mathrm{T}} \end{bmatrix},$$

$$\tilde{\Xi}_{22} = \begin{bmatrix} -\tau_0 \tilde{R}_1 & * \\ 0 & -\delta \tilde{R}_2 \end{bmatrix},$$

$$\Xi_{41}^{ij} = \begin{bmatrix} E_{ai} X^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & E_{adi} X^{\mathrm{T}} & 0 & 0 \\ E_{bi} Y_j & 0 & 0 & 0 \end{bmatrix}, \quad \Xi_{44} = \begin{bmatrix} -\mu_1 I & * & * \\ 0 & -\mu_2 I & * \\ 0 & 0 & -\mu_3 I \end{bmatrix}$$

and

$$\begin{aligned}
\tilde{\Gamma}_{11}^{ij} &= \tilde{Q} + \tilde{N}_{1ij} + \tilde{N}_{1ij}^{\mathrm{T}} + A_i X^{\mathrm{T}} + X A_i^{\mathrm{T}} + B_i Y_j + Y_j^{\mathrm{T}} B_i^{\mathrm{T}} + \mu_1 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 D_{adi} D_{adi}^{\mathrm{T}} \\
&\quad + \mu_3 D_{bi} D_{bi}^{\mathrm{T}}, \\
\tilde{\Gamma}_{21}^{ij} &= \tilde{N}_{2ij} - \tilde{T}_{1ij}^{\mathrm{T}} + X A_{di}^{\mathrm{T}} + \rho_2 A_i X^{\mathrm{T}} + \rho_2 B_i Y_j + \mu_1 \rho_2 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_2 D_{adi} D_{adi}^{\mathrm{T}} \\
&\quad + \mu_3 \rho_2 D_{bi} D_{bi}^{\mathrm{T}}, \\
\tilde{\Gamma}_{22}^{ij} &= -\tilde{T}_{2ij} - \tilde{T}_{2ij}^{\mathrm{T}} + \rho_2 X A_{di}^{\mathrm{T}} + \rho_2 A_{di} X^{\mathrm{T}} + \mu_1 \rho_2^2 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_2^2 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_2^2 D_{bi} D_{bi}^{\mathrm{T}}, \\
\tilde{\Gamma}_{31}^{ij} &= -\tilde{N}_{1ij}^{\mathrm{T}} + \tilde{N}_{3ij} + \tilde{T}_{1ij}^{\mathrm{T}} + \rho_3 A_i X^{\mathrm{T}} + \rho_3 B_i Y_j + \mu_1 \rho_3 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_3 D_{adi} D_{adi}^{\mathrm{T}} \\
&\quad + \mu_3 \rho_3 D_{bi} D_{bi}^{\mathrm{T}}, \\
\tilde{\Gamma}_{32}^{ij} &= -\tilde{N}_{2ij}^{\mathrm{T}} + \tilde{T}_{2ij}^{\mathrm{T}} - \tilde{T}_{3ij} + \rho_3 A_{di} X^{\mathrm{T}} + \mu_1 \rho_2 \rho_3 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_2 \rho_3 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_2 \rho_3 D_{bi} D_{bi}^{\mathrm{T}}, \\
\tilde{\Gamma}_{33}^{ij} &= -\tilde{Q} - \tilde{N}_{3ij} - \tilde{N}_{3ij}^{\mathrm{T}} + \tilde{T}_{3ij} + \tilde{T}_{3ij}^{\mathrm{T}} + \mu_1 \rho_3^2 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_3^2 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_3^2 D_{bi} D_{bi}^{\mathrm{T}},
\end{aligned}$$

