A Note on the *abc* Conjecture

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Abstract

In this paper we have generalized the *abc* conjecture for integers and extended Mason's theorem to polynomials in $\mathbb{C}^m$ by use of Nevanlinna's value distribution theory. © 2002 Wiley Periodicals, Inc.

1 Introduction

Mason [12, 13, 14] started a recent trend of studies by discovering a new relation among polynomials as follows: Let $f(z)$ be a polynomial with coefficients in an algebraically closed field κ of characteristic 0, and let $\overline{n}(1/f)$ denote the number of distinct zeros of f.

THEOREM 1.1 (Mason's Theorem) [11] *Let $a(z)$, $b(z)$, and $c(z)$ be relatively prime polynomials in κ, and not all constants, such that $a + b = c$. Then*

$$\max\{\deg(a), \deg(b), \deg(c)\} \leq \overline{n}\left(\frac{1}{abc}\right) - 1 .$$

Let a be a nonzero integer. Define the radical of a to be the product of the distinct prime factors of a:

$$\overline{n}\left(\frac{1}{a}\right) = \prod_{p|a} p .$$

There is a classical analogy between polynomials and integers. Although Osgood [17] did notice a similarity between the two in the Nevanlinna defect relation and in Roth's theorem, Vojta gave a much deeper analysis of the situation and compared the theory of heights in number theory with the second main theorem of Nevanlinna theory. Under that analogy, $\overline{n}(1/f)$ of a polynomial f corresponds to $\log \overline{n}(1/a)$ of an integer a. Influenced by Mason's theorem and considerations of Szpiro and Frey, Masser and Oesterlé formulated the so-called *abc* conjecture for integers as follows:

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CONJECTURE 1.2 *Given $\varepsilon > 0$, there exists a number $C(\varepsilon)$ having the following property: For any nonzero relatively prime integers a, b, and c such that $a+b=c$,*

$$\max\{|a|, |b|, |c|\} \leq C(\varepsilon)\overline{n}\left(\frac{1}{abc}\right)^{1+\varepsilon}.$$

This conjecture is a consequence of Vojta's conjecture [23]. In this paper, we first extend Mason's theorem as follows:

THEOREM 1.3 *Let $f_1, f_2, \ldots, f_n$ $(n \geq 2)$ be linearly independent polynomials in $\mathbb{C}^m$. Put $f_0 = f_1 + f_2 + \cdots + f_n$ and assume that* $\dim I \leq m-2$, *where*

$$I = \left\{z \in \mathbb{C}^m : f_0(z) = f_1(z) = \cdots = f_n(z) = 0\right\}.$$

Then the inequalities

$$\max_{0\leq j\leq n}\{\deg(f_j)\} \leq \sum_{k=0}^{n} n_w\left(\frac{1}{f_k}\right) - l\,, \tag{1.1}$$

$$\max_{0\leq j\leq n}\{\deg(f_j)\} \leq n_l\left(\frac{1}{f_0 f_1 \cdots f_n}\right) - l\,, \tag{1.2}$$

hold, where l and w denote the index and the Wronskian degree of $f_1, f_2, \ldots, f_n$, respectively.

For the symbols and notation, please refer to Section 2. When $m = 1$ and $k = 2$, our result reduces to Mason's theorem. A non-Archimedean version of Theorem 1.3 with a stronger hypothesis is given by Hu and Yang [9].

Let a be a nonzero integer. Then

$$a = \pm p_1^{i_1} p_2^{i_2} \cdots p_s^{i_s} \tag{1.3}$$

holds for distinct primes $p_1, p_2, \ldots, p_s$ and positive integers $i_1, i_2, \ldots, i_s$. Define

$$n_k\left(\frac{1}{a}\right) = \prod_{\nu=1}^{s} p_\nu^{\min\{i_\nu, k\}}.$$

Then Theorem 1.3 can be translated into a problem in number theory.

CONJECTURE 1.4 *Let a_j $(j = 0, 1, \ldots, n)$ be nonzero integers with $n \geq 2$ such that the greatest common divisor of $a_0, a_1, \ldots, a_n$ is 1,*

$$a_1 + a_2 + \cdots + a_n = a_0\,, \tag{1.4}$$

and no proper subsum of the left-hand side of (1.4) *is equal to* 0. *Then for $\varepsilon > 0$, there exists a number $C = C(n, \varepsilon)$ such that*

$$\max_{0\leq j\leq n}\{|a_j|\} \leq C\left(\prod_{i=0}^{n} n_w\left(\frac{1}{a_i}\right)\right)^{1+\varepsilon}, \tag{1.5}$$

$$\max_{0\leq j\leq n}\{|a_j|\} \leq Cn_l\left(\frac{1}{a_0 a_1 \cdots a_n}\right)^{1+\varepsilon} \tag{1.6}$$

hold for $w = n-1$ and $l = n(n-1)/2$.

If $n = 2$, this corresponds to the well-known *abc* conjecture. Some special cases of Conjecture 1.4 are given in [8, 9, 10]. Moreover, as noted by Cherry [4], Conjecture 1.4 might have some relation to the following conjecture due to Votja [24] (or see [23]).

