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Nonlinearly weighted convex risk measure and its application

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ABSTRACT

We propose a new class of risk measures which satisfy convexity and monotonicity, two well-accepted axioms a reasonable and realistic risk measure should satisfy. Through a nonlinear weight function, the new measure can flexibly reflect the investor's degree of risk aversion, and can control the fat-tail phenomenon of the loss distribution. A realistic portfolio selection model with typical market frictions taken into account is established based on the new measure. Real data from the Chinese stock markets and American stock markets are used for empirical comparison of the new risk measure with the expected shortfall risk measure. The in-sample and out-of-sample empirical results show that the new risk measure and the corresponding portfolio selection model can not only reflect the investor's risk-averse attitude and the impact of different trading constraints, but can find robust optimal portfolios, which are superior to the corresponding optimal portfolios obtained under the expected shortfall risk measure.

1. Introduction

Risk is an asymmetric concept related to downside outcomes, and any realistic way of measuring risk should consider upside and downside results differently. Meanwhile, risk-averse investors mainly concern the large losses. Due to these facts, there has been a great momentum in research on quantile-based risk measures in the last decade since the introduction of the value-at-risk (VaR). Unfortunately, VaR often produces overly prudent market risk assessments (Pérignon and Smith 2010a,b). VaR, when calculated using scenarios, is often non-convex, non-smooth as a function of investment positions and is, therefore, difficult to optimize. For these reasons, VaR is said to be seductive but dangerous (Bender, 1995).

To provide a consistent measure of risk, Artzner et al. (1999) introduces the notion of coherent risk measure, which is further extended by Delbaen (2002) and Artzner et al. (2007) to more general setups. A measure is called a coherent risk measure only if it satisfies the following four axioms: subadditivity, positive homogeneity, monotonicity and translation invariance. It is easy to demonstrate that VaR does not provide coherency. Extensive research has been aroused since the appearance of this new definition. Typical measures of this class include the expected shortfall (ES) defined in Acerbi and Tasche (2002) and the conditional value-at-risk (CVaR) developed in Rockafellar and Uryasev (2002). The most general theoretical result about this kind of measures is a complete space of coherent measures spectrally generated in Acerbi (2002).

Although extensive motivation has been given in Artzner et al. (1999) and Delbaen (2002), "coherency" has not fully taken hold in the community concerned with applications. The main obstacle arises from the axioms required in the definition of coherent risk measure. Positive homogeneity assumes that the risk grows in proportion to the volume of the portfolio. Nevertheless, a number of papers (Ding et al., 2009; Chung and Hrazdil, 2010) have recently shown that financial portfolios that experience positive liquidity shocks generally outperform those that experience negative liquidity shocks, that is to say, if liquidity cannot be assured in the market (which is often the situation in stock markets), the risk of a financial portfolio might increase in a nonlinear way with respect to the volume of the portfolio. Meanwhile, positive homogeneity is not necessarily desirable because it corresponds to the linear utility and a rational investor will not accept this kind of utility functions. Actually, laboratory experiments (Bosch-Domènech and Silvestre, 2006) have indicated that investors become more risk averse in face of large investment loss. All these facts suggest that the axiom of positive homogeneity should be relaxed. The corresponding notion of convex risk measures is thus introduced in

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Föllmer and Schied (2002) by replacing positive homogeneity and subadditivity with the weaker property of convexity. Convex risk measures have become a powerful tool in financial risk management (Lüthi and Doege, 2005).

Coherent risk measures and convex risk measures are not widely accepted in the practical finance and insurance community due to the translation invariance axiom required in both risk measures. Translation invariance deals with the effect of adding a constant to the random loss. This axiom has been questioned by many researchers. The problem with this axiom may lie in the confusion over the definition of loss. In Artzner et al. (1999), a loss refers to a negative outcome in terms of the cash flow or wealth of the portfolio; whereas it is assumed by many practitioners that a loss is the negative return rates of a portfolio, or a shortfall relative to some pre-specified target. Therefore, when the practitioners discuss the risk measure in terms of return rates, they have difficulty giving translation invariance a reasonable explanation. On the other hand, as demonstrated in Dhaene et al. (2003), translation invariance means that the way in which we allocate the economic capital among sub-companies within a financial conglomerate is irrelevant. Apparently, all the practical work done nowadays on capital allocation assumes incoherent risk measures. For these reasons, Rockafellar et al. (2006) stipulates an alternative class of risk functionals, called deviation measures, which do not require the translation invariance axiom. In addition, some recent downside risk measures with promising practical effects, such as the power CVaR (PCVaR) and the one-sided risk measures used in new-type performance ratios (Farinelli et al., 2008), do not require translation invariance, either.

Then, what necessary conditions should a reasonable risk measure satisfy? The reasonable risk measure here does not mean the "perfect" measure. In fact, a perfect measure does not exist in reality. Based on the above review and the "best practice" rules in finance and insurance (Dhaene et al., 2003), we assume that any reasonable, realistic risk measure should satisfy two axioms: convexity and monotonicity. These two axioms have been well-accepted by both academicians and practitioners. Based on this assumption, we propose in this paper a new risk measure which is convex and monotone. As a further extension to the current convex risk measures, the new measure can suitably describe the investor's degree of risk aversion and can be used to find robust optimal portfolios in practice.

In the following, we use a simple example to explain the motivation of our new risk measure. The discrete return rates and corresponding probabilities of stocks A and B in the example are given in Table 1.

A short calculation shows that $E(\mathrm{A}) = E(\mathrm{B}) = 0.0166$ and $ES_{0.05}(\mathrm{A}) = ES_{0.05}(\mathrm{B})$, where E stands for the expectation operator, ES_α stands for the ES measure at the confidence level $1-\alpha$. These results tell us that it is impossible to distinguish between A and B under ES_α. However, the highly risk-averse investor will select stock B since the risk-averse investor refuses to invest on the stocks with large losses rather than those with relatively small losses (Hwang and Satchell, 2010). Most existing coherent risk measures, such as ES and CVaR, cannot reflect this character since they treat all the losses beyond VaR equally and only take into account the linear probability weighting combination of those losses. In order to reflect the investor's risk profile, different weights should be assigned to different losses. This suggests that we should introduce a suitable weight function to subtly reflect the investor's attitudes toward investment losses. Meanwhile, if a risk measure is to be used in the portfolio selection, the investor's main interest should be in its consistency with his/her preference. Inspired by the above analyses and considerations, a class of nonlinear weight functions are introduced in our new risk measure.

Table 1
Return rates and corresponding probabilities of stocks A and B.

Return rate of A	Probability of A	Return rate of B	Probability of B
−2.00	0.03	−1.505	0.04
−0.02	0.02	−0.02	0.01
0.03	0.90	0.03	0.90
1.00	0.05	1.00	0.05

Most existing theoretical papers about risk measures (Acerbi, 2002; Fischer, 2003; Rockafellar et al., 2006) fail to consider the application of the risk measures in making optimal investment decision, needless to say the realistic portfolio selection problem with multiple market frictions taken into account. For real applications, what we mainly concern with is the practicality of the proposed risk measure in finding robust and efficient portfolios. In this application-oriented research, we will show a concrete way to compute our new measure and to use it to find the optimal portfolio. Especially, we will establish a portfolio selection model with multiple market friction constraints and use it to find robust portfolios in a real application.

This paper is organized as follows: Section 2 gives the definition of the new risk measure and studies its properties; in Section 3, two methods are presented to estimate the risk measure and the stabilities of the two methods are compared; based on the proposed measure, a realistic portfolio optimization model including typical market frictions is established in Section 4; theoretical results are then illustrated in Section 5 by using empirical inputs from Chinese stock markets and American stock markets; Section 6 presents our conclusions.

2. Definition and properties of the new risk measure

Generally speaking, risk measurement can be thought of as quantification of the characteristics of the future investment uncertainty. Risk in the static framework can be treated as a real-valued random variable X on some probability space $(\Omega, \mathcal{F}, P)$. X can represent the uncertain wealth, the rate of return, or the shortfall relative to the expectation or a benchmark in the future. In this paper, X denotes the random return rate of some asset or portfolio at a future point of time. Empirical research strongly supports that asymmetry and fat tails exist in the financial asset return distribution in real financial markets (Leland, 1999). We assume that the investor's main interest is in the lower tail of the loss distribution. Therefore, it is assumed throughout this paper that the α-quantile (of X) $x_\alpha \leqslant 0$ while $\alpha \ll 0.5$.

To overcome the drawbacks of the existing measures discussed in the previous section, we introduce a class of risk measures by nonlinearly penalizing large negative returns. Our new measure can flexibly reflect the investor's risk aversion level, and it is convex and monotonic. These properties are of vital importance for reasonably measuring the investment risk and for finding robust portfolios. At a tail probability level $\alpha \in (0,1)$, which in practical applications is something like $\alpha = 0.01$ or 0.05, our new risk measure is defined as follows.

Definition 1. (*Weighted expected shortfall, WES for short*). For the real random return rate X with $E[X^-] < \infty$, the new risk measure, called the weighted expected shortfall at a given tail level α, is defined as

$$WES_\alpha(X) = \alpha^{-1}(w(x_\alpha)x_\alpha(P[X \leqslant x_\alpha] - \alpha) - E[w(X)X1_{\{X \leqslant x_\alpha\}}]),$$

where $x_\alpha = \inf\{x \in R : P[X \leqslant x] \geqslant \alpha\}$, and $w(x)$ is a monotonically non-increasing function of x. Moreover, $w(x)$ is positive and convex for $x \leqslant 0$, and non-negative and concave for $x > 0$.

Obviously, if $w(X) \equiv 1$, $WES_\alpha(X) = ES_\alpha(X)$. Therefore, WES_α generalizes the construction of ES_α. That's why we call the new risk measure weighted expected shortfall.

WES_α improves ES_α by treating different losses below VaR_α individually, it enables that a large loss will contribute more to the value of WES_α than a comparatively small loss. On the other hand, through the selection of the weight function $w(x)$, WES_α treats small losses and large losses below VaR_α in an asymmetric manner. The more convex the $w(x)$ is when $x \leqslant 0$, the more risk-averse the investor is. Meanwhile, the more concave the $w(x)$ is when $x > 0$, the less the investor cares about earnings. Therefore, the choice of $w(x)$ depends on the investor's attitude towards risk. The four typical weight functions satisfying requirements in Definition 1 are $w(x) = \exp(-\lambda x)(\lambda \geqslant 0)$, $\exp(-(1+x))$, $(1-x)^\beta$ and $(\beta - x)^\beta$ ($\beta > 1$) for $x \leqslant 0$, and $w(x) = 0$ for $x > 0$.

The term $w(x_\alpha)x_\alpha(P[X \leqslant x_\alpha] - \alpha)$ in Definition 1 can be interpreted as the "exceeding" part to be subtracted from the expected value $E[w(X)X1_{\{X\leqslant x_\alpha\}}]$ when X has a discontinuous distribution or has a jump at x_α. In this case, the total probability for outcomes with $X \leqslant x_\alpha$ is probably larger than α. If the distribution of X is continuous, then $P[X \leqslant x_\alpha] = \alpha$, and the term $w(x_\alpha)x_\alpha(P[X \leqslant x_\alpha] - \alpha)$ vanishes.

In the following, we explain and illustrate the advantage of our risk measure by using the example given in Table 1. If we select $w(x) = \exp(-0.01x)$ for $x \leqslant 0$, and $w(x) = 0$ for $x > 0$, we get $WES_{0.05}(A) = 1.2322 > WES_{0.05}(B) = 1.2263$. Therefore, the investor would prefer B to A under WES_α. This is consistent with risk-averse investors' risk intuition: the stock that brings in large negative returns with high probability is easier to be rejected than otherwise. WES_α can thus characterize the investor's risk-averse attitude more elaborately than ES_α.

To demonstrate the favorable features of WES_α, we first introduce the following notation:

$$1^\alpha_{\{X\leqslant x\}} = \begin{cases} 1_{\{X\leqslant x\}}, & P[X = x] = 0 \\ 1_{\{X\leqslant x\}} + \frac{\alpha - P[X\leqslant x]}{P[X=x]} 1_{\{X=x\}}, & P[X = x] > 0 \end{cases}$$

for any $x \in R$. It is not difficult to see that $1^\alpha_{\{X\leqslant x_\alpha\}} \in [0,1]$ and

$$E\left[1^\alpha_{\{X\leqslant x_\alpha\}}\right] = \alpha, \tag{1}$$

$$\alpha^{-1}E\left[w(X)X1^\alpha_{\{X\leqslant x_\alpha\}}\right] = \bar{x}_{(\alpha)}. \tag{2}$$

Therefore, we can rewrite $WES_\alpha(X)$ as

$$WES_\alpha(X) = -\frac{1}{\alpha}E\left[w(X)X1^\alpha_{\{X\leqslant x_\alpha\}}\right]. \tag{3}$$

The following proposition shows that $WES_\alpha(X)$ is convex with respect to X.