$$\tilde{\Gamma}_{41}^{ij} = \tilde{P} + \tilde{N}_{4ij} - X + \rho_4 A_i X^{\mathrm{T}} + \rho_4 B_i Y_j + \mu_1 \rho_4 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_4 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_4 D_{bi} D_{bi}^{\mathrm{T}},$$
$$\tilde{\Gamma}_{42}^{ij} = -\tilde{T}_{4ij} - \rho_2 X + \rho_4 A_{di} X^{\mathrm{T}} + \mu_1 \rho_2 \rho_4 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_2 \rho_4 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_2 \rho_4 D_{bi} D_{bi}^{\mathrm{T}},$$
$$\tilde{\Gamma}_{43}^{ij} = -\tilde{N}_{4ij} + \tilde{T}_{4ij} - \rho_3 X + \mu_1 \rho_3 \rho_4 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_3 \rho_4 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_3 \rho_4 D_{bi} D_{bi}^{\mathrm{T}},$$
$$\tilde{\Gamma}_{44}^{ij} = \tau_0 \tilde{R}_1 + 2\delta \tilde{R}_2 - \rho_4 X - \rho_4 X^{\mathrm{T}} + \mu_1 \rho_4^2 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_4^2 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_4^2 D_{bi} D_{bi}^{\mathrm{T}},$$

$1 \leqslant i < j \leqslant n$, *then system* (8) *with the control law* $u(t) = \sum_{i=1}^{n} \mu_i(z(t)) K_i x(t)$ *is asymptotically stable when* $\tau(t)$ *satisfies Assumption* 1.

Proof. Denote $M_2 = \rho_2 M_1$, $M_3 = \rho_3 M_1$, $M_4 = \rho_4 M_1$, so we can see $\rho_4 \neq 0$ and M_1 is nonsingular from (30) and (31). Pre and post-multiplying both side of (30) with $diag(X\ X\ X\ X\ X\ X\ I\ I\ I)$ and both side of (17) with $diag(X\ X\ X\ X\ X\ X\ X\ I\ I\ I\ I\ I\ I)$ and their transpose, respectively, defining new variables $X = M_1^{-1}$, $\tilde{P} = XPX^{\mathrm{T}}$, $\tilde{Q} = XQX^{\mathrm{T}}$, $\tilde{R}_1 = XR_1X^{\mathrm{T}}$, $\tilde{R}_2 = XR_2X^{\mathrm{T}}$, $\tilde{N}_{kij} = XN_{kij}X^{\mathrm{T}}$, $Y_j = K_jX^{\mathrm{T}}$, $\tilde{T}_{kij} = XT_{kij}X^{\mathrm{T}}(k = 1, 2, 3, 4, i, j = 1, 2, \ldots, n)$ and $\mu_i = \varepsilon_i^{-1}(i = 1, 2, 3)$, we can obtain (34) and (35), respectively, by using Schur complements. It is easy to see that (30) and (31), respectively, imply (34) and (35). Therefore, in terms of Theorem 1, we can complete the proof. □

Similarly, we can obtain the following result based on Corollary 1.

Corollary 2. *For given scalars* $\tau > 0$, $\rho_i (i = 2, 3)$, $\rho_3 \neq 0$, *if there exist matrices* $\tilde{P} > 0$, $\tilde{Q} > 0$, $\tilde{R} > 0$, $\tilde{N}_{kij}$, $\tilde{T}_{kij}$, X *and* $Y_j (i, j = 1, 2, \ldots, n, k = 1, 2, 3)$ *with appropriate dimensions and scalars* μ_1, μ_2 *and* μ_3 *such that*

$$\begin{bmatrix} \tilde{\Omega}_{11}^{ii} & * & * \\ \tilde{\Omega}_{21}^{ii} & -\tau\tilde{R} & * \\ \Xi_{41}^{ii} & 0 & \Xi_{44} \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} \tilde{\Omega}_{11}^{ij} + \tilde{\Omega}_{11}^{ji} & * & * & * & * \\ \tilde{\Omega}_{21}^{ij} & -\tau\tilde{R} & * & * & * \\ \tilde{\Omega}_{21}^{ji} & 0 & -\tau\tilde{R} & * & * \\ \Xi_{41}^{ij} & 0 & 0 & \Xi_{44} & * \\ \Xi_{41}^{ji} & 0 & 0 & 0 & \Xi_{44} \end{bmatrix} < 0, \quad 1 \leqslant i < j \leqslant n, \tag{37}$$