CONJECTURE 1.5 *Let X be a smooth complete variety over a global field k of characteristic 0, let D be a normal crossing divisor on X, let $\mathcal{K}$ denote the canonical line sheaf on X, let $\mathcal{A}$ be a big line sheaf on X, let $r \in \mathbb{Z}_{>0}$, and let $\epsilon > 0$. Then there exists a proper Zariski-closed subset $Z \subsetneqq X$ such that*

$$h_{\mathcal{K}}(P) + m(D, P) \le d_k(P) + \epsilon\, h_{\mathcal{A}}(P) + O(1)$$

for all $P \in X(\bar{k}) \setminus Z$ with $[k(P) : k] \le r$.

Furthermore, Vojta showed in [24] that the above conjecture can be used to derive the following conjecture:

CONJECTURE 1.6 *If $a_0, a_1, \dots, a_n$ are nonzero integers such that $a_1 + a_2 + \cdots + a_n = a_0$ and the greatest common divisor of $a_0, a_1, \dots, a_n$ is 1, then*

$$\max\{|a_0|, |a_1|, \dots, |a_n|\} \le C\overline{n}\left(\frac{1}{a_0 a_1 \cdots a_n}\right)^{1+\epsilon}$$

for all $a_0, a_1, \dots, a_n$ as above outside a proper Zariski-closed subset of the hyperplane $x_1 + \cdots + x_n = x_0$ in $\mathbb{P}^n$.

2 Preliminaries

We will apply Nevanlinna theory to prove Theorem 1.3. Some notation and terminology will be needed. Let $\mathbb{Z}_+$ be the set of nonnegative integers. Let $z = (z_1, z_2, \dots, z_m)$ be the natural coordinate system of $\mathbb{C}^m$. For a multi-index $\nu = (i_1, i_2, \dots, i_m) \in \mathbb{Z}_+^m$, write

$$|\nu| = i_1 + i_2 + \cdots + i_m \quad \text{and} \quad \partial_{z_j} = \frac{\partial}{\partial z_j}, \quad \partial^\nu = (\partial_{z_1})^{i_1}(\partial_{z_2})^{i_2}\cdots(\partial_{z_m})^{i_m}.$$

Fix multi-indices $\nu_i \in \mathbb{Z}_+^m$ with $|\nu_i| > 0$ $(i = 1, 2, \dots, n)$. Denote the *Wronski determinant* of the meromorphic functions $f_0, f_1, \dots, f_n$ in $\mathbb{C}^m$ with respect to these multi-indices by

$$\mathbf{W}(f_0, f_1, \dots, f_n) = \mathbf{W}_{\nu_1\nu_2\cdots\nu_n}(f_0, f_1, \dots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ \partial^{\nu_1} f_0 & \partial^{\nu_1} f_1 & \cdots & \partial^{\nu_1} f_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{\nu_n} f_0 & \partial^{\nu_n} f_1 & \cdots & \partial^{\nu_n} f_n \end{vmatrix},$$

and define

$$\mathbf{S}(f_0, f_1, \dots, f_n) = \mathbf{S}_{\nu_1\nu_2\cdots\nu_n}(f_0, f_1, \dots, f_n) = \frac{\mathbf{W}(f_0, f_1, \dots, f_n)}{f_0 f_1 \cdots f_n}.$$

LEMMA 2.1 *Let $f_0, f_1, \dots, f_n$ be linearly independent meromorphic functions in $\mathbb{C}^m$. Write $f = (f_0, f_1, \dots, f_n)$. Then for $i = 1, 2, \dots, n$, there are multi-indices $\nu_i \in \mathbb{Z}_+^m$, $0 < |\nu_i| \leq i$, such that $f, \partial^{\nu_1} f, \partial^{\nu_2} f, \dots, \partial^{\nu_n} f$ are $\mathbb{C}$-linearly independent, and for each $\nu \in \mathbb{Z}_+^m$ satisfying $|\nu| < \max_{1 \leq i \leq n} |\nu_i|$,*

$$\partial^{\nu} f \in \mathbb{C} f + \sum_{|\nu_i| \leq |\nu|} \mathbb{C} \partial^{\nu_i} f ;$$

that is, $\partial^{\nu} f$ lies in the $\mathbb{C}$-span of f and the ν_i-order partial derivatives of f of no higher order. For such multi-indices, then

$$\mathbf{W}(hf_0, hf_1, \dots, hf_n) = h^{n+1} \mathbf{W}(f_0, f_1, \dots, f_n)$$

holds for any nonzero meromorphic function h on $\mathbb{C}^m$.