Proposition 1 (*Convexity of WES*). *Given any $\gamma \in [0,1]$ and two random return rates X and Y on $(\Omega, \mathcal{F}, P)$ with $E[X^-] < \infty$ and $E[Y^-] < \infty$, respectively, we have*

$$WES_\alpha(\gamma X + (1-\gamma)Y) \leqslant \gamma WES_\alpha(X) + (1-\gamma)WES_\alpha(Y).$$

Proof. Since the investor mainly concerns with the left tail distribution of random return rates and $\alpha \ll 0.50$, we have $x_\alpha \leqslant 0$ and $y_\alpha \leqslant 0$. Let $Z = \gamma X + (1-\gamma)Y$, we can derive from (1) and (3) that

$$\begin{aligned}
&\alpha(\gamma WES_\alpha(X) + (1-\gamma)WES_\alpha(Y) - WES_\alpha(Z)) \\
&= E\left[w(Z)Z1^\alpha_{\{Z\leqslant z_\alpha\}} - \gamma w(X)X1^\alpha_{\{X\leqslant x_\alpha\}} - (1-\gamma)w(Y)Y1^\alpha_{\{Y\leqslant y_\alpha\}}\right] \\
&\geqslant E\Big[(\gamma w(X) + (1-\gamma)w(Y))(\gamma X + (1-\gamma)Y)1^\alpha_{\{Z\leqslant z_\alpha\}} \\
&\quad - \gamma w(X)X1^\alpha_{\{X\leqslant x_\alpha\}} - (1-\gamma)w(Y)Y1^\alpha_{\{Y\leqslant y_\alpha\}}\Big] \\
&= E\left\{\gamma Xw(X)\left[1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{X\leqslant x_\alpha\}}\right] + (1-\gamma)Yw(Y)\left[1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{Y\leqslant y_\alpha\}}\right]\right\} \\
&\quad + E\left\{[\gamma(1-\gamma)(X-Y)(w(Y) - w(X))]1^\alpha_{\{Z\leqslant z_\alpha\}}\right\} \\
&\geqslant E\left\{\gamma Xw(X)\left[1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{X\leqslant x_\alpha\}}\right] + (1-\gamma)Yw(Y)\left[1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{Y\leqslant y_\alpha\}}\right]\right\} \\
&\geqslant E\Big\{\gamma x_\alpha w(x_\alpha)\left[1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{X\leqslant x_\alpha\}}\right] \\
&\quad + \left(1-\gamma)y_\alpha w(y_\alpha)\left[1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{Y\leqslant y_\alpha\}}\right]\right\} \\
&= \gamma x_\alpha w(x_\alpha)(\alpha - \alpha) + (1-\gamma)y_\alpha w(y_\alpha)(\alpha - \alpha) = 0.
\end{aligned}$$

The first inequality can be illustrated as follows. For any $\omega \in \Omega$, if $Z(\omega) \leqslant z_\alpha$,

$$w(Z(\omega))Z(\omega) \geqslant (\gamma w(X(\omega)) + (1-\gamma)w(Y(\omega)))(\gamma X(\omega) + (1-\gamma)Y(\omega))$$

because $w(\cdot)$ is convex if $Z(\omega) \leqslant 0$ and is concave if $Z(\omega) > 0$. On the other hand, $1^\alpha_{\{Z(\omega)\leqslant z_\alpha\}} = 0$ if $Z(\omega) > z_\alpha$. Therefore, for all $\omega \in \Omega$, we have

$$\begin{aligned}
(w(Z(\omega))Z(\omega))1^\alpha_{\{Z(\omega)\leqslant z_\alpha\}} \geqslant\ & (\gamma w(X(\omega)) + (1-\gamma)w(Y(\omega)))(\gamma X(\omega) \\
& + (1-\gamma)Y(\omega))1^\alpha_{\{Z(\omega)\leqslant z_\alpha\}}.
\end{aligned}$$

The second inequality holds because the monotonic and non-increasing properties of $w(\cdot)$ ensure that $(X - Y)(w(Y) - w(X)) \geqslant 0$.

The third inequality relies on the non-negative, monotonically non-increasing properties of $w(\cdot)$ and the following two formulae:

$$\begin{cases} 1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{X\leqslant x_\alpha\}} \geqslant 0, & X > x_\alpha \\ 1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{X\leqslant x_\alpha\}} \leqslant 0, & X \leqslant x_\alpha \end{cases}$$

and

$$\begin{cases} 1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{Y\leqslant y_\alpha\}} \geqslant 0, & Y > y_\alpha, \\ 1^\alpha_{\{Z\leqslant z_\alpha\}} - 1^\alpha_{\{Y\leqslant y_\alpha\}} \leqslant 0, & Y \leqslant y_\alpha. \end{cases}$$

In fact, it can be easily deduced from $x_\alpha \leqslant 0$ that: for every $\omega \in \Omega$, $x_\alpha w(x_\alpha) \leqslant X(\omega)w(X(\omega))$ if $X(\omega) > x_\alpha$, and $X(\omega)w(X(\omega)) \leqslant x_\alpha w(x_\alpha)$ if $X(\omega) \leqslant x_\alpha$. So, for all $\omega \in \Omega$, we have

$$X(\omega)w(X(\omega))\left[1^\alpha_{\{Z(\omega)\leqslant z_\alpha\}} - 1^\alpha_{\{X(\omega)\leqslant x_\alpha\}}\right] \geqslant x_\alpha w(x_\alpha)\left[1^\alpha_{\{Z(\omega)\leqslant z_\alpha\}} - 1^\alpha_{\{X(\omega)\leqslant x_\alpha\}}\right].$$

Similarly, we can prove that

$$Y(\omega)w(Y(\omega))\left[1^\alpha_{\{Z(\omega)\leqslant z_\alpha\}} - 1^\alpha_{\{Y(\omega)\leqslant y_\alpha\}}\right] \geqslant y_\alpha w(y_\alpha)\left[1^\alpha_{\{Z(\omega)\leqslant z_\alpha\}} - 1^\alpha_{\{Y(\omega)\leqslant y_\alpha\}}\right].$$

By now, we have shown that $WES_\alpha(\gamma X + (1-\gamma)Y) \leqslant \gamma WES_\alpha(X) + (1-\gamma)WES_\alpha(Y)$, that is, $WES_\alpha(X)$ is convex with respect to X. □

As an illustration to Proposition 1, we consider a simple example of two stocks. The discrete return rates of stocks A and B and their corresponding probabilities in different cases are given in Table 2.

If we choose $w(x) = \exp(-0.01x)$ for $x \leqslant 0$ and $w(x) = 0$ for $x > 0$, the values of $WES_{0.05}$ for stocks A, B and the portfolio $\frac{1}{3}A + \frac{2}{3}B$ are 0.21, 0.21 and 0.15, respectively. This shows that the risk of the considered portfolio is smaller than the risk of each of its component stocks. That is, $WES_{0.05}$ captures the notion of diversification. As the most characteristic feature of a risk measure, the convexity

Table 2
Return rates and corresponding probabilities of stocks A and B in different cases.

Event	Return rate of A	Return rate of B	Probability
Event 1	−0.29	0.01	0.03
Event 2	−0.09	0.01	0.02
Event 3	0.01	−0.29	0.03
Event 4	0.01	−0.09	0.02
Event 5	0.01	0.01	0.90

of WES_α is not only essential for investment strategy selection in stock markets, but also useful for the financial accounting management in banks and insurance companies. For instance, in the case of internal capital budget, since WES_α satisfies convexity, the head office can be sure that, when they set a risk limit to each individual company division, the risk of the whole company should not exceed the total of the individual risk limits. With this understanding, the head office is able to decentralize the risk constraint and control the risk with ease. Convexity of a risk measure also plays a role in the portfolio optimization. This advantage of WES_α and its practicability will be illustrated in Sections 4 and 5.

We further consider the following example to examine the properties of WES_α. The logarithms of the return rates X and Y of stocks A and B follow the Student t-distributions with degrees of freedom being 2 and 10, respectively. Suppose that $\alpha = 0.05$ and the weight function is chosen as $w(x) = \exp(-10x)$ if $x \leqslant 0$ and $w(x) = 0$ if $x > 0$, we can easily get $WES_{0.05}(X) = 1.83\text{E}+04$, $WES_{0.05}(2X) = 1.6698\text{E}+008 \neq 2WES_{0.05}(X)$, $WES_{0.05}(Y) = 7.92\text{E}+03$, and $WES_{0.05}(X+Y) = 2.30\text{E}+07 > WES_{0.05}(X) + WES_{0.05}(Y)$. These results show that WES_α is neither subadditive nor positive homogeneous.

The monotonicity of WES_α is demonstrated by Proposition 2 as follows.

Proposition 2 (*Monotonicity of WES*). *WES_α is a monotonic and non-increasing function. That is, given two random return rates X and Y on $(\Omega, \mathcal{F}, P)$ with $E[X^-] < \infty$ and $E[Y^-] < \infty$, we have $WES_\alpha(Y) \leqslant WES_\alpha(X)$ if $X \leqslant Y$.*

The economic interpretation of this property is straightforward. By monotonicity, we mean that if the return rate of a portfolio Y is always at least as high as that of X, Y cannot be riskier than X. Therefore, the financial position with higher returns should be less risky than the corresponding positions with lower returns. The monotonicity of WES_α is certainly a reasonable feature.

Another self-evident property of WES_α related to Proposition 2 is its monotonicity with respect to α. That is, the smaller α is, the greater the risk is.

In the financial industry, there is a growing demand to deal with random returns with discontinuous distributions, such as portfolios of non-traded loans (purely discrete distributions) or portfolios containing derivatives (mixture of continuous and discrete distributions). One trouble with most existing tail risk measures, such as VaR, tail conditional expectation and worst conditional expectation (Artzner et al., 1999), is that when applied to discontinuous distributions, these tail risk measures are extremely sensitive to the small changes in the confidence level. In other words, they are not continuous with respect to the confidence level. In contrast, WES_α is continuous with respect to α. Hence, regardless of the underlying return distribution, one can be sure that the risk measured by WES_α will not change dramatically when there is a switch in the confidence level $1-\alpha$ by, say, some base points. In order to establish this property, we first derive an alternatively integral representation of WES_α. By using the argument similarly to that in Acerbi and Tasche (2002), we can easily prove Proposition 3.

Proposition 3. *If X is a random variable of the return rate on $(\Omega, \mathcal{F}, P)$ with $E[X^-] < \infty$, and $\alpha \in (0,1)$ is fixed, then*

$$WES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha w(x_u)x_u\,du,$$

where x_u is defined as that in Definition 1.

Remark. From this expression, we can establish the strictly monotonically decreasing property of WES_α. For any two random return rates X and Y, we have $x_u \leqslant 0$ and $y_u \leqslant 0$ almost surely for $0 \leqslant u \leqslant \alpha$ since $\alpha \ll 0.5$. Consequently, if $X < Y$ almost surely, for any positive weight function $w(\cdot)$, we have the expression $x_u w(x_u) < y_u w(y_u)$ for any $0 \leqslant u \leqslant \alpha$. This expression and the expression in Proposition 3 clearly show that $WES_\alpha(Y) < WES_\alpha(X)$.

Moreover, the following desired continuity result can be easily obtained from the expression in Proposition 3.

Corollary 4. *If X is the random return rate with $E[X^-] < \infty$, the mapping $\alpha \mapsto WES_\alpha$ is continuous on $(0,1)$.*

What is more important, from the practical point of view, is that the weighted summation of our new risk measures is still a risk measure satisfying convexity and monotonicity. The following proposition explains this argument.

Proposition 5. *Suppose that ρ_i, $i = 1,2,\ldots,n$, are a group of risk measures defined by formula (3) with specific weight functions and probability levels, then for any $0 \leqslant a_i \leqslant 1, i = 1,2,\ldots,n$, with $\sum_{i=1}^n a_i = 1$, $\rho = \sum_{i=1}^n a_i\rho_i$ is still a risk measure which satisfies convexity and monotonicity.*

Proposition 5 shows that the convex combination of WES_αs with different α and/or $w(x)$ can also be adopted as a risk measure. As a result of this, the distribution characteristics of X can be reflected more comprehensively, and the advantages of different WES_αs can be utilized simultaneously.

Last but not least, we demonstrate the differentiability of WES_α with respect to the portfolio weight vector. For this purpose, suppose that there are n risky stocks whose random return rates are r_i, $1 \leqslant i \leqslant n$. Let $r = (r_1, r_2, \ldots, r_n)^T$ be the return rate vector and $x = (x_1, x_2, \ldots, x_n)^T$ be the corresponding portfolio weight vector, here and in the following, we use the superscript T to denote the transpose of a vector. Then.

Proposition 6. *If the joint distribution of return rates has a continuous and positive density, then*

$$\nabla_x WES_\alpha(x) = -\frac{1}{\alpha}E[(\nabla_x w(r^T x)r^T x + w(r^T x)r)1_{\{r^T x \leqslant q_\alpha(r^T x)\}}]. \quad (4)$$

This proposition is helpful for us to choose a proper algorithm for solving the derived portfolio selection problem, where WES_α is adopted to control the investment risk. We will explain this point in Sections 4 and 5.

3. Estimation of WES_α

Two methods used to estimate WES_α are discussed in this section. Their efficiency and stability are examined through simulation and empirical comparisons.

3.1. The historical method

The complete distribution information of X is extremely difficult to obtain in practice. The distribution of X is usually approximated through its discrete samples or scenarios. Nowadays, a popular way to estimate ES_α is by L-statistics, which satisfy important asymptotic properties as shown in Pflug and Wozabal (2010). Inspired by this characteristic of L-statistics, we consider here how to estimate WES_α using L-statistics.

Suppose that M return rate samples of X are denoted by $X_1, X_2, \ldots$, and X_M. Sort $X_m (1 \leqslant m \leqslant M)$ in the increasing order:

$$X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(M)},$$

where $X_{(m)}$ is the mth order statistic of $X_1, X_2, \ldots, X_M$.

If αM is an integer, that is, $\alpha M \in N$, WES_α can be directly estimated by the weighted mean of worst αM return samples according to Definition 1. That is,

$$W\widehat{ES}_\alpha(X) = -\frac{1}{\alpha M}\left(\sum_{m=1}^{\alpha M} w(X_{(m)})X_{(m)}\right).$$

If $\alpha M \notin N$, according to the definition of WES_α, we estimate that

$$\begin{aligned} &W\widehat{ES}_\alpha(X) \\ &= -\frac{1}{\alpha M}\left(\sum_{m=1}^{[\alpha M]} w(X_{(m)})X_{(m)} + (\alpha M - [\alpha M])w(X_{([\alpha M]+1)})X_{([\alpha M]+1)}\right), \end{aligned} \tag{5}$$

where $[y]$ denotes the lower integer part of the real number y. The term $(\alpha M - [\alpha M])X_{[\alpha M]+1}w(X_{[\alpha M]+1})$ represents a fraction of the $([\alpha M]+1)$th smallest return sample. Obviously, this term plays a minor role when M is extremely large.