where

$$\tilde{\Omega}_{11}^{ij} = \begin{bmatrix} \tilde{\Pi}_{11}^{ij} & * & * \\ \tilde{\Pi}_{21}^{ij} & \tilde{\Pi}_{22}^{ij} & * \\ \tilde{\Pi}_{31}^{ij} & \tilde{\Pi}_{32}^{ij} & \tilde{\Pi}_{33}^{ij} \end{bmatrix}, \quad \tilde{\Omega}_{21}^{ij} = [\tau\tilde{N}_{1ij}^{\mathrm{T}}\ \ \tau\tilde{N}_{2ij}^{\mathrm{T}}\ \ \tau\tilde{N}_{3ij}^{\mathrm{T}}], \quad \Xi_{41}^{ij}, \Xi_{44}$$

are as given in Theorem 2 *and*

$$\begin{aligned}
\tilde{\Pi}_{11}^{ij} &= \tilde{Q} + \tilde{N}_{1ij} + \tilde{N}_{1ij}^{\mathrm{T}} + A_i X^{\mathrm{T}} + X A_i^{\mathrm{T}} + B_i Y_j + Y_j^{\mathrm{T}} B_i^{\mathrm{T}} + \mu_1 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 D_{adi} D_{adi}^{\mathrm{T}} \\
&\quad + \mu_3 D_{bi} D_{bi}^{\mathrm{T}}, \\
\tilde{\Pi}_{21}^{ij} &= \tilde{N}_{2ij} - \tilde{N}_{1ij}^{\mathrm{T}} + X A_{di}^{\mathrm{T}} + \rho_2 A_i X^{\mathrm{T}} + \rho_2 B_i Y_j + \mu_1 \rho_2 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_2 D_{adi} D_{adi}^{\mathrm{T}} \\
&\quad + \mu_3 \rho_2 D_{bi} D_{bi}^{\mathrm{T}}, \\
\Pi_{22}^{ij} &= -\tilde{Q} - \tilde{N}_{2ij} - \tilde{N}_{2ij}^{\mathrm{T}} + \rho_2 A_{di} X^{\mathrm{T}} + \rho_2 X A_{di}^{\mathrm{T}} + \mu_1 \rho_2^2 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_2^2 D_{adi} D_{adi}^{\mathrm{T}} \\
&\quad + \mu_3 \rho_2^2 D_{bi} D_{bi}^{\mathrm{T}}, \\
\Pi_{31}^{ij} &= \tilde{P} + \tilde{N}_{3ij} + \rho_3 A_i X^{\mathrm{T}} + \rho_3 B_i Y_j - X + \mu_1 \rho_3 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_3 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_3 D_{bi} D_{bi}^{\mathrm{T}}, \\
\Pi_{32}^{ij} &= -\tilde{N}_{3ij} + \rho_3 A_{di} X^{\mathrm{T}} - \rho_2 X + \mu_1 \rho_2 \rho_3 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_2 \rho_3 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_2 \rho_3 D_{bi} D_{bi}^{\mathrm{T}}, \\
\Pi_{33}^{ij} &= -\rho_3 X - \rho_3 X + \tau \tilde{R} + \mu_1 \rho_3^2 D_{ai} D_{ai}^{\mathrm{T}} + \mu_2 \rho_3^2 D_{adi} D_{adi}^{\mathrm{T}} + \mu_3 \rho_3^2 D_{bi} D_{bi}^{\mathrm{T}},
\end{aligned}$$

$1 \leqslant i < j \leqslant n$, *then system* (8) *with the control law* $u(t) = \sum_{i=1}^{n} \mu_i(z(t)) K_i x(t)$ *is asymptotically stable.*

Remark 2. From the process of the proofs of Theorems 1 and 2, it can be easily found that the information of time-derivative of time delay $\tau(t)$ is not used. That is to say, there is no supplementary requirement for $\dot{\tau}(t)$, which means that our method can deal with the systems with any fast time-varying delay case.

In the following, we will give an algorithm for Theorem 2 to get the maximum $\bar{\tau}$.

Algorithm 1. First, for given $\underline{\tau}$, replace τ_0 and δ with $\frac{1}{2}(\bar{\tau} + \underline{\tau})$ and $\frac{1}{2}(\bar{\tau} - \underline{\tau})$ in (34) and (35), then using the following steps, the maximum $\bar{\tau}_{\max}$ and the corresponding feedback gain $K_j\,(j = 1, 2, \ldots, n)$ can be obtained.