Under the conditions of Lemma 2.1, A. Vitter [22] proved earlier that there exist multi-indices $\nu_i \in \mathbb{Z}_+^m$ $(1 \leq i \leq n)$ such that

$$\mathbf{W}_{\nu_1 \nu_2 \cdots \nu_n}(f_0, f_1, \dots, f_n) \not\equiv 0 ,$$

and a complete proof of the lemma was then proved by H. Fujimoto [5]. For the readers' interests, we also refer to Ye [26] for a simpler and more direct proof of the lemma. For the multi-indices $\nu_i \in \mathbb{Z}_+^m$ in Lemma 2.1, the integers

$$l = |\nu_1| + |\nu_2| + \cdots + |\nu_n|$$

and $w = |\nu_n|$ will be called the (*Wronskian*) *index* and the *Wronskian degree* of the family $\{f_0, f_1, \dots, f_n\}$, respectively. Obviously, the numbers w and l satisfy the following properties:

$$1 \leq w \leq n \leq l \leq \frac{n(n+1)}{2}, \quad w = n, \ l = \frac{n(n+1)}{2} \ \ (\text{if } m = 1) . \tag{2.1}$$

Let $z = (z_1, z_2, \dots, z_m)$ be the natural coordinate system in $\mathbb{C}^m$ and set

$$\tau(z) = \|z\|^2 = \sum_{j=1}^{m} |z_j|^2 , \quad \upsilon = dd^c \tau, \ \sigma = d^c \log \tau \wedge (dd^c \log \tau)^{m-1} ,$$

where $d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$. If $A \subseteq \mathbb{C}^m$ and if $r \geq 0$, define

$$A[r] = \{x \in A : \tau(x) \leq r^2\} , \quad A\langle r \rangle = \{x \in A : \tau(x) = r^2\} .$$

Let μ be a function on $\mathbb{C}^m$ such that $\dim S \leq m - 1$, where $S = \operatorname{supp} \mu$. Fix a number $r_0 > 0$ and write

$$n_{\mu}(t) = t^{2-2m} \int\limits_{S[t]} \mu \upsilon^{m-1} dt , \quad t > 0 ,$$

$$N_{\mu}(r) = \int_{r_0}^{r} n_{\mu}(t) \frac{dt}{t} , \qquad r > r_0 > 0 ,$$

if the integrals exist.

Let f be a meromorphic function on $\mathbb{C}^m$, and let μ_f^a be the a-valued multiplicity of f for a point $a \in \mathbb{P}^1$; see [21]. Then the valence function of f for a is defined by

$$N\left(r, \frac{1}{f-a}\right) = N_{\mu_f^a}(r),$$

where here and from now on we denote $\frac{1}{f-a}$ by f if $a = \infty$. Also define the truncated multiplicity $\mu_{f,k}^a (k \in \mathbb{Z}_+)$ by

$$\mu_{f,k}^a = \min\{k, \mu_f^a\}.$$

Write

$$n_k\left(r, \frac{1}{f-a}\right) = n_{\mu_{f,k}^a}(r), \quad N_k\left(r, \frac{1}{f-a}\right) = N_{\mu_{f,k}^a}(r),$$

and as usual denote

$$\overline{n}\left(r, \frac{1}{f-a}\right) = n_1\left(r, \frac{1}{f-a}\right), \quad \overline{N}\left(r, \frac{1}{f-a}\right) = N_1\left(r, \frac{1}{f-a}\right).$$

Then we have

$$n_k\left(\infty, \frac{1}{f-a}\right) = \lim_{r\to\infty} n_k\left(r, \frac{1}{f-a}\right) = \lim_{r\to\infty} \frac{N_k\left(r, \frac{1}{f-a}\right)}{\log r}.$$

If there is no confusion, we will abbreviate

$$n_k\left(\frac{1}{f-a}\right) := n_k\left(\infty, \frac{1}{f-a}\right), \quad \overline{n}\left(\frac{1}{f-a}\right) := n_1\left(\infty, \frac{1}{f-a}\right).$$

For $x \geq 0$, we set

$$x^+ = \max\{1, x\}, \quad \log^+ x = \log x^+;$$

then the compensation function of f for a is defined by

$$m\left(r, \frac{1}{f-a}\right) = \int_{\mathbb{C}^m\langle r\rangle} \log^+ \frac{1}{|f-a|}\sigma.$$

The Nevanlinna characteristic function of f is defined by

$$T(r, f) = N(r, f) + m(r, f),$$

which is increasing for r. Then f is nonconstant if and only if $T(r, f) \to \infty$ as $r \to \infty$, and rational if and only if

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} < \infty.$$

If $f \not\equiv 0$, the *Jensen formula*

$$N\left(r, \frac{1}{f}\right) - N(r, f) = \int_{\mathbb{C}^m\langle r\rangle} \log|f|\sigma - \int_{\mathbb{C}^m\langle r_0\rangle} \log|f|\sigma \tag{2.2}$$

holds; cf. Stoll [21]. A deeper result in value distribution theory (or Nevanlinna theory) is the second main theorem:

THEOREM 2.2 *Let f be a nonconstant meromorphic function in $\mathbb{C}^m$ and let $a_1, a_2, \ldots, a_q$ be distinct values in the Riemannian sphere $\mathbb{P}^1$. Then*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right) - N_{\mathrm{ram}}(r,f) + \log\left\{\left(\frac{\rho}{r}\right)^{2m-1}\frac{T(R,f)}{\rho - r}\right\} + O(1)$$

holds for any $r_0 < r < \rho < R$, where $N_{\mathrm{ram}}(r,f)$ is the valence function of the ramification divisor of f.