As we know, the consistency is an essential cornerstone to support the actual application of any estimation method. Fortunately, we can establish the strong consistency of the estimation (5) as follows.

Proposition 7. *Suppose that $\alpha \in (0,1)$ is fixed and X is a real-valued random variable satisfying $E[Xw(X)] < \infty$. If $(X_1, X_2, \ldots, X_M, \ldots)$ is an independent sequence of random variables which have the same distribution function F as that of X, we then have*

$$W\widehat{ES}_\alpha(X) - WES_\alpha(X) \rightarrow_{a.s.} 0 \quad \text{as } M \to \infty.$$

Proof. Let $Y = Xw(X)$, and suppose that the distribution function of Y is $\overline{F}$. Then, the empirical distribution function of the corresponding samples $Y_1, Y_2, \ldots, Y_M$ is

$$\overline{F}_M(y) \equiv \frac{1}{M}\sum_{m=1}^{M} 1_{(-\infty,y]}(Y_m) \quad \text{for } -\infty < y < \infty.$$

Assume that $Y_{(m)}(1 \leqslant m \leqslant M)$ is the mth order statistic of $Y_1, Y_2, \ldots, Y_M$. As argued before, we have $x_\alpha \leqslant 0$. In addition, $w(x)$ is a non-negative and monotonically non-increasing function of x. Therefore, we obtain $Y_{(m)} = w(X_{(m)})X_{(m)}$ for $1 \leqslant m \leqslant [\alpha M]+1$ and $Y_{(m')} \geqslant Y_{([\alpha M]+1)}$ for $[\alpha M]+1 \leqslant m' \leqslant M$.

According to (5), we have

$$\begin{aligned} W\widehat{ES}_\alpha(X) &= -\frac{1}{\alpha M}\left(\sum_{m=1}^{[\alpha M]} Y_{(m)} + (\alpha M - [\alpha M])Y_{([\alpha M]+1)}\right) \\ &= \frac{1}{M}\left(\sum_{m=1}^{[\alpha M]}\left(-\frac{1}{\alpha}Y_{(m)}\right) + \frac{[\alpha M]-\alpha M}{\alpha}Y_{([\alpha M]+1)} + \sum_{m=[\alpha M]+2}^{M} 0 \cdot Y_{(m)}\right) \\ &= \int_0^1 \overline{F}_M^{-1}(t)J_M(t)dt, \end{aligned}$$

where

$$J_M(t) = \begin{cases} -\frac{1}{\alpha}, & 0 \leqslant t \leqslant \frac{[\alpha M]}{M}, \\ \frac{[\alpha M]-\alpha M}{\alpha}, & \frac{[\alpha M]}{M} < t \leqslant \frac{[\alpha M]+1}{M}, \\ 0, & \frac{[\alpha M]+1}{M} < t \leqslant 1. \end{cases}$$

Meanwhile, according to Proposition 3, we have

$$\begin{aligned} WES_\alpha(X) &= -\frac{1}{\alpha}\int_0^\alpha w(x_u)x_u\,du = \int_0^\alpha -\frac{1}{\alpha}y_u\,du + \int_\alpha^1 0 \cdot y_u\,du \\ &= \int_0^1 \overline{F}^{-1}(t)J(t)dt, \end{aligned}$$

where

$$J(t) = \begin{cases} -\frac{1}{\alpha}, & 0 \leqslant t \leqslant \alpha \\ 0, & \alpha < t \leqslant 1. \end{cases}$$

The above expressions can easily verify that Proposition 7 is essentially a special case of Theorem 3.1 in Van Zwet (1980) with $0 = t_0 < \alpha = t_1 < t_2 = 1$ and $p_1 = p_2 = \infty$. Consequently, the desired strong consistency of estimation (5) follows from above expressions for $W\widehat{ES}_\alpha$, WES_α and Theorem 3.1 in Van Zwet (1980). □

Estimation (5) does not rely on any distribution assumption about X, which is extremely important for the practical financial risk management since estimation (5) can drastically reduce the mathematical complexity of the problem. Nevertheless, one might need a large number of samples to estimate ES_α in a stable way. It is shown in Inui and Kijima (2005) that the Richardson extrapolation method can speed up the convergence of the historical method. Therefore, we now combine the historical method with the Richardson extrapolation method to estimate WES_α.

3.2. The Richardson extrapolation method

We try to enhance the efficiency of the WES_α estimation by using the Richardson extrapolation method. For simplicity, we directly use the following second-order Richardson extrapolation formula (Inui and Kijima, 2005) to get a better WES_α estimation result.

$$t_n = 0.5s_n - 4.0s_{n+1} + 4.5s_{n+2}, \tag{6}$$

where $\{s_n\}$ is a sequence that converges to the true WES_α value. According to the Richardson extrapolation theory, the sequence $\{t_n\}$ converges to the true WES_α value faster than $\{s_n\}$.

To test the actual effect of the extrapolation formula (6), we carry out numerical experiments by assuming that X follows the log-normal distribution and t-distributions with degree of freedom (DF for short) being 3, 30 and 300, respectively. The smaller the DF is, the fatter the tail of the distribution will be. Meanwhile, $\{s_n\}$ in (6) is chosen from the values obtained with the historical method. In this simulation, we set the probability levels α to be 0.01 and 0.05, respectively. The nonlinear weight function is $w(x) = \exp(-\lambda x)$ with specific λ if $x \leqslant 0$, and $w(x) \equiv 0$ if $x > 0$. The results obtained using (6) are compared with the theoretical values and the results obtained from the historical method, respectively, to show the effect of the extrapolation formula (6). Our simulation procedure is as follows:

- *S1*. Re-sample the three different (non-duplicated) groups of samples from the log-normal distribution or t-distribution with the sample size being $n_1 = 100$, $n_2 = 200$ and $n_3 = 300$, respectively, and calculate WES_α by using (5) for each case.
- *S2*. Corresponding to n_1, n_2 and n_3, repeat *S1* for 1000 times to calculate the mean of WES_α, which are denoted as s_n, s_{n+1} and s_{n+2}, respectively. Here, s_{n+2} is taken as an estimate for WES_α by the historical method.
- *S3*. Apply the Richardson extrapolation formula (6) to obtain t_n, which is used as an estimate for WES_α by the Richardson extrapolation method.
- *S4*. Repeat *S1–S3* for 1000 times and calculate the mean values of s_n and t_n, respectively.

Due to the space limitation, we only present in Table 3 the results obtained for the samples from t-distributions with $\alpha = 0.01$, $\lambda = 0.1$ and different DFs. Other results have similar patterns. In Table 3, WES^*, $W\widehat{ES}_\alpha$ and $W\widehat{ES}_\alpha^{(2)}$ stand for the actual WES_α value, the WES_α value estimated by the historical method, and the WES_α value estimated by the Richardson extrapolation method, respectively. Bias is calculated as the ratio of the difference of the mean value of WES_α obtained by the relevant method and the WES^* to the WES^*. That is,

Table 3
WES estimates with $\alpha = 0.01$ and $\lambda = 0.1$.

	WES^*	$W\widehat{ES}_\alpha$	$W\widehat{ES}_\alpha^{(2)}$
t(DF = 3)			
Mean	1.1007	1.097	1.1008
Bias (%)		−0.34	0.0091
t(DF = 30)			
Mean	1.032	1.0249	1.0319
Bias (%)		−0.00688	−0.000097
t(DF = 300)			
Mean	1.0192	1.0122	1.0189
Bias (%)		−0.00687	−0.00029

$$\text{Bias} = \frac{\text{Mean of } WES_\alpha - WES^*}{WES^*}.$$

The bigger the number is, the more biased the estimate is.

We can see from Table 3 that the Richardson extrapolation method is useful for finding accurate WES_α estimates for symmetrically distributed returns, since it works well to adjust the bias. Nevertheless, it is well-recognized that return distributions are skewed with fat tails. To examine the performance of the Richardson extrapolation method in practice, we carry out empirical tests in the next subsection.

3.3. Comparison of the historical method with the Richardson extrapolation method

We use the daily return rate data of the S&P 500 Index and the Dow Jones Industrial Average Index to empirically examine the stability and efficiency of the two estimation techniques. The former consists of 11,800 return rates from 2 January, 1962 to 3 April, 2009; the later consists of 20,000 return rates from 1 October, 1928 to 3 April, 2009.

By setting the time-step to be 100, we randomly sample n return rate data, with n increasing from 100 to 11,800 for the S&P 500 Index and from 100 to 20,000 for the Dow Jones Index, respectively. Then we calculate WES_α by using (5) and (6), respectively. To compare the accuracy and robustness of the two estimation methods, we examine the variation of the estimated WES_αs with the sample size and parameter combination. Concretely, we choose $\alpha = 0.01$ and 0.05, $\lambda = 0$, 0.1, 10 and 50 for λ in $w(x) = \exp(-\lambda x)$ if $x \leqslant 0$, and $w(x) \equiv 0$ if $x > 0$, respectively. For comparison, we draw the scatter plot and the fitted curve of WES_α obtained by the two estimation methods in the same graph. Since the results for the Dow Jones Index are similar to those for the S&P 500 Index, we only show in Figs. 1–8 the empirical results for the S&P 500 Index. In each figure, for reference, we especially show the estimates of WES_α obtained for $n = 600$ by using the historical method and the Richardson extrapolation method, respectively.

The following observations can be derived from these figures: with the increase of the sample size, the stability of the estimates by the historical method significantly increases, compared with the stability of the estimates by the Richardson extrapolation method; in general, the bias of the historical method is smaller than that of the Richardson extrapolation method; for almost all the cases, the estimates by the historical method converge to the actual WES_α value faster than those by the Richardson extrapolation method; the accuracy and stability of the Richardson extrapolation method are more or less the same as those of the historical method only when the sample size and α are small.

In summary, the Richardson extrapolation method may be more efficient than the historical method in the estimation of WES_α when the return rate distribution is symmetric or when both the sample size and α are small. Nevertheless, these situations usually do not occur in reality. Therefore, considering the extra computational effort required by the Richardson extrapolation method, the usual historical method is more suitable for the estimation of WES_α in practice.

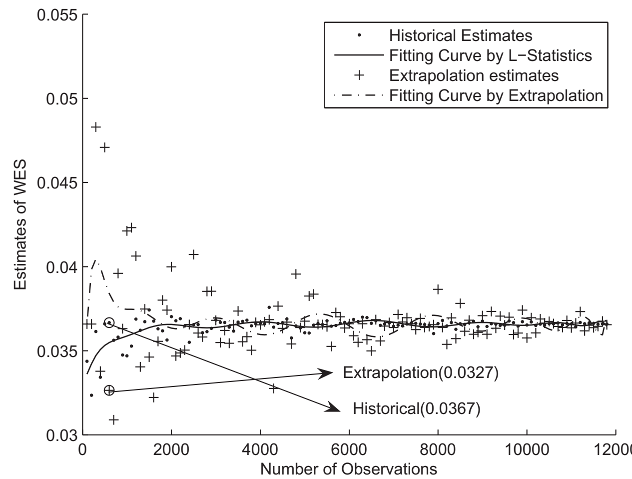

Fig. 1. S&P 500 with $\alpha = 0.01$ and $\lambda = 0$.

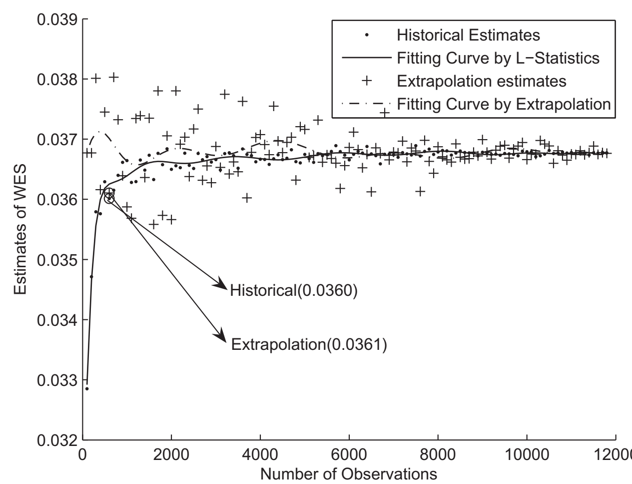

Fig. 2. S&P 500 with $\alpha = 0.01$ and $\lambda = 0.1$.

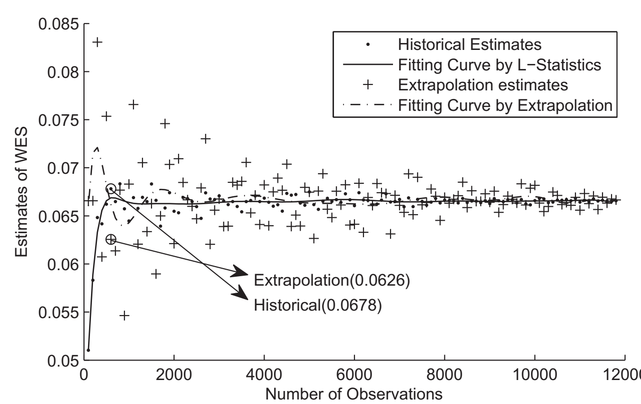

Fig. 3. S&P 500 with $\alpha = 0.01$ and $\lambda = 10$.