Step 1: Set the variable range of ρ_k as $-\bar{\rho}_k \leqslant \rho_k \leqslant \bar{\rho}_k$, $\bar{\rho}_k > 0(k = 2, 3, 4)$ and choose a small constant ε as the step. Find the maximum allowable value of $\bar{\tau}$ satisfying (34), (35) and solve the feedback gain $K_j = Y_j X^{-\mathrm{T}}(j = 1, 2, \ldots, n)$. Set $\bar{\tau}^* = \bar{\tau}$ and $K_j^* = K_j$.
Step 2: Set $\rho_2 = -\bar{\rho}_2$, $\rho_3 = -\bar{\rho}_3$, $\rho_4 = -\bar{\rho}_4 + \varepsilon$. If $\rho_4 > \bar{\rho}_4$, go to Step 3. Otherwise, find the maximum allowable value of $\bar{\tau}$ satisfying (34), (35) and solve the feedback gain $K_j = Y_j X^{-\mathrm{T}}$. If $\bar{\tau} > \bar{\tau}^*$, set $\bar{\tau}^* = \bar{\tau}$, $K_j^* = K_j$, $\rho_k^* = \rho_k$. Repeats Step 2 until $\rho_4 > \bar{\rho}_4$.
Step 3: Set $\rho_2 = -\bar{\rho}_2, \rho_3 = -\bar{\rho}_3, +\varepsilon$, $\rho_4 = -\bar{\rho}_4$, if $\rho_3 > \bar{\rho}_3$, go to Step 4. Otherwise, go to Step 2.
Step 4: Set $\rho_2 = -\bar{\rho}_2 + \varepsilon, \rho_3 = -\bar{\rho}_3$, $\rho_4 = -\bar{\rho}_4$, if $\rho_2 > \bar{\rho}_2$, go to Step 5. Otherwise, go to Step 2.
Step 5: Set $\bar{\tau}_{\max} = \bar{\tau}^*$, $K_j = K_j^*$, output $\bar{\tau}_{\max}$, K_j and ρ_k^*, then stop. Then $\bar{\tau}_{\max}$ is the maximum $\bar{\tau}$.

Remark 3. From Algorithm 1 we can see that when $\rho_k = \rho_k^*(k = 2, 3, 4)$, we can get the local optimal value of $\bar{\tau}_{\max}$ for the searching interval $[-\bar{\rho}_k, \bar{\rho}_k]$. Obviously, if $\bar{\rho}_k$ are given bigger, the searching range of $[-\bar{\rho}_k, \bar{\rho}_k]$ are larger, then we are more likely to get a bigger $\bar{\tau}_{\max}$, but it can increase the computation at the same time.

4. Numerical examples

Example 1. Consider a system with the following rules

Rule 1: If $x_1(t)$ is W_1, then

$$\dot{x}(t) = A_1 x(t) + A_{d1} x(t - \tau(t)). \tag{38}$$

Rule 2: If $x_1(t)$ is W_2, then

$$\dot{x}(t) = A_2 x(t) + A_{d2} x(t - \tau(t)) \tag{39}$$

and the membership functions for rule 1 and rule 2 are

$$\begin{aligned} \mu_1(z(t)) &= \frac{1}{1+\exp(-2x_1(t))}, \\ \mu_2(z(t)) &= 1 - \mu_1(z(t)), \end{aligned} \tag{40}$$

where A_i and $A_{di}(i = 1, 2)$ are given as Example 1 in [19]:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.$$

For the constant delay case, i.e., $\tau(t) = \tau$, by using Corollary 1 in [10], we can get the upper bound of τ is 1.00. Setting $\bar{B}_i = 0$ and using Corollary 1 in our paper, we can obtain the upper bound of τ as $\overline{\tau}_{\max} = 1.597$. Obviously, our result is less conservative than that obtained by the method in [10].

When the time delay is fast time-varying, by using Theorem 1 in our paper, we can get the results as shown in Table 1.