Theorem 2.2 was first obtained by R. Nevanlinna [16] for meromorphic functions of one variable and extended to the case of holomorphic curves into a higher-dimensional complex projective space $\mathbb{P}^n$ by H. Cartan [3]. W. Stoll [20, 21] unified Nevanlinna and Cartan theory by proving the second main theorem for meromorphic mappings of several variables into $\mathbb{P}^n$. The refined estimate of the error term in the above theorem was proved by Z. Ye [26].

LEMMA 2.3 *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ are linearly independent meromorphic functions in $\mathbb{C}^m$ such that*

$$f_1 + f_2 + \cdots + f_n \equiv 1 . \tag{2.3}$$

Then for $1 \le j \le n$, $R > \rho > r > r_0$,

$$\begin{aligned} T(r, f_j) \le N(r, f_j) &+ \sum_{k=1}^{n}\left\{N\left(r, \frac{1}{f_k}\right) - N(r, f_k)\right\} \\ &+ N(r, \mathbf{W}) - N\left(r, \frac{1}{\mathbf{W}}\right) + l\log\left\{\left(\frac{\rho}{r}\right)^{2m-1}\frac{T(R)}{\rho - r}\right\} + O(1), \end{aligned} \tag{2.4}$$

where $\mathbf{W} = \mathbf{W}_{\nu_1\nu_2\cdots\nu_{n-1}}(f_1, f_2, \ldots, f_n) \not\equiv 0$ is the Wronskian determinant,

$$n - 1 \le l = |\nu_1| + |\nu_2| + \cdots + |\nu_{n-1}| \le \frac{n(n-1)}{2},$$

and where

$$T(r) = \max_{1\le k\le n}\{T(r, f_k)\}.$$

Lemma 2.3 is due to Nevanlinna [16] for the case of one variable. For the case of several variables, see Hu and Yang [6, 7]. Here we used the refined estimates of the error terms of the second main theorem obtained by Z. Ye [26]. Take $\varepsilon > 0$ and put

$$R = r + \frac{r}{(\log T(r))^{1+\varepsilon}}, \quad \rho = \frac{R+r}{2} = r + \frac{r}{2(\log T(r))^{1+\varepsilon}}. \tag{2.5}$$

By applying a well-known lemma of calculus due to Borel (cf. [26]), there is a constant $C > 1$ such that

$$\text{(2.6)} \qquad \| \quad T(R) = T\left(r + \frac{r}{(\log T(r))^{1+\varepsilon}}\right) \leq CT(r)\,,$$

where the symbol $\|$ at the front of an inequality means that the inequality holds as $r \to \infty$ except for a possible set of finite linear measure. On the other hand, for all large r,

$$\text{(2.7)} \qquad \frac{\rho}{r} = O(1)\,, \quad \frac{1}{\rho - r} = \frac{2(\log T(r))^{1+\varepsilon}}{r}\,.$$

Hence one obtains the estimate

$$\text{(2.8)} \quad \| \quad \log\left\{\left(\frac{\rho}{r}\right)^{2m-1}\frac{T(R)}{\rho - r}\right\} < \log T(r) + (1+\varepsilon)\log\log T(r) - \log r + O(1)\,.$$

THEOREM 2.4 *Let $f_1, f_2, \ldots, f_n$ be linearly independent entire functions in $\mathbb{C}^m$. Put $f_0 = f_1 + f_2 + \cdots + f_n$ and assume that* $\dim I \leq m - 2$, *where*

$$I = \{z \in \mathbb{C}^m : f_0(z) = f_1(z) = \cdots = f_n(z) = 0\}\,.$$

Then for $j = 1, 2, \ldots, n$, the inequalities

$$T\left(r, \frac{f_j}{f_0}\right) < \sum_{i=0}^{n} N_w\left(r, \frac{1}{f_i}\right) - N_{\mu_{(f_0, f_j)}}(r) + l \log\left\{\left(\frac{\rho}{r}\right)^{2m-1}\frac{T(R)}{\rho - r}\right\} + O(1)\,,$$

$$T\left(r, \frac{f_j}{f_0}\right) < N_l\left(r, \frac{1}{f_0 \ldots f_n}\right) - N_{\mu_{(f_0, f_j)}}(r) + l \log\left\{\left(\frac{\rho}{r}\right)^{2m-1}\frac{T(R)}{\rho - r}\right\} + O(1)\,,$$

hold for $r_0 < r < \rho < R$, where l and w are, respectively, the index and the Wronskian degree of the family $\{f_1, f_2, \ldots, f_n\}$, $\mu_{(f_0, f_j)}$ is the multiplicity of the zero divisor $D_{(f_0, f_j)}$ of (f_0, f_j), and

$$T(r) = \max_{1 \leq i \leq n}\left\{T\left(r, \frac{f_i}{f_0}\right)\right\}.$$