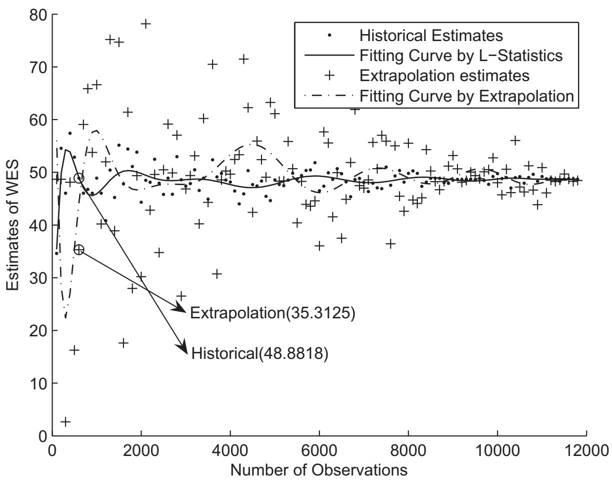

Fig. 4. S&P 500 with $\alpha = 0.01$ and $\lambda = 50$.

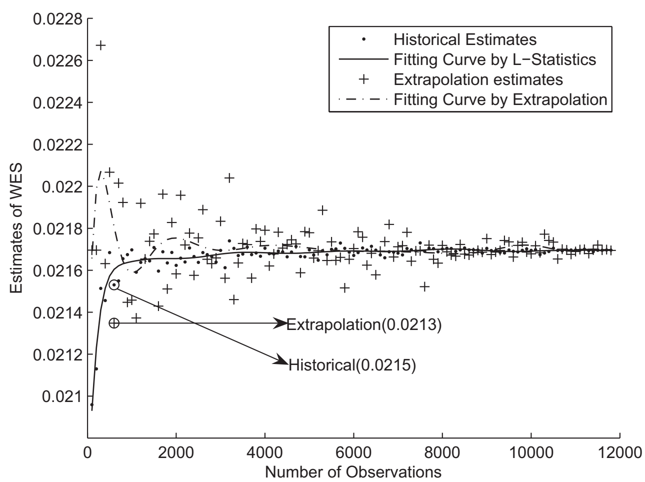

Fig. 5. S&P 500 with $\alpha = 0.05$ and $\lambda = 0$.

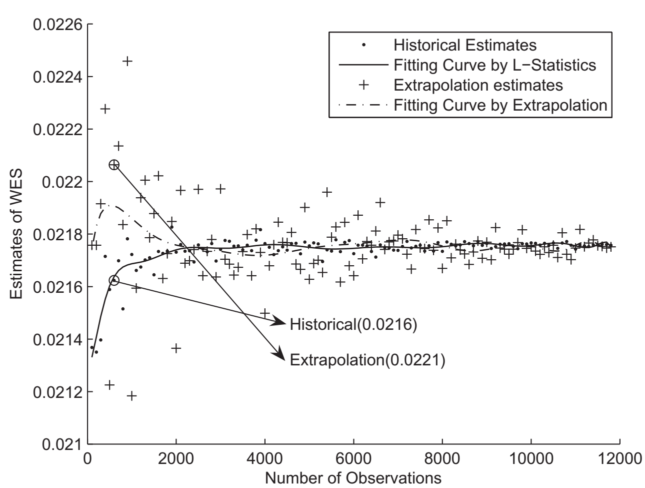

Fig. 6. S&P 500 with $\alpha = 0.05$ and $\lambda = 0.1$.

4. The realistic portfolio selection model under WES_α

A portfolio selection model is established here to demonstrate the application of the proposed new risk measure WES_α in the

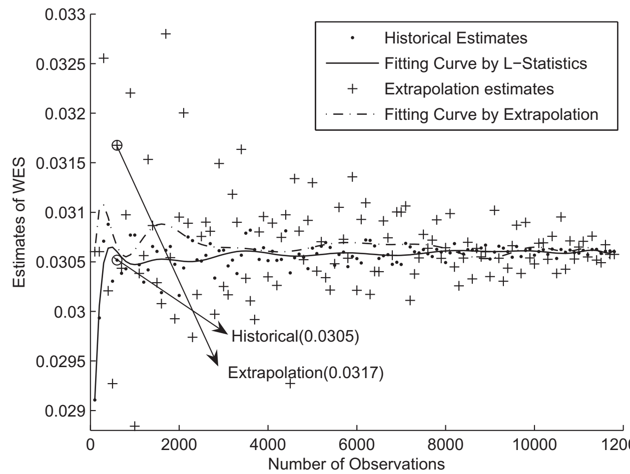

Fig. 7. S&P 500 with $\alpha = 0.05$ and $\lambda = 10$.

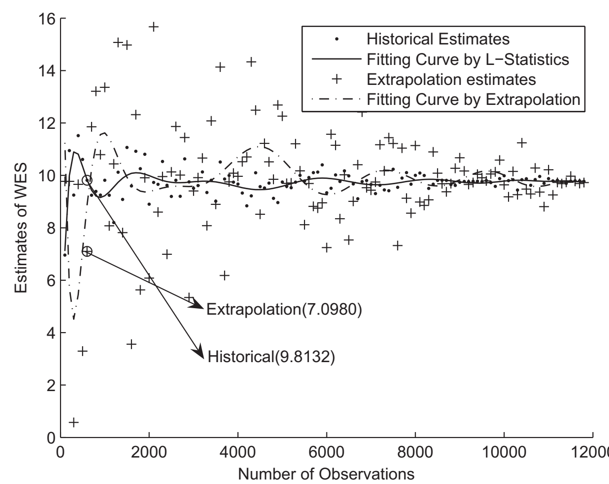

Fig. 8. S&P 500 with $\alpha = 0.05$ and $\lambda = 50$.

optimal and robust investment decision-making. In order to obtain greater realism in our portfolio selection model, several market frictions have to be taken into account simultaneously.

Suppose that there exist n risky assets offering random return rates and a riskless asset offering a fixed return rate. The investor allocates his/her wealth among n risky assets and the riskless asset and tries to minimize the risk for his/her portfolio return rate after taxes and transaction costs are deduced. It is assumed that taxes have to be paid on both ordinary income and capital gains. It is also assumed that dividends and transaction costs on risky assets are paid at the end of the investment period and are known to the investor with certainty at the beginning of the investment period. To make our exposition easier to follow, we list all the notations that will appear hereafter as follows:

d_i dividend yield on risky asset i, equal to the monetary dividend divided by the current price
k_i (per unit trade) transaction cost rate of the ith risky asset, $k_i \geqslant 0$, $i = 1, \ldots, n$
r_i holding period rate of return on risky asset i ($i = 1, \ldots, n$)
r_{n+1} holding period rate of return on the riskless asset
t_g marginal capital gains tax rate for the investor

t_0 marginal ordinary income tax rate for the investor
x_i proportion of wealth that the investor will invest on the ith risky asset ($i = 1,\ldots,n$) or the riskless asset ($i = n + 1$)
x_i^0 proportion of wealth that the investor already holds in the ith risky asset ($i = 1,\ldots,n$) or the riskless asset ($i = n + 1$)

With the above notations, the total capital gains on portfolio $x = (x_1,x_2,\ldots,x_n,x_{n+1})^T$ is then $\sum_{i=1}^n r_i x_i$, and the total ordinary income on the portfolio is expressed as $\sum_{i=1}^n d_i x_i + r_{n+1}x_{n+1}$.

Transaction cost is an important factor for an investor to take into consideration in the portfolio selection. Ignoring transaction cost in a portfolio selection model often leads to an inefficient portfolio in practice. Here, we assume that the transaction cost c_i of the ith risky asset is a V-shaped function of the difference between the old portfolio $x^0 = (x_1^0,x_2^0,\ldots,x_n^0,x_{n+1}^0)^T$ and the new portfolio x:

$$c_i = k_i|x_i - x_i^0|, \quad i = 1,2,\ldots,n.$$

Thus the total transaction cost is $\sum_{i=1}^n c_i = \sum_{i=1}^n k_i|x_i - x_i^0|$.

The net return rate after excluding the tax and transaction cost on the portfolio is

$$g(x) \doteq (1-t_g)\sum_{i=1}^n r_i x_i + (1-t_0)\left(\sum_{i=1}^n d_i x_i + r_{n+1}x_{n+1}\right) - \sum_{i=1}^n k_i|x_i - x_i^0| = \sum_{i=1}^n R_i x_i + R_{n+1}x_{n+1} - \sum_{i=1}^n k_i|x_i - x_i^0|, \quad (7)$$

where $R_i \doteq (1-t_g)r_i + (1-t_0)d_i$ is the after-tax rate of return on risky asset $i(i = 1,2,\ldots,n)$, and $\overline{R}_{n+1} = R_{n+1} \doteq (1-t_0)r_{n+1}$ is the after-tax rate of return on the riskless asset. The expected net return after excluding the tax and transaction cost on the portfolio is

$$G(x) \doteq E[g(x)] = \sum_{i=1}^{n+1} \bar{R}_i x_i - \sum_{i=1}^n k_i|x_i - x_i^0|,$$

where $\overline{R}_i \doteq E[R_i] = (1-t_g)\bar{r}_i + (1-t_0)d_i$ is the expected after-tax rate of return on risky asset i and $\bar{r}_i = E[r_i]$ is the expected value of r_i, $i = 1,2,\ldots,n$.

Let

$$x_i' = (x_i - x_i^0)^+, \quad x_i'' = (x_i - x_i^0)^-, \quad i = 1,2,\ldots,n,$$

and $x' = (x_1',\ldots,x_n')^T, x'' = (x_1'',\ldots,x_n'')^T$. Then we can re-express $g(x)$ and $G(x)$ as follows:

$$g(x,x',x'') \doteq \sum_{i=1}^{n+1} R_i x_i - \sum_{i=1}^n k_i(x_i' + x_i''), \quad (8)$$

$$G(x,x',x'') \doteq \sum_{i=1}^{n+1} \overline{R}_i x_i - \sum_{i=1}^n k_i(x_i' + x_i'').$$

Aside from the taxes and transaction costs, other typical constraints (Chen and Wang, 2007, 2008), such as the upper and lower bounds on the proportion of the wealth that the investor will invest on a certain asset, should also be considered so that real characteristics of the practical investment environment can be modeled. The constraint on the lower and upper bounds, denoted as $\underline{x}_i$ and $\bar{x}_i$, respectively, can be written as

$$\underline{x}_i \leqslant x_i \leqslant \bar{x}_i, \quad i = 1,2,\ldots,n+1.$$

The above constraint is partly imposed because of the institutional restrictions.

In many security markets in the world, for example the Chinese stock markets, short sales are forbidden and borrowing is either strictly restricted or rather costly. Here, we consider the portfolio optimization problem in the case of no-short sales of assets and no borrowing. The investor wants to minimize the investment risk measured by WES_α while ensuring a certain level of return on investment at the end of the investment period. Given $\alpha \in (0,1)$ and a target portfolio return rate r_p, the corresponding portfolio optimization problem can be described as the following stochastic program:

$$\begin{aligned} \min \quad & WES_\alpha(g(x,x',x'')) \\ \text{s.t.} \quad & E[g(x,x',x'')] \geqslant r_p, \\ & \sum_{i=1}^{n+1} x_i = 1, \\ & x_i' = (x_i - x_i^0)^+, \quad i = 1,2,\ldots,n, \\ & x_i'' = (x_i - x_i^0)^-, \quad i = 1,2,\ldots,n, \\ & \underline{x}_i \leqslant x_i \leqslant \bar{x}_i, \quad i = 1,2,\ldots,n+1, \\ & x_i \geqslant 0, \quad i = 1,2,\ldots,n+1. \end{aligned} \quad (9)$$

where $WES_\alpha(g(x,x',x''))$ is the portfolio risk. The constraint $\sum_{i=1}^{n+1} x_i = 1$ implies that the initial budget is fully invested among risky and riskless assets, and the last group of constraints $x_i \geqslant 0$, $i = 1,2,\ldots,n+1$, are no-short selling constraints for risky assets and the no-borrowing constraint for the riskless asset.

It should be pointed out that when $t_g = 0$ (or $t_0 = 0$), our model includes the case where no taxes are imposed on capital gains (or ordinary income). When $k_i = 0$ for $i = 1,2,\ldots,n$, our model includes the case where there are no transaction costs. In addition, when the variable x_{n+1} is eliminated from the model (9), the model also includes the case where all the investment is put on risky assets.

Due to the simultaneous occurrence of the order statistics and the expectation operator in the objective function and the first inequality constraint, we cannot directly use existing algorithms to solve this stochastic programming problem. For this reason, problem (9) will be transformed into a tractable nonlinear programming problem by sufficiently utilizing the properties of WES_α, the problem background as well as the discrete approximation to the joint return rate distribution of risky assets. On the other hand, as an extension to the current discussion on the implementation of convex risk measures, model (9) and the solution method to be introduced provide a concrete framework for the practical application of convex risk measures, which improve theoretical studies in Föllmer and Schied (2002) and Lüthi and Doege (2005) from the application point of view.

We first consider the computation of $WES_\alpha(g(x,x',x''))$. Based on the empirical results in Section 3, we use the L-statistics method to compute $WES_\alpha(g(x,x',x''))$ so that a good balance can be achieved between the stability of estimation and the overall efficiency of our portfolio selection method. The advantage of this computing is that we do not make any assumption on the distribution of the return rate vector; instead, we work directly with the historical data. Meanwhile, through a series of transformations, we can reformulate the optimization problem (9) as a smooth convex optimization problem, which can be efficiently solved by using current nonlinear programming algorithms.