Example 2. Consider a fuzzy system with time-delay

$$\begin{array}{lll} \text{Rule 1:} & \text{If} & (x_2(t)/0.5) \text{ is about } 0, \\ & \text{then} & \dot{x}(t) = (A_1 + \Delta A_1)x(t) + (A_{d1} + \Delta A_{d1})x(t - \tau(t)) + B_1 u(t); \\ \text{Rule 2:} & \text{If} & (x_2(t)/0.5) \text{ is about } \pi \text{ or } -\pi, \\ & \text{then} & \dot{x}(t) = (A_2 + \Delta A_2)x(t) + (A_{d2} + \Delta A_{d2})x(t - \tau(t)) + B_2 u(t), \end{array} \tag{41}$$

where

$$\begin{aligned} &A_1 = \begin{bmatrix} 0 & 1 \\ 0.1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0.1 & -0.5 - 1.5\beta \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ &A_{d1} = A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, \quad \beta = \frac{0.01}{\pi}, \quad D_{ai} = D_{adi} = \begin{bmatrix} -0.03 & 0 \\ 0 & 0.03 \end{bmatrix}, \\ &E_{ai} = \begin{bmatrix} -0.15 & 0.2 \\ 0 & 0.04 \end{bmatrix}, \quad E_{adi} = \begin{bmatrix} -0.05 & -0.35 \\ 0.08 & -0.45 \end{bmatrix}, \ i = 1, 2. \end{aligned}$$

and β is used to avoid system matrices being singular.

Table 1

$\underline{\tau}$	0	0.4	0.8	1.0	1.2
$\bar{\tau}_{\max}$	0.721	0.883	1.093	1.211	1.336

Table 2

$\underline{\tau}$	$\bar{\tau}_{\max}$	Feedback gain K_1	Feedback gain K_2
0	7.0355	$[-13.9297 \quad -54.9242]$	$[-13.9297 \quad -55.9468]$
0.5	7.5354	$[-18.2104 \quad -55.0267]$	$[-18.2104 \quad -56.0492]$
1	8.0354	$[-21.8139 \quad -55.0597]$	$[-21.8139 \quad -56.1183]$

Table 3

$\underline{\tau}$	$\bar{\tau}_{\max}$	Feedback gain K_1	Feedback gain K_2
0	6.3	$[-5.2618 \quad -21.8583]$	$[-5.2618 \quad -23.3535]$
0.5	6.8	$[-7.7011 \quad -27.8193]$	$[-7.7011 \quad -27.8193]$
1	7.3	$[-11.5810 \quad -35.2849]$	$[-11.5810 \quad -36.7801]$

The membership functions are set as Example 1 in [7],

$$\begin{aligned}
&\mu_1(z(t)) = \left(1 - \frac{1}{1+\exp\{-3((x_2/0.5)-(\pi/2))\}}\right) \times \frac{1}{1+\exp\{-3((x_2/0.5)+(\pi/2))\}},\\
&\mu_2(z(t)) = 1 - \mu_1(z(t)).
\end{aligned} \tag{42}$$

When $\tau(t)$ is time-invariant and $\Delta A_i = \Delta A_{di} = 0$, (41) is just the case of Example 1 in [7]. The maximal allowable value of time delay obtained in [7] is 3.7836.

Using Algorithm 1 and Theorem 2 with $\Delta A_i = \Delta A_{di} = 0$ and choosing $\rho_2 = 0.1$, $\rho_3 = -0.1$, $\rho_3 = 14$, we can get the results for different $\underline{\tau}$, as shown in Table 2.

From the above computation, it can be seen that our results are much less conservative than those in [7] even for the case when $\tau(t)$ is fast time-varying delay. When the parameter uncertainties are concerned, we can get the result with $\rho_2 = 0.1$, $\rho_3 = -0.1$ and $\rho_3 = 14$, as shown in Table 3.

5. Conclusion

In this paper, we have investigated the delay-dependent stability and controller design problems of uncertain nonlinear systems with time-varying delay via T–S fuzzy modeling. A new method for controller design has been provided by introducing some free-weighing matrices and employing the lower bound of the time-varying delay $\tau(t)$. The maximum allowable value of the time delay and the feedback gain can be obtained by solving a set of linear matrix inequalities. Two numerical examples have shown that our results are less conservative than the existing ones.

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