PROOF: Applying Theorem 2.3 to $f_1/f_0, f_2/f_0, \ldots, f_n/f_0$, for $1 \leq j \leq n$, we obtain

$$\begin{aligned} T\left(r, \frac{f_j}{f_0}\right) \leq \sum_{k=1}^{n} N\left(r, \frac{f_0}{f_k}\right) - \sum_{k \neq j} N\left(r, \frac{f_k}{f_0}\right) + N(r, \mathbf{W}) \\ - N\left(r, \frac{1}{\mathbf{W}}\right) + l \log\left\{\left(\frac{\rho}{r}\right)^{2m-1}\frac{T(R)}{\rho - r}\right\} + O(1)\,, \end{aligned}$$

where $\mathbf{W} = \mathbf{W}(f_1/f_0, f_2/f_0, \ldots, f_n/f_0)$ is the Wronskian of $f_1/f_0, f_2/f_0, \ldots, f_n/f_0$. Note that

$$\mathbf{W} = \mathbf{W}\left(\frac{f_1}{f_0}, \frac{f_2}{f_0}, \ldots, \frac{f_n}{f_0}\right) = \frac{\mathbf{W}(f_1, f_2, \ldots, f_n)}{f_0^n}\,.$$

Abbreviate $\mathbf{W}_1 = \mathbf{W}(f_1, f_2, \dots, f_n)$. By Jensen's formula, we obtain easily

$$N\left(r, \frac{f_0}{f_k}\right) - N\left(r, \frac{f_k}{f_0}\right) = N\left(r, \frac{1}{f_k}\right) - N\left(r, \frac{1}{f_0}\right),$$

$$N(r, \mathbf{W}) - N\left(r, \frac{1}{\mathbf{W}}\right) = nN\left(r, \frac{1}{f_0}\right) - N\left(r, \frac{1}{\mathbf{W}_1}\right),$$

and hence, for $1 \le j \le n$, we obtain

$$\tag{2.9} \begin{aligned} T\left(r, \frac{f_j}{f_0}\right) \le \sum_{k=0}^{n} N\left(r, \frac{1}{f_k}\right) - N\left(r, \frac{1}{\mathbf{W}_1}\right) + N\left(r, \frac{f_0}{f_j}\right) \\ - N\left(r, \frac{1}{f_j}\right) + l \log\left\{\left(\frac{\rho}{r}\right)^{2m-1} \frac{T(R)}{\rho - r}\right\} + O(1)\,. \end{aligned}$$

By simple observation, we find

$$\tag{2.10} N\left(r, \frac{1}{f_j}\right) - N\left(r, \frac{f_0}{f_j}\right) = N_{\mu_{(f_0, f_j)}}(r)\,.$$

Thus Theorem 2.4 follows from (2.9), (2.10), and the following estimates:

$$\tag{2.11} \sum_{k=0}^{n} \mu_{f_k}^0 - \mu_{\mathbf{W}_1}^0 \le \sum_{k=0}^{n} \mu_{f_k, w}^0\,,$$

$$\tag{2.12} \sum_{k=0}^{n} \mu_{f_k}^0 - \mu_{\mathbf{W}_1}^0 \le \mu_{f_0 \dots f_n, l}^0\,.$$

To prove (2.11), it is sufficient to show that (2.11) holds for any $z_0 \in \mathbb{C}^m - I$. Then $\mu_{f_i}^0(z_0) = 0$ for some $i \in \{0, 1, \dots, n\}$. Note that

$$\mathbf{W}_1 = \mathbf{W}(f_1, f_2, \dots, f_{i-1}, f_0, f_{i+1}, \dots, f_n)\,.$$

Hence expanding the determinant shows that for $k \ne i$, $1 \le j \le n-1$,

$$\mu_{\partial^{\nu_j} f_k}^0(z_0) \ge \mu_{f_k}^0(z_0) - \mu_{f_k, |\nu_j|}^0(z_0) \ge \mu_{f_k}^0(z_0) - \mu_{f_k, w}^0(z_0)\,,$$

and, hence,

$$\mu_{\mathbf{W}_1}^0(z_0) \ge \sum_{k \ne i} \left\{\mu_{f_k}^0(z_0) - \mu_{f_k, w}^0(z_0)\right\};$$

that is,

$$\begin{aligned} \sum_{k=0}^{n} \mu_{f_k}^0(z_0) - \mu_{\mathbf{W}_1}^0(z_0) &= \sum_{k \ne i} \mu_{f_k}^0(z_0) - \mu_{\mathbf{W}_1}^0(z_0) \\ &\le \sum_{k \ne i} \mu_{f_k, w}^0(z_0) = \sum_{k=0}^{n} \mu_{f_k, w}^0(z_0)\,. \end{aligned}$$

Inequality (2.12) can be obtained similarly by comparing the zero multiplicity of $f_0 f_1 \dots f_n$ and $\mathbf{W}_1$. $\square$

3 Proof of Theorem 1.3

Let P be a nonconstant polynomial in $\mathbb{C}^m$. We can easily prove the following relation:

$$T(r, P) = \deg(P) \log r + O(1) . \tag{3.1}$$

If Q is another nonconstant polynomial in $\mathbb{C}^m$ such that P and Q are coprime, we can also prove that

$$T\left(r, \frac{P}{Q}\right) = \max\{\deg(P), \deg(Q)\} \log r + O(1) \tag{3.2}$$

holds. An effective divisor D on $\mathbb{C}^m$ is said to be *algebraic* if D is the zero divisor of a polynomial. For the multiplicity ν of an effective divisor on $\mathbb{C}^m$, define

$$n_\nu(\infty) = \lim_{r\to\infty} n_\nu(r) = \lim_{r\to\infty} \frac{N_\nu(r)}{\log r} .$$

The following fact is proven in Rutishauser [18] and Stoll [20].