As the full distribution of r_i is rarely known in reality, the expected return rate $\bar{r}_i$ is often estimated by the average of a set of data $\{r_{im}, m = 1,2,\ldots,M\}$, where r_{im} can be either the observed historical sample or the forecast return rate for security i at time m. Therefore, we set $\bar{r}_i = \frac{1}{M}\sum_{m=1}^M r_{im}$. The portfolio risk $WES_\alpha(g(x,x',x''))$ can thus be estimated as

$$\widehat{WES}_\alpha(g(x,x',x'')) = -\frac{1}{\alpha M}\left(\sum_{m=1}^{[\alpha M]} w(g(x,x',x'')_{(m)})g(x,x',x'')_{(m)} + (\alpha M - [\alpha M])w(g(x,x',x'')_{([\alpha M]+1)})g(x,x',x'')_{([\alpha M]+1)}\right),$$

where

$$g(x,x',x'')_m = \sum_{i=1}^{n}[(1-t_g)r_{im} + (1-t_0)d_i]x_i + (1-t_0)r_{n+1}x_{n+1} - \sum_{i=1}^{n} k_i(x'_i + x''_i)$$

and $g(x,x',x'')_{(m)}$ is the mth order statistic of portfolio return rates $g(x,x',x'')_m$, $m = 1,2,\ldots,M$. Similarly, $G(x,x',x'')$ can be estimated by

$$\widehat{G}(x,x',x'') = \sum_{i=1}^{n}\left[(1-t_g)\left(\frac{1}{M}\sum_{m=1}^{M} r_{im}\right) + (1-t_0)d_i\right]x_i + (1-t_0)r_{n+1}x_{n+1} - \sum_{i=1}^{n} k_i(x'_i + x''_i) = \sum_{i=1}^{n+1}\widehat{\overline{R}}_i x_i - \sum_{i=1}^{n} k_i(x'_i + x''_i),$$

where

$$\widehat{\overline{R}}_i \doteq (1-t_g)\left(\frac{1}{M}\sum_{m=1}^{M} r_{im}\right) + (1-t_0)d_i, \quad i = 1,2,\ldots,n,$$

is an estimate of the expected after-tax rate of return on risky asset i and $\widehat{\overline{R}}_{n+1} \doteq (1-t_0)r_{n+1}$.

With the above expressions, problem (9) can be transformed into the following realistic portfolio optimization problem:

$$\begin{aligned}
\min \quad & W\widehat{ES}_\alpha(g(x,x',x'')) && (10)\\
\text{s.t.} \quad & \widehat{G}(x,x',x'') \geqslant r_p, && (11)\\
& \sum_{i=1}^{n+1} x_i = 1, && (12)\\
& x'_i = (x_i - x_i^0)^+, \quad i = 1,2,\ldots,n, && (13)\\
& x''_i = (x_i - x_i^0)^-, \quad i = 1,2,\ldots,n, && (14)\\
& \underline{x}_i \leqslant x_i \leqslant \bar{x}_i, \quad i = 1,2,\ldots,n+1, && (15)\\
& x_i \geqslant 0, \quad i = 1,2,\ldots,n+1. && (16)
\end{aligned}$$

This programming problem is difficult to solve because of the presence of the order statistics $g(x,x',x'')_{(m)}$, $m = 1,2,\ldots,M$, and the non-smooth constraints (13) and (14). Fortunately, constraints (13) and (14) can be equivalently re-expressed as

$$x_i = x'_i - x''_i + x_i^0, \quad x'_i x''_i = 0, \quad x'_i \geqslant 0, \quad x''_i \geqslant 0, \quad i = 1,2,\ldots,n.$$

Furthermore, we can prove that the above portfolio optimization problem (10)–(16) is equivalent to the following optimization problem:

$$\begin{aligned}
\min \quad & W\widehat{ES}_\alpha(g(x,x',x'')) && (17)\\
\text{s.t.} \quad & \widehat{G}(x,x',x'') \geqslant r_p, && (18)\\
& \sum_{i=1}^{n+1} x_i = 1, && (19)\\
& x_i = x'_i - x''_i + x_i^0, \quad i = 1,2,\ldots,n, && (20)\\
& x'_i \geqslant 0, \quad x''_i \geqslant 0, \quad i = 1,2,\ldots,n, && (21)\\
& \underline{x}_i \leqslant x_i \leqslant \bar{x}_i, \quad i = 1,2,\ldots,n+1, && (22)\\
& x_i \geqslant 0, \quad i = 1,2,\ldots,n+1, && (23)
\end{aligned}$$

The equivalence between the problem (10)–(16) and the problem (17)–(23) can be established if we can show that $x'_i x''_i = 0$ for any i ($i = 1,2,\ldots,n$) in the optimal solution of the problem (17)–(23). Suppose that, on the contrary, $x'_{i_0} x''_{i_0} > 0$ for some i_0 ($1 \leqslant i_0 \leqslant n$) in the optimal solution (x,x',x'') of the problem (17)–(23). Without loss of generality, we assume that $x'_{i_0} \geqslant x''_{i_0} > 0$. Then, according to (8), we get

$$g(x,x',x'') = \sum_{i=1}^{n+1} R_i x_i - \sum_{i=1, i\neq i_0}^{n} k_i(x'_i + x''_i) - k_{i_0}\left(x'_{i_0} - x''_{i_0}\right) - 2k_{i_0}x''_{i_0},$$

$$\widehat{G}(x,x',x'') = \sum_{i=1}^{n+1}\widehat{\overline{R}}_i x_i - \sum_{i=1, i\neq i_0}^{n} k_i(x'_i + x''_i) - k_{i_0}\left(x'_{i_0} - x''_{i_0}\right) - 2k_{i_0}x''_{i_0}.$$

We can construct a new solution vector $(\bar{x},\bar{x}',\bar{x}'')$ by only changing the i_0th components of x' and x'' in (x,x',x'') into $x'_{i_0} - x''_{i_0}$ and 0, respectively. It is obvious that $(\bar{x},\bar{x}',\bar{x}'')$ satisfies all the linear constraints in problem (17)–(23) and $g(\bar{x},\bar{x}',\bar{x}'') > g(x,x',x'')$ and $\widehat{G}(\bar{x},\bar{x}',\bar{x}'') > \widehat{G}(x,x',x'')$. This means that $(\bar{x},\bar{x}',\bar{x}'')$ is another feasible solution to problem (17)–(23). Meanwhile, by applying the conclusion in the Remark after Proposition 3 to the discretization form of WES_α, that is $W\widehat{ES}_\alpha$, we get that $W\widehat{ES}_\alpha(g(\bar{x},\bar{x}',\bar{x}'')) < W\widehat{ES}_\alpha(g(x,x',x''))$. This contradicts the optimality of (x,x',x'') to the problem (17)–(23).

It is natural from the financial point of view that the problem (10)–(16) is equivalent with the problem (17)–(23). Due to the transaction cost, any rational investor will either buy or sell, but will not do both. This means that $x'_i x''_i = 0$, $i = 1,2,\ldots,n$, will automatically hold in the optimal solution of the problem (17)–(23).

Problem (17)–(23) needs to be further transformed to avoid the order statistics $g(x,x',x'')_{(m)}$ ($m = 1,2,\ldots,M$) in the objective function. Since M is usually rather large, αM is often an integer in practice. Fortunately, in this situation, we have.

Proposition 8. *When αM is an integer,*

$$\sum_{m=1}^{[\alpha M]} w(g(x,x',x'')_{(m)})g(x,x',x'')_{(m)} + (\alpha M - [\alpha M])w(g(x,x',x'')_{([\alpha M]+1)})g(x,x',x'')_{([\alpha M]+1)}$$

equals the optimal value of the following linear optimization problem

$$\begin{aligned}
-\min \quad & \alpha t M + \sum_{i=1}^{M} y_i \\
\text{s.t.} \quad & w(g(x,x',x'')_i)g(x,x',x'')_i + t + y_i \geqslant 0, \quad i = 1,2,\ldots,M, && (24)\\
& y_i \geqslant 0, \quad i = 1,2,\ldots,M.
\end{aligned}$$

Proof. First, since αM is an integer, $(\alpha M - [\alpha M])w(g(x,x',x'')_{([\alpha M]+1)})g(x,x',x'')_{([\alpha M]+1)}$ disappears. Let $m' = \min\{m : g(x,x',x'')_{(m)} > 0\}$, we have

$$w(g(x,x',x'')_{(m)})g(x,x',x'')_{(m)} \geqslant w(g(x,x',x'')_{(m'-1)})g(x,x',x'')_{(m'-1)}$$

for $m = m', m'+1,\ldots,M$ because $w(\cdot)$ is always non-negative and monotonically non-increasing; since $\alpha \ll 0.50$, we have $g(x,x',x'')_\alpha \leqslant 0$, from which we get $g(x,x',x'')_{([\alpha M])} \leqslant 0$. These two facts mean that

$$w(g(x,x',x'')_{(1)})g(x,x',x'')_{(1)} \leqslant \cdots \leqslant w(g(x,x',x'')_{([\alpha M])})g(x,x',x'')_{([\alpha M])} \leqslant \cdots \leqslant w(g(x,x',x'')_{(m'-1)})g(x,x',x'')_{(m'-1)} < 0.$$

Meanwhile, if $g(x,x',x'') \leqslant 0$, $w(g(x,x',x''))g(x,x',x'')$ is a non-decreasing function of $g(x,x',x'')$. All the above conclusions ensure that

$$\sum_{m=1}^{[\alpha M]} w(g(x,x',x'')_{(m)})g(x,x',x'')_{(m)} + (\alpha M - [\alpha M])w(g(x,x',x'')_{([\alpha M]+1)})g(x,x',x'')_{([\alpha M]+1)}$$

equals the optimal value of the following linear program:

$$\begin{aligned} \min \quad & \sum_{i=1}^{M} w(g(x,x',x'')_i)g(x,x',x'')_i z_i \\ \text{s.t.} \quad & \sum_{i=1}^{M} z_i = \alpha M, \quad 0 \leqslant z_i \leqslant 1, \ \ i = 1,2,\ldots,M. \end{aligned} \tag{25}$$

According to the strong duality theorem for linear programs, the optimum of the problem (25) equals the optimal value of the following linear programming problem:

$$\begin{aligned} \max \quad & (\alpha M)\lambda + \sum_{i=1}^{M} (-1.0)\lambda_i \\ \text{s.t.} \quad & \lambda - \lambda_i \leqslant w(g(x,x',x'')_i)g(x,x',x'')_i, \quad i = 1,2,\ldots,M, \\ & \lambda_i \geqslant 0, \quad i = 1,2,\ldots,M, \end{aligned}$$

where the dual variables λ and $\lambda_i (1 \leqslant i \leqslant M)$ correspond to the equality constraint $\sum_{i=1}^{M} z_i = \alpha M$ and the inequality constraints $-z_i \geqslant -1 (1 \leqslant i \leqslant M)$, respectively.

Let $t = -\lambda$ and $y_i = \lambda_i$. Then, we obtain

$$\begin{aligned} \max \quad & -\alpha t M - \sum_{i=1}^{M} y_i \\ \text{s.t.} \quad & w(g(x,x',x'')_i)g(x,x',x'')_i + t + y_i \geqslant 0, \quad i = 1,2,\ldots,M, \\ & y_i \geqslant 0, \quad i = 1,2,\ldots,M. \end{aligned} \tag{26}$$

Considering that $\max(\theta) = -\min(-\theta)$, we can see that problem (26) is actually the problem (24). □

Since

$$\min -\frac{1}{\alpha M}\left(-\min\left(\alpha t M + \sum_{i=1}^{M} y_i\right)\right)$$

is equivalent to

$$\min t + \frac{1}{\alpha M}\sum_{i=1}^{M} y_i,$$

according to Proposition 8, problem (17)–(23) can be re-formulated as

$$\min \quad t + \frac{1}{\alpha M}\sum_{i=1}^{M} y_i \tag{27}$$

$$\text{s.t.} \quad w(g(x,x',x'')_i)g(x,x',x'')_i + t + y_i \geqslant 0, \quad i = 1,2,\ldots,M, \tag{28}$$

$$\widehat{G}(x,x',x'') \geqslant r_p, \tag{29}$$

$$\sum_{i=1}^{n+1} x_i = 1, \tag{30}$$

$$x_i = x'_i - x''_i + x_i^0, \quad i = 1,2,\ldots,n, \tag{31}$$

$$x'_i \geqslant 0, \ x''_i \geqslant 0, \quad i = 1,2,\ldots,n, \tag{32}$$

$$\underline{x}_i \leqslant x_i \leqslant \bar{x}_i, \quad i = 1,2,\ldots,n+1, \tag{33}$$

$$x_i \geqslant 0, \quad i = 1,2,\ldots,n+1, \tag{34}$$

$$y_i \geqslant 0, \quad i = 1,2,\ldots,M. \tag{35}$$

What's more important, by using the same method as that in the proof of Proposition 1, we can prove that $w(g(x,x',x'')_i)g(x,x',x'')_i$ is concave with respect to (x,x',x'') for the nonlinear weight function $w(x)$ specified in Definition 1. Therefore, the problem (27)–(35) is a convex optimization problem. This and Proposition 6 ensure that the problem (27)–(35) can be easily solved by using typical smooth convex programming algorithms, for example, the sequential quadratic programming method. Our portfolio selection model is thus more practical and easier to solve than the models under the convex risk measures in Föllmer and Schied (2002) and Lüthi and Doege (2005).

5. Empirical results

The detailed impact of different risk measures from WES_α and different market frictions on the optimal portfolio selection will be investigated empirically in this section by using real trade data. In the following experiments, the nonlinear weight function is chosen to be $w(x) = \exp(-\lambda x)$ with specific λ if $x \leqslant 0$ and $w(x) \equiv 0$ if $x > 0$. Here $\lambda > 0$ can reflect the risk-averse degree. This weight function is selected because it is highly plausible and easy to interpret. For simplicity, we set the marginal capital gains tax rate to be equal to the marginal ordinary income tax rate. According to (7), we can directly use the daily return rate with dividend re-invested to take the dividend yield into account. Concretely, r_{im} is chosen as the daily rate of return of security i on day m with dividend re-invested. Without loss of generality, we assume that the investor only holds cash at the beginning of the investment period, i.e., $x_i^0 = 0$, $i = 1,2,\ldots,n$, which means that $x''_i = 0$ and $x_i = x'_i$.

To demonstrate the practical effect of WES_α on the optimal portfolio selection, results obtained using our new measure is compared with the corresponding results obtained using ES_α. Since ES_α corresponds to the special WES_α with $w(x) \equiv 1$, the corresponding portfolio selection model under ES_α can be directly derived from the problem (27)–(35) by setting $w(\cdot) \equiv 1$ in (28). Actually, the resulting optimization problem is simply a linear programming problem.

The effects of different risk measures characterized by the risk-averse coefficient λ, different transaction cost ratios k, and different target rates of return r_p on the optimal portfolio selection is examined in succession from the following perspectives: the diversification of the optimal portfolio, the return rate, the risk magnitude and the performance of the optimal portfolio. The portfolio diversification is examined by the number of stocks actually included in the optimal portfolio and the Herfindahl index (H-index) of concentration (Silver, 1985). H-index can measure concentration and diversification at both the aggregate and market levels. The lower the H-index is, the better the diversification of the portfolio is. In order to comprehensively compare the performance of the optimal portfolio from different angles, we select a group of performance ratios to evaluate its performance. The examined ratios include the return/WES_α (R/Risk) ratio, the return/ES_α (R/ES) ratio, the return/Power CVaR (R/PCVaR) ratio as well as two newly introduced two-sided variability ratios (the generalized Rachev (G-Rachev) ratio and the Farinelli–Tibiletti (F–T) ratio). Here, the power CVaR is defined as

$$PCVaR_{(\alpha,q)}(X) = E[(-X)^q | X \leqslant -VaR_\alpha(X)],$$

where X denotes the return rate of the portfolio and $VaR_\alpha(X) = -\inf\{x \in R : P[X \leqslant x] \geqslant \alpha\}$. The F–T ratio is defined as the ratio of the properly weighted favorable events to the unfavorable ones:

$$\text{the F–T ratio} = \frac{E^{\frac{1}{p}}[(X^+)^p]}{E^{\frac{1}{q}}[(X^-)^q]}.$$

The G-Rachev ratio is defined as the ratio of the power CVaR of the opposite of the excess return at a given confidence level to the power CVaR of the excess return at another confidence level:

$$\text{the G-Rachev ratio} = \frac{E[(X^+)^\gamma | X \geqslant -VaR_\alpha(X)]}{E[(X^-)^\delta | X \leqslant -VaR_\beta(X)]}.$$

Please refer to papers such as Farinelli et al. (2006, 2008) about the detailed definition and computation of these performance ratios. Since all these ratios are the reward-to-risk indices, the higher the ratio is, the more efficient the corresponding portfolio will be.

In order to show the practicality, super-performance and robustness of our new risk measure and the corresponding portfolio selection model for optimal investment decision-making, we carry out empirical studies by using trade data from both the emerging stock markets, the Chinese stock markets, and the advanced stock markets, the American stock markets. Moreover, in addition to the usual in-sample test, the out-of-sample comparison is also done to further illustrate the super-performance and robustness of our new risk measure and the corresponding portfolio selection model. The detailed empirical results are presented in the following two subsections.

All the empirical tests are carried out by using the Lenovo Personal Computer with Pentium 4, CPU 2.40 GHz and 3.00 GB Memory. The function **fmincon** in the optimization toolbox of Matlab 7.0 is used to solve each convex optimization problem (27)–(35).

5.1. The Chinese stock markets

A riskless asset and 30 risky stocks are randomly selected as our investment universe from all the A-share stocks in Chinese stock markets. Daily return rates with dividend re-invested of these stocks in 600 trading days from May 31, 2004 to November 16, 2006 are used to determine the values of the parameters in the problem (27)–(35). In the experiments, we set $p = \gamma = 2$, $q = \delta = 5$, $\alpha = \beta = 0.05$, $t_g = t_0 = 0.00001$, and $r_{n+1} = 0.00007$. Moreover, to find out the out-of-sample performance and robustness of the optimal portfolios determined using the above data, we examine the optimal portfolios' returns, risks and performance ratios 1 week (5 days, OS-5) after and 4 weeks (20 days, OS-20) after the final date November 16, 2006, respectively. Both in-sample results, IS-600, and out-of-sample results, OS-5 and OS-20, are presented in each test.

First, we consider the influence of different risk measures specified by λ in the weight function $w(x)$ on the optimal portfolio selection. Considering the practical situation of Chinese stock markets, we set $k_i = 0.0003$ for all the chosen stocks. Since the return rates of these stocks in the adopted sample period are very low, we set the required return rate to $r_p = 0.0016$. Shown in Table 4 are the characteristics of the optimal portfolios obtained using different risk measures $WES_{0.05}$ corresponding to $\lambda = 0.1$, 10, 20, 40 and 60, and $ES_{0.05}$, respectively. Tabled here are only those stocks with non-zero holdings. For convenience, the column with $\lambda = 0$ in Table 4 is used to present results obtained using $ES_{0.05}$.

The five optimal portfolios have the same return rate 0.0016, which equals the required rate of return. When λ increases from 0 to 60, the risk value of the corresponding optimal portfolio monotonically increases, which means that the larger the λ is, the more suitable the corresponding measure is for conservative investors to adopt. We naturally deduce from these risk values that the R/Risk ratio monotonically decreases with the increasing of λ.

Table 4
Optimal portfolios and their characteristics under different risk measures characterized by the risk-averse coefficient λ–Chinese stocks.

λ	0	0.1	10	20	40	60
000002	0.0350	0.0362	0.0307	0.0307	0.0349	0.0392
600018	0.0917	0.0915	0.0978	0.0971	0.1027	0.1056
600028	0.0442	0.0439	0.0545	0.0541	0.0490	0.0466
600030	0.0002	0.00004	0	0	0	0
600033	0.0098	0.0097	0.0016	0.0039	0.0093	0.0106
600036	0.1296	0.1291	0.1023	0.1063	0.0968	0.0793
000033	0.0129	0.0129	0.0151	0.0167	0.0080	0.0044
000527	0.0600	0.0607	0.0723	0.0637	0.0674	0.0776
000538	0.1685	0.1686	0.1670	0.1691	0.1679	0.1662
000568	0.0730	0.0726	0.0903	0.0968	0.1075	0.1318
600151	0.0011	0.0013	0.0022	0.0019	0.0012	0
600177	0.0738	0.0735	0.0663	0.0597	0.0551	0.0388
Riskless	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000
IS-600						
Return	0.0016	0.0016	0.0016	0.0016	0.0016	0.0016
Risk	0.0167	0.0167	0.0201	0.0243	0.0359	0.0535
Stk.no.	13	13	12	12	12	11
H-index	0.1615	0.1613	0.1598	0.1605	0.1610	0.1638
R/Risk	0.0959	0.0957	0.0794	0.0658	0.0446	0.0300
R/ES	0.0959	0.0959	0.0955	0.0955	0.0950	0.0938
R/PCVaR	3.64E+05	3.64E+05	4.03E+05	4.19E+05	4.63E+05	5.26E+05
G-Rachev	3.27E+04	3.27E+04	3.61E+04	3.69E+04	4.07E+04	4.55E+04
F–T	0.8655	0.8649	0.8825	0.8823	0.8967	0.9075
OS-5						
Return	0.00908	0.00908	0.00913	0.0089	0.00915	0.00914
Risk	0.0042	0.0042	0.0040	0.0042	0.0043	0.0036
R/Risk	2.1567	2.1670	2.2649	2.1001	2.1546	2.5105
R/ES	2.1567	2.1679	2.3545	2.2716	2.4949	3.0118
R/PCVaR	6.88E+09	7.04E+09	1.04E+10	9.57E+09	1.38E+10	3.55E+10
G-Rachev	1.66E+08	1.71E+08	2.39E+08	2.12E+08	3.03E+08	7.43E+08
F–T	3.5218	3.5444	3.7285	3.5789	3.8666	4.5574
OS-20						
Return	0.0075	0.0075	0.0076	0.0076	0.0078	0.0079
Risk	0.0224	0.0224	0.0268	0.0329	0.0498	0.0655
R/Risk	0.3350	0.3344	0.2839	0.2313	0.1571	0.1211
R/ES	0.3350	0.3352	0.3524	0.3552	0.3678	0.3992
R/PCVaR	1.34E+06	1.34E+06	1.62E+06	1.68E+06	1.80E+06	2.56E+06
G-Rachev	5.30E+04	5.32E+04	6.23E+04	6.46E+04	6.90E+04	9.71E+04
F–T	1.1015	1.1026	1.1339	1.1412	1.1658	1.2485

Here and in the following tables, Stk.no. is the number of stocks actually invested, H-index is the value of the Herfindahl index, Risk is the value of the corresponding risk measure, Return is the return rate, R/Risk, R/ES and R/PCVaR are the ratios of the return value to the associated risk value, ES and power CVaR, respectively. G-Rachev stands for the generalized Rachev ratio and F–T is the Farinelli–Tibiletti ratio.

The number of the stocks actually included in the optimal portfolio and the H-index value presented in Table 4 indicate that the diversification of the optimal portfolio decreases with the increase in λ. This decrease in diversification is due to the fact that the risk-averse degree monotonically increases with λ and the investor will control the investment risk by concentrating his/her investment on a few superior stocks.

To further show the advantage of our new risk measure, we examine the performances of the optimal portfolios under four common performance measures R/ES, R/PCVaR, G-Rachev and F–T ratios. Naturally, the $ES_{0.05}$ optimal portfolio has the largest R/ES ratio; for the other optimal portfolios, the R/ES ratio decreases with the increase in λ. The R/PCVaR ratio and G-Rachev ratio uniformly and monotonically increase with respect to λ. Although oscillating slightly, the F–T ratio basically monotonically increases with respect to λ. This is due to the fact that our new risk measure mainly concerns the lower tail risk while the F–T ratio focuses on the positive and negative deviations from 0 simultaneously. These results demonstrate that the performance of the optimal portfolio obtained using $WES_{0.05}$ is better than that using the current popular coherent risk measure $ES_{0.05}$. Generally speaking, the larger the λ is, the more significantly the performance of the optimal portfolio will improve.

The 1-week-after and 4-week-after tests in Table 4 show that $WES_{0.05}$ optimal portfolios almost surely have higher returns than the $ES_{0.05}$ optimal portfolio. What's more important, the out-of-sample performance of the $WES_{0.05}$ optimal portfolios is superior to that of the $ES_{0.05}$ optimal portfolio under almost all the performance ratios (R/ES, R/PCVaR, G-Rachev and F–T). Usually, the bigger the λ is, the better the out-of-sample performance is. Therefore, the performances of the $WES_{0.05}$ optimal portfolios for the in-sample and out-of-sample tests often significantly increase with the increase in λ, when compared with those of the $ES_{0.05}$ optimal portfolio.

The superior performance and robustness of our portfolio selection model (27)–(35) can be explained as follows: as an extension to ES_α, our new risk measure considers losses below VaR_α unequally. It can flexibly describe the investor's specific risk-averse attitude. The penalty for large losses strictly increases with the increase in λ, and the asymmetry of WES_α thus monotonically increases with respect to λ. These modifications are of vital importance for the improvement of the performance and robustness of the resulting optimal portfolio. In addition to reflecting the investor's degree of risk aversion, the nonlinear weight function $w(x)$ can also help us to control the fat-tail phenomenon, that is, the occurrence of extreme losses.

Table 5
The characteristics of optimal portfolios under different risk measures and different transaction cost ratios–Chinese stocks.