PROPOSITION 3.1 *An effective divisor D on $\mathbb{C}^m$ is algebraic if and only if the counting function $n_\nu(r)$ of multiplicity ν of D is bounded. Moreover, if $n_\nu(\infty) = n < \infty$, then D is the divisor of a polynomial of degree n.*

PROOF OF THEOREM 1.3: Combining Theorem 2.4 with the estimates of the error term in Section 2 implies

$$\begin{aligned} \| \quad T\left(r, \frac{f_j}{f_0}\right) < \sum_{k=0}^{n} N_w\left(r, \frac{1}{f_k}\right) - N_{\mu_{(f_0, f_j)}}(r) \\ + O(\log^+ T(r)) - l \log r + O(1) \end{aligned} \tag{3.3}$$

holds for $j = 1, 2, \ldots, n$, where

$$T(r) = \max_{1\le j\le n} \left\{ T\left(r, \frac{f_j}{f_0}\right)\right\} = O(\log r) .$$

We can choose polynomials f_{j0}, f_{0j}, and h_j such that f_{j0} and f_{0j} are coprime, and

$$f_j = h_j f_{j0} , \quad f_0 = h_j f_{0j} .$$

Note that $\mu_{(f_0, f_j)} = \mu^0_{h_j}$. Thus (3.3) and (3.2) imply

$$\max\{\deg(f_{j0}), \deg(f_{0j})\} \le \sum_{k=0}^{n} n_w\left(\infty, \frac{1}{f_k}\right) - n\left(\infty, \frac{1}{h_j}\right) - l . \tag{3.4}$$

By Proposition 3.1, we have

$$n\left(\infty, \frac{1}{h_j}\right) = \deg(h_j) . \tag{3.5}$$

Hence by (3.4) and (3.5), we obtain

$$\max\{\deg(f_j), \deg(f_0)\} \leq \sum_{k=0}^{n} n_w\left(\infty, \frac{1}{f_k}\right) - l\,, \quad j = 1, 2, \ldots, n\,. \tag{3.6}$$

Then (1.1) follows from (3.6), and (1.2) can be proven similarly. □

4 Notes on Theorem 1.3

If $m = 1$ and $n = 2$, Theorem 1.3 yields a theorem of Mason; see [11, 13, 23]. For $0 \leq i < j \leq n$, set

$$\xi_{ij} = (\xi_{ij,1}, \xi_{ij,2}, \ldots, \xi_{ij,n-1}) = (f_0, f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n)\,,$$

$$\xi'_{ij} = \begin{pmatrix} \partial_{z_1}\xi_{ij,1} & \partial_{z_1}\xi_{ij,2} & \cdots & \partial_{z_1}\xi_{ij,n-1} \\ \partial_{z_2}\xi_{ij,1} & \partial_{z_2}\xi_{ij,2} & \cdots & \partial_{z_2}\xi_{ij,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{z_m}\xi_{ij,1} & \partial_{z_m}\xi_{ij,2} & \cdots & \partial_{z_m}\xi_{ij,n-1} \end{pmatrix},$$

$$\gamma = \max_{z \in \mathbb{C}^m} \max_{0 \leq i < j \leq n} \operatorname{rank}\left(\xi'_{ij}(z)\right). \tag{4.1}$$

If $\gamma = n - 1$, then we can take $w = 1$ and $l = n - 1$ in Theorem 1.3. Generally, we have

$$1 \leq w \leq n - \gamma, \quad n - 1 \leq l \leq \gamma + \frac{(n - \gamma - 1)(n - \gamma + 2)}{2}\,. \tag{4.2}$$

For any positive integer k, $a \in \mathbb{P}^1$, and any meromorphic function f on $\mathbb{C}^m$, note that

$$n_k\left(\frac{1}{f - a}\right) \leq k\overline{n}\left(\frac{1}{f - a}\right).$$

Theorem 1.3 immediately yields the following fact:

COROLLARY 4.1 *Let $f_1, f_2, \ldots, f_n$, $n \geq 2$, be linearly independent polynomials in $\mathbb{C}^m$. Put $f_0 = f_1 + f_2 + \cdots + f_n$ and assume that* $\dim I \leq m - 2$, *where*

$$I = \{z \in \mathbb{C}^m : f_0(z) = f_1(z) = \cdots = f_n(z) = 0\}\,.$$

Then the following inequalities hold:

$$\max_{0 \leq j \leq n} \{\deg(f_j)\} \leq w \sum_{k=0}^{n} \overline{n}\left(\frac{1}{f_k}\right) - l\,, \tag{4.3}$$

$$\max_{0 \leq j \leq n} \{\deg(f_j)\} \leq l\overline{n}\left(\frac{1}{f_0 f_1 \cdots f_n}\right) - l\,, \tag{4.4}$$

where l and w denote the index and the Wronskian degree of $f_1, f_2, \ldots, f_n$, respectively.