Risk measures	k	0	0.0001	0.0002	0.0003	0.00035
IS-600 (WES)	Return	0.0016	0.0016	0.0016	0.0016	0.0016
	Risk	0.0182	0.0186	0.0193	0.0201	0.0206
	Stk.no.	11	11	11	12	12
	H-index	0.1566	0.1607	0.1559	0.1598	0.1644
	R/Risk	0.0881	0.0858	0.0828	0.0794	0.0778
	R/ES	0.1040	0.1019	0.0990	0.0956	0.0941
	R/PCVaR	7.79E+05	6.20E+05	4.82E+05	4.03E+05	3.43E+05
	G-Rachev	5.82E+04	4.78E+04	4.03E+04	3.61E+04	3.14E+04
	F–T	0.9110	0.8850	0.8820	0.8825	0.8644
IS-600 (ES)	Return	0.0016	0.0016	0.0016	0.0016	0.0016
	Risk	0.0154	0.0157	0.0162	0.0167	0.0170
	Stk.no.	11	11	11	13	13
	H-index	0.1558	0.1548	0.1557	0.1615	0.1637
	R/ES	0.1041	0.1017	0.0990	0.0959	0.0942
	R/PCVaR	7.49E+05	5.99E+05	4.63E+05	3.64E+05	3.23E+05
	G-Rachev	5.62E+04	4.62E+04	3.87E+04	3.27E+04	3.00E+04
	F–T	0.9078	0.8802	0.8732	0.8660	0.8610
OS-5 (WES)	Return	0.0116	0.0105	0.0101	0.0091	0.0085
	Risk	0.0037	0.0035	0.0038	0.0040	0.0044
	R/Risk	3.1420	2.9912	2.6438	2.2649	1.9261
	R/ES	3.2560	3.0940	2.7430	2.3545	2.0088
	R/PCVaR	2.02E+10	2.37E+10	1.49E+10	1.04E+10	6.40E+09
	G-Rachev	5.31E+08	6.06E+08	3.70E+08	2.39E+08	1.47E+08
	F–T	4.9020	4.8332	4.2985	3.7285	3.3138
OS-5 (ES)	Return	0.0115	0.0108	0.0102	0.0091	0.0087
	Risk	0.0037	0.0035	0.0038	0.0042	0.0042
	R/ES	3.1086	3.0750	2.6860	2.1567	2.0813
	R/PCVaR	1.64E+10	2.02E+10	1.31E+10	6.88E+09	6.92E+09
	G-Rachev	4.32E+08	5.25E+08	3.30E+08	1.66E+08	1.69E+08
	F–T	4.7024	4.7655	4.2329	3.5218	3.4887
OS-20 (WES)	Return	0.0080	0.0076	0.0078	0.0076	0.0073
	Risk	0.0322	0.0276	0.0290	0.0268	0.0255
	R/Risk	0.2475	0.2740	0.2680	0.2839	0.2877
	R/ES	0.3180	0.3418	0.3373	0.3524	0.3541
	R/PCVaR	8.04E+05	1.43E+06	1.20E+06	1.62E+06	1.91E+06
	G-Rachev	3.17E+04	5.56E+04	4.71E+04	6.23E+04	7.41E+04
	F–T	1.0115	1.1094	1.0832	1.1339	1.1625
OS-20 (ES)	Return	0.0078	0.0078	0.0078	0.0075	0.0074
	Risk	0.0248	0.0242	0.0234	0.0224	0.0215
	R/ES	0.3158	0.3227	0.3322	0.3350	0.3440
	R/PCVaR	8.39E+05	9.39E+05	1.12E+06	1.34E+06	1.60E+06
	G-Rachev	3.26E+04	3.76E+04	4.40E+04	5.30E+04	6.35E+04
	F–T	1.0072	1.0449	1.0703	1.1015	1.1385

Next, we investigate how different transaction cost ratios affect the diversification of the optimal portfolio and its performance. To make comparisons on the same basis, we fix $r_p = 0.0016$ and consider five different transaction cost ratios k_i $(1 \leqslant i \leqslant n) = k = 0$, 0.0001, 0.0002, 0.0003 and 0.00035, respectively. Due to the space limitation, we only report in Table 5 the results for $WES_{0.05}$ with $\lambda = 10$ and $ES_{0.05}$.

We first examine the in-sample results IS-600 (WES) and IS-600 (ES). When k increases, the return rate of the optimal portfolio is always equal to the required return rate 0.0016, while the value of risk monotonically increases. Accordingly, the R/Risk ratio decreases. Similarly, the values of the performance ratios R/ES, R/PCVaR, G-Rachev and F–T decrease with the increase in k. Therefore, the performance of the optimal portfolio deteriorates with the increasing of the transaction cost ratio. Meanwhile, as k increases, the number of stocks really included in the optimal portfolio increases by, at most, 1 or 2, while the corresponding value of the H-index basically increases. Consequently, the diversification effect of the optimal portfolio degenerates. These results are rather reasonable. The investment cost is a monotonically increase function of k, therefore, in order to ensure the required investment return rate and to control risk, the investor has to concentrate his/her investment on the stocks with relatively higher returns but higher risks. Due to the investor's decision, the risk of the optimal portfolio increases and its performance deteriorates with the increase in k. Our results also confirm the existing empirical conclusion that the portfolio performance and diversification often decrease with the increase in the transaction cost.

The portfolio risk basically increases and the performance ratios (R/Risk, R/ES, R/PCVaR, G-Rachev and F–T) almost surely decrease monotonically with the increase in k in the 1-week out-of-sample tests (OS-5), which agrees with the in-sample results. Therefore, the conclusions for the in-sample case still hold for the 1-week out-of-sample situation. Nevertheless, probably due to the bull market season, the return rates of the optimal portfolios for the four groups of out-of-sample cases, OS-5 (WES), OS-5 (ES), OS-20 (WES) and OS-20 (ES), are all higher than the required return rate 0.0016. Interestingly, the risks of the optimal portfolios for the out-of-sample tests after 4 weeks, OS-20 (WES) and OS-20 (ES), basically decrease with the increase in k, while all the corresponding performance ratios R/Risk (R/ES), R/PCVaR, G-Rachev and F–T often increase with respect to k.

Last but not least, by comparing the results obtained under the transaction cost ratios given in Table 5, we can easily find that the performance ratios obtained under the $WES_{0.05}$ $(\lambda = 10)$ optimal portfolios for both in-sample and out-of-sample cases are almost

Table 6
The characteristics of optimal portfolios under different risk measures and different required return rates–Chinese stocks.

Risk measures	r_p	0.00152	0.00155	0.00158	0.0016	0.00162
IS-600 (WES)	Return	0.00152	0.00155	0.00158	0.0016	0.00162
	Risk	0.0193	0.0196	0.0199	0.0201	0.0203
	Stk.no.	11	11	11	12	12
	H-index	0.1549	0.1570	0.1591	0.1598	0.1627
	R/Risk	0.0786	0.0791	0.0794	0.0794	0.0797
	R/ES	0.0939	0.0947	0.0954	0.0955	0.0962
	R/PCVaR	4.70E+05	4.37E+05	4.14E+05	4.03E+05	3.53E+05
	G-Rachev	4.12E+04	3.85E+04	3.69E+04	3.61E+04	3.18E+04
	F–T	0.8866	0.8816	0.8812	0.8825	0.8656
IS-600 (ES)	Return	0.00152	0.00155	0.00158	0.0016	0.00162
	Risk	0.0161	0.0163	0.0165	0.0167	0.0168
	Stk.no.	11	11	12	13	13
	H-index	0.1588	0.1586	0.1619	0.1615	0.1622
	R/ES	0.0941	0.0949	0.0956	0.0959	0.0962
	R/PCVaR	4.18E+05	3.90E+05	3.76E+05	3.64E+05	3.45E+05
	G-Rachev	3.61E+04	3.44E+04	3.39E+04	3.27E+04	3.12E+04
	F–T	0.8576	0.8599	0.8706	0.8655	0.8637
OS-5 (WES)	Return	0.0101	0.0097	0.0093	0.0091	0.0088
	Risk	0.0039	0.0040	0.0042	0.0040	0.0043
	R/Risk	2.6090	2.4240	2.2278	2.2649	2.0607
	R/ES	2.7085	2.5187	2.3185	2.3545	2.1472
	R/PCVaR	1.38E+10	1.17E+10	9.13E+09	1.04E+10	7.52E+09
	G-Rachev	3.39E+08	2.82E+08	2.10E+08	2.39E+08	1.80E+08
	F–T	4.2220	3.9813	3.6564	3.7285	3.5397
OS-5 (ES)	Return	0.0101	0.0098	0.0094	0.0091	0.0088
	Risk	0.0039	0.0039	0.0042	0.0042	0.0042
	R/ES	2.5599	2.5057	2.2314	2.1567	2.1124
	R/PCVaR	1.06E+10	1.05E+10	7.09E+09	6.88E+09	6.94E+09
	G-Rachev	2.80E+08	2.75E+08	1.76E+08	1.66E+08	1.67E+08
	F–T	4.1447	4.0858	3.6250	3.5218	3.4863
OS-20 (WES)	Return	0.0077	0.0076	0.0076	0.0076	0.0074
	Risk	0.0297	0.0285	0.0278	0.0268	0.0270
	R/Risk	0.2600	0.2683	0.2731	0.2839	0.2758
	R/ES	0.3289	0.3367	0.3411	0.3524	0.3427
	R/PCVaR	1.08E+06	1.27E+06	1.39E+06	1.62E+06	1.54E+06
	G-Rachev	4.23E+04	4.96E+04	5.41E+04	6.23E+04	6.11E+04
	F–T	1.0609	1.0898	1.1034	1.1339	1.1297
OS-20 (ES)	Return	0.0074	0.0075	0.0075	0.0075	0.0075
	Risk	0.0229	0.0229	0.0225	0.0224	0.0220
	R/ES	0.3252	0.3293	0.3320	0.3350	0.3415
	R/PCVaR	1.19E+06	1.20E+06	1.29E+06	1.34E+06	1.46E+06
	G-Rachev	4.72E+04	4.80E+04	5.09E+04	5.30E+04	5.79E+04
	F–T	1.0756	1.0842	1.0923	1.1015	1.1216

always significantly larger than the corresponding performance ratios obtained under the $ES_{0.05}$ optimal portfolio. Naturally, the superiority and robustness of $WES_{0.05}$ optimal portfolios for the 1 week out-of-sample tests are better than those in the out-of-sample tests after 4 weeks. All the above findings strongly demonstrate that our new risk measure shows better performance and robustness than $ES_{0.05}$ in finding optimal portfolios.

Finally, the impact of different target rates of return r_p on the optimal portfolio selection is investigated. Five different target return rates are selected here, and the associated results are presented in Table 6. Since the sample return rates of the selected stocks are low, we set the transaction cost ratio k_i ($i = 1,2,\ldots,n$) to 0.0003 to ensure that meaningful optimal portfolios can be found for large target return rates. Due to the space limitation, we report here the results obtained with the risk measures $WES_{0.05}$ ($\lambda = 10$) and $ES_{0.05}$, respectively.

Both the return rate and risk value of the optimal portfolio for the in-sample tests, IS-600 (WES) and IS-600 (ES), monotonically increase with the increase in r_p. Nonetheless, as the return rate increases faster than that of the risk, the R/Risk and R/ES ratios monotonically increase with respect to r_p. On the other hand, the R/PCVaR and G-Rachev ratios uniformly and monotonically decrease with respect to r_p. The F–T ratio oscillates slightly, for the same reason as the F–T ratio in Table 4 does. These conclusions confirm the well-known fact that portfolios with higher returns are almost surely accompanied by higher risks, measured by any reasonable risk measure. Meanwhile, an obvious financial explanation for the above variation in the performance ratio is that, to achieve the required high return rate, the investor tends to concentrate his/her investment on the stocks with relatively high return rate but large risk. For this reason, the diversifications of the optimal portfolios determined under the two risk measures basically decrease with the increase in r_p, due to the monotonic increase of the H-index and the small change in the number of stocks actually invested.

For the out-of-sample results, due to the superiority of $WES_{0.05}$, the return rates of the optimal portfolios for the 1-week cases (OS-5 (WES) and OS-5 (ES)) and the 4-week cases (OS-20 (WES) and OS-20 (ES)) are all higher than the corresponding required return rates. Just as the variation tendencies in the in-sample cases, the portfolio risks in the 1-week out-of-sample cases basically increase with respect to r_p, and the performance ratios (R/Risk, R/ES, R/PCVaR, G-Rachev and F–T) almost always monotonically decrease with respect to r_p. The portfolio risks in the out-of-sample tests after 4 weeks basically decrease with the increase in r_p, which results in the monotonic increasing of the R/Risk and R/ES ratios, and more importantly, the R/PCVaR, G-Rachev and F–T ratios with respect to r_p. These interesting changes are similar to what we have found in the out-of-sample test after 4 weeks in Table 5. These phenomena further show the super-performance and robustness of $WES_{0.05}$ in the optimal investment decision-making.

When the performance of the optimal portfolios obtained using $WES_{0.05}$ is compared with that of the corresponding optimal portfolios obtained using $ES_{0.05}$, conclusions similar to those derived

Table 7
Optimal portfolios and their characteristics under different risk measures characterized by the risk-averse coefficient λ–American stocks.

λ	0	0.1	20	60	80	100
DUK	0.0898	0.0909	0.0933	0.0974	0.0991	0.1052
GD	0.0582	0.0565	0.052	0.0459	0.0437	0.0414
GS	0.0491	0.0512	0.0513	0.0491	0.0502	0.0438
MET	0.0335	0.0359	0.0401	0.0424	0.044	0.0538
NKE	0	0	0	0.0024	0.0018	0.0004
RTP	0.0695	0.0679	0.0665	0.066	0.062	0.058
TRI	0.0085	0.0079	0.0078	0.0076	0.0083	0.004
UNH	0.0291	0.0303	0.0304	0.0296	0.0293	0.0287
WYE	0.0923	0.0904	0.0849	0.0796	0.0797	0.0763
PEP	0.0288	0.029	0.0308	0.0316	0.03	0.0301
BA	0.0577	0.0558	0.0548	0.0549	0.057	0.0671
MCD	0.1233	0.1232	0.1237	0.1246	0.1236	0.1221
SBUX	0.0602	0.0612	0.0645	0.0688	0.0713	0.069
Riskless	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000
IS-600						
Return	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005
Risk	0.0094	0.0095	0.0115	0.0171	0.0209	0.0256
Stk.no.	13	13	13	14	14	14
H-index	0.1422	0.1419	0.1416	0.1417	0.1418	0.1427
R/Risk	0.0530	0.0529	0.0435	0.0293	0.0239	0.0196
R/ES	0.0530	0.0530	0.0530	0.0529	0.0529	0.0529
R/PCVaR	4.36E+06	4.38E+06	4.43E+06	4.49E+06	4.53E+06	4.58E+06
G-Rachev	2.65E+05	2.67E+05	2.71E+05	2.79E+05	2.82E+05	2.82E+05
F–T	0.8017	0.8044	0.8080	0.8180	0.8194	0.8134
OS-5						
Return	0.0006	0.0006	0.0007	0.0008	0.0008	0.0008
Risk	0.0035	0.0035	0.0038	0.0044	0.0044	0.0042
R/Risk	0.1735	0.1798	0.1818	0.1747	0.1726	0.2009
R/ES	0.1735	0.1798	0.1951	0.216	0.2263	0.2729
R/PCVaR	1.10E+09	1.15E+09	1.23E+09	1.39E+09	1.72E+09	3.10E+09
G-Rachev	1.64E+07	1.66E+07	1.69E+07	1.77E+07	2.13E+07	3.58E+07
F–T	0.9674	0.973	0.989	1.0088	1.0365	1.1512
OS-10						
Return	0.0004	0.0004	0.0005	0.0005	0.0005	0.0005
Risk	0.0065	0.0066	0.0075	0.0097	0.0113	0.0121
R/Risk	0.0642	0.0667	0.0643	0.0555	0.0474	0.0411
R/ES	0.0642	0.0667	0.0734	0.0824	0.0805	0.0780
R/PCVaR	3.64E+07	3.60E+07	3.87E+07	4.42E+07	4.15E+07	4.67E+07
G-Rachev	7.69E+05	7.29E+05	7.32E+05	7.73E+05	7.26E+05	9.97E+05
F–T	0.564	0.560	0.567	0.578	0.570	0.670

from the previous two situations in Tables 4 and 5 can be obtained for both in-sample and out-of-sample cases. Concretely, for any given r_p, the performance ratio, under most examined performance measures, of the optimal portfolio determined under $WES_{0.05}$ is usually higher than that of the corresponding optimal portfolio obtained with $ES_{0.05}$. Therefore, our new risk measure is better than $ES_{0.05}$ in helping investors to find more efficient and robust investment strategy.

5.2. The American stock markets

In this subsection, we further investigate the performance and robustness of our new risk measure and the corresponding portfolio selection model in advanced stock markets. As comparison, a riskless asset and 30 risky stocks are randomly selected from the American stock markets as our investment universe. Daily returns with dividend re-invested on these stocks in 600 trading days, from May 28, 2004 to October 13, 2006, are used to determine the values of the parameters in the problem (27)–(35). In the experiments, we set $p = \gamma = 2$, $q = \delta = 5$, $\alpha = \beta = 0.05$, $t_g = t_0 = 0.00001$, and $r_{n+1} = 0.00007$.

To check the out-of-sample performance and robustness of the optimal portfolios determined using the above data and under different situations, we examine their return rates, risks and performance indices 1-week (5 days, OS-5) after and 2-week (10 days, OS-10) after the final date October 13, 2006, respectively. Both the in-sample results, IS-600, and the out-of-sample results, OS-5 and OS-10, are given in each following table.

We examine in succession the influence of the risk-averse coefficient, transaction cost ratio and target return rate on the configuration and performance of the optimal portfolio in the American stock markets. The detailed results are given in Tables 7–9, respectively. By comparing the results in Tables 4–6 with the corresponding results in Tables 7–9, respectively, we find similar variation patterns and draw similar conclusions with regard to the three parameters λ, k and r_p and the risk measures $WES_{0.05}$ and $ES_{0.05}$. Generally speaking, the performance of the $WES_{0.05}$ optimal portfolio in the American stock markets increases with the increase in λ, while the performance of the $WES_{0.05}$ optimal portfolio tends to deteriorate with the increase in k or r_p. For any given value of each of the three parameters, the $WES_{0.05}$ optimal portfolio almost surely shows better performance than the $ES_{0.05}$ optimal portfolio under considered performance ratios.

Considering the space limitation, we will not analyze in detail the empirical results in Tables 7–9 which are similar to those in Tables 4–6. Here, we just point out some new features deduced from

Table 8
The characteristics of optimal portfolios under different risk measures and different transaction cost ratios–American stocks.

Risk measures	k	0.0002	0.0003	0.00035	0.0004	0.00045
IS-600 (WES)	Return	0.0005	0.0005	0.0005	0.0005	0.0005
	Risk	0.0102	0.0115	0.0123	0.0132	0.0145
	Stk.no.	16	13	13	10	10
	H-index	0.1383	0.1416	0.149	0.1514	0.1626
	R/Risk	0.0587	0.053	0.0502	0.0471	0.0437
	R/ES	0.0587	0.053	0.0502	0.0472	0.0437
	R/PCVaR	7.51E+06	4.43E+06	3.22E+06	2.35E+06	1.63E+06
	G-Rachev	3.72E+05	2.71E+05	2.26E+05	1.82E+05	1.42E+05
	F–T	0.8038	0.8080	0.8190	0.8134	0.8070
IS-600 (ES)	Return	0.0005	0.0005	0.0005	0.0005	0.0005
	Risk	0.0085	0.0094	0.0100	0.0106	0.0114
	Stk.no.	16	13	12	11	10
	H-index	0.1394	0.1422	0.1484	0.1513	0.1623
	R/ES	0.0587	0.0530	0.0502	0.0471	0.0437
	R/PCVaR	7.26E+06	4.36E+06	3.23E+06	2.35E+06	1.62E+06
	G-Rachev	3.57E+05	2.65E+05	2.24E+05	1.81E+05	1.41E+05
	F–T	0.7935	0.8017	0.8140	0.8121	0.8050
OS-5 (WES)	Return	0.0012	0.0007	0.0005	0.0004	0.0007
	Risk	0.0023	0.0038	0.0046	0.0052	0.0065
	R/Risk	0.5265	0.1818	0.1048	0.0784	0.1025
	R/ES	0.5506	0.1951	0.1141	0.0862	0.115
	R/PCVaR	2.19E+10	1.23E+09	3.49E+08	1.73E+08	1.05E+08
	G-Rachev	2.33E+08	1.69E+07	7.56E+06	5.32E+06	2.23E+06
	F–T	1.6366	0.9890	0.8632	0.8447	0.7472
OS-5 (ES)	Return	0.0012	0.0006	0.0005	0.0004	0.0007
	Risk	0.0024	0.0035	0.0043	0.0048	0.0058
	R/ES	0.4865	0.1735	0.1064	0.0836	0.1169
	R/PCVaR	1.40E+10	1.10E+09	3.23E+08	1.60E+08	1.05E+08
	G-Rachev	1.53E+08	1.64E+07	7.32E+06	4.91E+06	2.21E+06
	F–T	1.5500	0.9674	0.8512	0.8298	0.7503
OS-10 (WES)	Return	0.0007	0.0005	0.0004	0.0007	0.0010
	Risk	0.0069	0.0075	0.0071	0.0077	0.0065
	R/Risk	0.1008	0.0643	0.0570	0.0875	0.1559
	R/ES	0.1139	0.0734	0.0646	0.1001	0.1749
	R/PCVaR	8.22E+07	3.87E+07	4.17E+07	4.85E+07	1.60E+08
	G-Rachev	1.33E+06	7.32E+05	1.09E+06	1.05E+06	2.86E+06
	F–T	0.7225	0.5670	0.6273	0.6831	0.8052
OS-10 (ES)	Return	0.0007	0.0004	0.0004	0.0007	0.0010
	Risk	0.0059	0.0065	0.0064	0.0067	0.0058
	R/ES	0.1143	0.0642	0.0646	0.1000	0.1745
	R/PCVaR	9.36E+07	3.64E+07	3.95E+07	4.90E+07	1.56E+08
	G-Rachev	1.55E+06	7.69E+05	1.02E+06	1.05E+06	2.79E+06
	F–T	0.7440	0.5640	0.6208	0.679	0.8062

Table 9
The characteristics of optimal portfolios under different risk measures and different required return rates–American stocks.

Risk measures	r_p	0.00035	0.0005	0.00065	0.00075	0.0009
IS-600 (WES)	Return	0.00035	0.0005	0.00065	0.00075	0.0009
	Risk	0.0092	0.0115	0.0158	0.0194	0.0264
	Stk.no.	17	13	10	10	6
	H-index	0.1311	0.1417	0.1178	0.1172	0.1800
	R/Risk	0.0381	0.0435	0.0411	0.0386	0.0341
	R/ES	0.0448	0.0530	0.0531	0.0522	0.0498
	R/PCVaR	8.35E+06	4.43E+06	1.49E+06	7.80E+05	3.08E+05
	G-Rachev	4.74E+05	2.71E+05	1.20E+05	7.49E+04	3.68E+04
	F–T	0.789	0.808	0.821	0.821	0.808
IS-600 (ES)	Return	0.00035	0.0005	0.00065	0.00075	0.0009
	Risk	0.0078	0.0094	0.0122	0.0144	0.0181
	Stk.no.	17	13	10	10	6
	H-index	0.1311	0.1422	0.1182	0.1156	0.1774
	R/ES	0.0448	0.0530	0.0531	0.0523	0.0498
	R/PCVaR	8.35E+06	4.36E+06	1.49E+06	7.75E+05	3.07E+05
	G-Rachev	4.73E+05	2.65E+05	1.20E+05	7.36E+04	3.66E+04
	F–T	0.789	0.802	0.819	0.814	0.805
OS-5 (WES)	Return	0.0017	0.0007	0.0005	0.0007	0.0012
	Risk	0.0012	0.0038	0.0060	0.0070	0.0136
	R/Risk	1.3812	0.1818	0.0901	0.1013	0.0870
	R/ES	1.4139	0.1951	0.1004	0.1147	0.1083
	R/PCVaR	7.48E+11	1.23E+09	1.17E+08	7.86E+07	7.62E+06
	G-Rachev	7.12E+09	1.69E+07	3.73E+06	2.56E+06	2.64E+05
	F–T	3.4769	0.9890	0.8688	0.8807	0.6728
OS-5 (ES)	Return	0.0016	0.0006	0.0005	0.0007	0.0011
	Risk	0.0012	0.0035	0.0055	0.0063	0.0112
	R/ES	1.3600	0.1735	0.0905	0.1057	0.0988
	R/PCVaR	6.41E+11	1.10E+09	9.90E+07	6.80E+07	6.21E+06
	G-Rachev	6.17E+09	1.64E+07	3.44E+06	2.41E+06	2.32E+05
	F–T	3.3335	0.9674	0.8532	0.8735	0.6582
OS-10 (WES)	Return	0.0008	0.0005	0.0008	0.0011	0.0022
	Risk	0.0078	0.0075	0.0090	0.0109	0.0136
	R/Risk	0.1074	0.0643	0.0933	0.0989	0.1617
	R/ES	0.1230	0.0734	0.1088	0.1186	0.2012
	R/PCVaR	5.85E+07	3.87E+07	3.07E+07	1.73E+07	1.42E+07
	G-Rachev	6.62E+05	7.32E+05	7.33E+05	4.47E+05	3.51E+05
	F–T	0.5650	0.5665	0.6993	0.7030	0.8145
OS-10 (ES)	Return	0.0008	0.0004	0.0008	0.0011	0.0021
	Risk	0.0068	0.0065	0.0078	0.0091	0.0112
	R/ES	0.1216	0.0642	0.1074	0.1159	0.1838
	R/PCVaR	5.75E+07	3.64E+07	2.93E+07	1.67E+07	1.15E+07
	G-Rachev	6.59E+05	7.69E+05	7.11E+05	4.46E+05	3.02E+05
	F–T	0.5646	0.5638	0.6960	0.7028	0.7967

these empirical results. The out-of-sample performance of the $WES_{0.05}$ optimal portfolio for both OS-5 and OS-10 is almost always superior to the performance of the corresponding $ES_{0.05}$ optimal portfolio. However, the uniformity and significance of the $WES_{0.05}$ optimal portfolios for the American stock markets are not so good as those for the Chinese stock market data. For instance, the R/PCVaR, G-Rachev and F–T ratios of the $WES_{0.05}$ optimal portfolio with $\lambda = 0.1$ for the 2-week-after results (OS-10) in Table 7 are smaller than the corresponding ratios of the $ES_{0.05}$ optimal portfolio. Meanwhile, the stability of the superior performance of the $WES_{0.05}$ optimal portfolios for the 2-week-after out-of -sample results is more or less the same as that of the $ES_{0.05}$ optimal portfolios, but is not so good as that for the Chinese stock market 4 weeks after the final sample date. These phenomena are easy to understand. With the buy-and-hold strategy, investors can hardly earn high returns constantly in advanced markets such as the American stock markets. Therefore, investors should adjust their investment strategy more frequently in advanced markets than in emerging markets.

In summary, the in-sample and out-of-sample results for both Chinese stock markets and American stock markets sufficiently show the practicality, efficiency and robustness of our new risk measure and the corresponding portfolio selection model. They can be useful for investors to make efficient and robust investment decisions in both emerging stock markets and developed stock markets.

6. Conclusion

By adaptively penalizing large losses using a nonlinear weight function, this paper introduces a new class of generalized convex risk measures (WES_α). The new measure satisfies convexity and monotonicity, which are well-accepted by academicians and practitioners as the two axioms a reasonable, realistic risk measure should satisfy. The well known fat-tail phenomenon and the asymmetry of the loss distribution can be suitably controlled through the proper selection of the weight function; the investor's risk attitude can be reflected elaborately; more importantly, due to its remarkable mathematical properties, our new risk measure can be more easily applied to practical investment decision-making than the existing convex measures.

A realistic portfolio selection model is established based on the proposed risk measure. This model takes into account typical trading frictions such as taxes and transaction costs. What's more important, instead of using the current solution methods for stochastic programs, our portfolio selection model is specifically transformed into a convex optimization problem, which can be easily solved. Detailed empirical results for both in-sample and out-of-sample cases with real trade data from the Chinese stock markets and American stock markets show that our new risk measure and the established portfolio selection model are helpful for the determination of a robustly optimal investment strategy. They can reasonably reflect the influence of different market constraints, and can be used to find optimal portfolios with specific characteristics such as the return-risk pattern. The optimal portfolio determined under WES_α is generally better than the corresponding optimal portfolio obtained under ES_α in terms of the diversification degree and typical performance measures. Thus, WES_α is more suitable for the optimal portfolio selection in financial management.

When applying our new risk measure to insurance, one might be interested in calculating WES_α for α, which is so small that $\alpha M < 1$. Then, an important issue is how to reasonably estimate WES_α by using the extreme value method. Meanwhile, we only examine the exponential-type weight function. Other weight functions can also be used in WES_α. What influences do different weight functions have on the practical risk control and portfolio selection? Can we establish some practical rules about the proper selection of the weight function? All these issues are left for future research.

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