For the case $m = 1$, the inequality (4.4) was obtained independently by J. F. Voloch [25] and by W. D. Brownawell and D. Masser [2]. Earlier R. C. Mason [15] derived this estimate with $l = \frac{1}{2}n(n-1)$ replaced by $l = 4^{n-1}$. J. Browkin and J. Brzeziński [1] conjectured that the sharp value of l in (4.4) is $2n-3$.

Now we remove the restriction on the linear independence of polynomials f_1, $f_2, \dots, f_n$. In what follows, we will use the notation

$$f_i \equiv 0 \mod \{f_{i_1}, f_{i_2}, \dots, f_{i_{s_i}}\} \tag{4.5}$$

to denote that $\{i_1, i_2, \dots, i_{s_i}\} \subset \{0, 1, \dots, n\} - \{i\}$ are distinct, $f_{i_1}, f_{i_2}, \dots, f_{i_{s_i}}$ linearly independent, and

$$f_i = \sum_{\alpha=1}^{s_i} c_\alpha f_{i_\alpha}, \quad c_\alpha \in \mathbb{C} - \{0\} \quad (1 \le \alpha \le s_i).$$

Define

$$I_i = \left\{z \in \mathbb{C}^m : f_i(z) = f_{i_1}(z) = \cdots = f_{i_{s_i}}(z) = 0\right\}.$$

COROLLARY 4.2 *For a fixed integer $n \ge 1$, let f_j $(j = 0, 1, \dots, n)$ be nonzero polynomials on $\mathbb{C}^m$ such that $f_1 + f_2 + \cdots + f_n = f_0$. Assume also that not all the f_j are constants, and that the f_j are pairwise relatively prime. Then*

$$\max_{0 \le j \le n} \{\deg(f_j)\} \le (d-1)\left(\overline{n}\left(\frac{1}{f_0 f_1 \cdots f_n}\right) - 1\right), \tag{4.6}$$

where d is the dimension of the vector space spanned by the f_i over $\mathbb{C}$.

PROOF: The proof of Corollary 4.2 proceeds by induction on n. For $n = 1$ it is obviously true since if $f_0 = f_1$ and f_0 and f_1 are relatively prime, then they both are constants. Assume the corollary is true for all cases n' with $2 \le n' < n$, and consider the case of $n+1$ polynomials. By the assumptions in Corollary 4.2, at least two of the f_i are nonconstant. Note that if two of the f_i are constants, then we may either eliminate them if their sum is zero or replace them by their sum when it is not zero. Then the inductive hypothesis could be applied to yield the desired result. Thus we may assume that at most one of the f_i is a constant. For each $i \in \{0, 1, \dots, n\}$, it is easy to show that

$$f_i \equiv 0 \mod \{f_{i_1}, f_{i_2}, \dots, f_{i_{s_i}}\}$$

for some $i_1, i_2, \dots, i_{s_i}$. Obviously, $d \ge s_i \ge 2$ and $\dim I_i \le m-2$ since the f_j are pairwise relatively prime. So by Corollary 4.1, we have

$$\max_{0 \le \alpha \le s_i} \{\deg(f_{i_\alpha})\} \le w_i \sum_{\alpha=0}^{s_i} \overline{n}\left(\frac{1}{f_{i_\alpha}}\right) - l_i, \tag{4.7}$$

where $i_0 = i$, and l_i and w_i denote the index and the Wronskian degree of f_{i_1}, f_{i_2}, $\dots, f_{i_{s_i}}$, respectively. Note that

$$1 \le w_i \le s_i - 1 \le d - 1, \quad l_i \ge s_i - 1.$$

Therefore, we obtain

$$
\begin{aligned}
\max_{0\le\alpha\le s_i}\{\deg(f_{i_\alpha})\} &\le w_i \sum_{\alpha=0}^{s_i} \overline{n}\left(\frac{1}{f_{i_\alpha}}\right) - l_i \\
&\le (s_i-1)\left(\sum_{\alpha=0}^{s_i} \overline{n}\left(\frac{1}{f_{i_\alpha}}\right) - 1\right) \\
&\le (d-1)\left(\sum_{k=0}^{n} \overline{n}\left(\frac{1}{f_k}\right) - 1\right) \\
&= (d-1)\left(\overline{n}\left(\frac{1}{f_0 f_1\cdots f_n}\right) - 1\right);
\end{aligned}
$$

that is, for each $i \in \{0, 1, \ldots, n\}$,

$$
\deg(f_i) \le (d-1)\left(\overline{n}\left(\frac{1}{f_0 f_1\cdots f_n}\right) - 1\right).
$$

Hence Corollary 4.2 is proven. □

Let $\deg_*(f_k)$ be the sum of the degrees of all distinct irreducible factors of f_k. Proposition 3.1 implies

$$
\overline{n}\left(\frac{1}{f_k}\right) \le \deg_*(f_k).
$$

Thus the following theorem of Shapiro and Sparer follows from Corollary 4.2.

THEOREM 4.3 [19] *For fixed integer $n \ge 2$, let f_j, $j = 0, 1, \ldots, n$, be polynomials on $\mathbb{C}^m$ such that $f_1 + f_2 + \cdots + f_n = f_0$. Assume also that the f_j are not all constant, and that the f_j are relatively prime by pairs. Then*

$$
\max_{0\le j\le n}\{\deg(f_j)\} \le (n-1)\left(\deg_*\left(\prod_{k=0}^{n} f_k\right) - 1\right). \tag{4.8}
$$

5 Notes on Conjecture 1.4

For integers, it is natural to ask how to define the number γ similar to (4.1). Here we approach this question as follows. Express a nonzero integer a in the form (1.3) and regard a as a polynomial of several variables $p_1, p_2, \ldots, p_s$ so that for a prime p, we can define

$$
\partial_p a = \begin{cases} \pm p_\alpha^{i_\alpha - 1} \prod_{\nu\ne\alpha} p_\nu^{i_\nu}, & p = p_\alpha,\ 1 \le \alpha \le s, \\ 0, & p \notin \{p_1, p_2, \ldots, p_s\}. \end{cases} \tag{5.1}
$$

If there are primes, say $p_1, p_2, \ldots, p_m$, satisfying the properties

$$
p \nmid a_j, \quad j = 0, 1, \ldots, n,\ p \notin \{p_1, p_2, \ldots, p_m\},
$$

in (4.1), replacing $f, z_1, z_2, \ldots, z_m$, respectively, by $a, p_1, p_2, \ldots, p_m$, we obtain the required number γ. If the number is equal to $n-1$, we might take $w=1$ and $l=n-1$ in Conjecture 1.4.

Here we study the example due to J. Browkin and J. Brzeziński [1]. For every $k \geq 0$, define a polynomial of positive integral coefficients by

$$f_k(z) = \prod_{j=1}^{k}(z+2-2\cos\alpha_j) = \sum_{j=0}^{k} s_j z^j\,, \quad \alpha_j = \frac{2\pi j}{2k+1}\,, \tag{5.2}$$

which satisfies (cf. [1])

$$\frac{x^{2k+1}-1}{x-1} = x^k f_k\left(\frac{(x-1)^2}{x}\right). \tag{5.3}$$

If in (5.3) we put $k=n-2$ and $x=-b/a$, then, in view of (5.2), one gets

$$a^{2n-3} + b^{2n-3} - \sum_{j=0}^{n-2} s_j (a+b)^{2j+1}(-ab)^{n-2-j} = 0\,. \tag{5.4}$$

If we choose $a = 2^i$, where $i > n-2$, and $b=-1$, then we have

$$a_1 + a_2 + \cdots + a_n = a_0\,,$$

where

$$a_{j+1} = s_j(2^i-1)^{2j+1}2^{i(n-2-j)}\ (0 \leq j \leq n-2)\,, \quad a_n = 1\,, \quad a_0 = 2^{i(2n-3)}\,.$$

Obviously, there is no proper subsum equal to zero. Since $a_n = 1$, the greatest common divisor of all a_j is 1. Therefore, the conditions in Conjecture 1.4 are satisfied. Now we have

$$M_n = \max_{0\leq j\leq n}\{|a_j|\} = a_0 = 2^{i(2n-3)}\,.$$

A positive integer $\chi_n \geq 2n-3$ exists such that

$$\begin{aligned} L_n := \prod_{i=0}^{n} n_{n-1}\left(\frac{1}{a_i}\right) &= 2^{n-2}\prod_{j=0}^{n-2} n_{n-1}\left(\frac{1}{s_j(2^i-1)^{2j+1}2^{i(n-2-j)}}\right) \\ &\geq 2^{(n-2)(n-2)}\prod_{j=0}^{n-2} n_{n-1}\left(\frac{1}{(2^i-1)^{2j+1}}\right) = 2^{(n-2)(n-2)}(2^i-1)^{\chi_n}\,. \end{aligned}$$

Since there are infinitely many $i > n-2$ such that the numbers 2^i-1 are relatively prime (e.g., all prime $i > n-2$), there exists a positive constant $C(n)$ that is independent of i such that

$$\frac{2^{i(2n-3)}}{2^{(n-2)(n-2)}(2^i-1)^{\chi_n}} \leq C(n)\,,$$

that is, $M_n \leq C(n)L_n$. We can also show that

$$M_n \leq C(n)n_{\frac{n(n-1)}{2}}\left(\frac{1}{a_0 a_1 \cdots a_n}\right)$$

holds for some constant $C(n)$. Thus Conjecture 1.4 holds for such a_j.

J. Browkin and J. Brzeziński [1] conjectured as follows:

CONJECTURE 5.1 *Let a_j $(j = 0, 1, \ldots, n)$ be nonzero integers with $n \geq 2$ such that the greatest common divisor of $a_0, a_1, \ldots, a_n$ is 1,*

$$a_1 + a_2 + \cdots + a_n = a_0\,, \tag{5.5}$$

and no proper subsum of (5.5) is equal to 0. Then for $\varepsilon > 0$, there exists a number $C = C(n, \varepsilon)$ such that

$$\max_{0\leq j\leq n} \{|a_j|\} \leq C\overline{n}\left(\frac{1}{a_0 a_1 \cdots a_n}\right)^{2n-3+\varepsilon}. \tag{5.6}$$

Further, Browkin and Brzeziński use the above example to show that the number $2n - 3$ is a sharp lower bound.

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