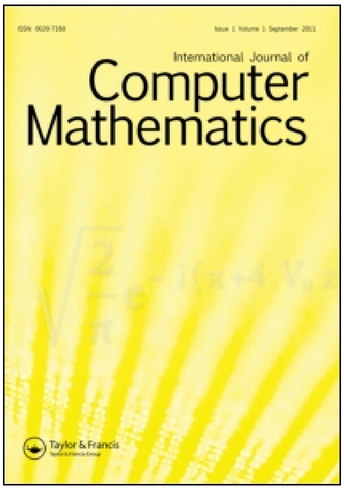

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A full discrete two-grid finite-volume method for a nonlinear parabolic problem

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A full discrete two-grid finite-volume method for a nonlinear parabolic problem

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A fully discrete two-grid finite-volume method (FVM) for a nonlinear parabolic problem is studied in this paper. This method involves solving a nonlinear parabolic equation on coarse mesh space and a linearized parabolic equation on fine grid. Both L^2 and H^1 norm error estimates of the standard FVM for the nonlinear parabolic problem are derived. Compared with the standard FVM, the two-level method is of the same order as the one-level method in the H^1-norm as long as the mesh sizes satisfy $h = \mathcal{O}(H^{3/2})$. However, the two-level method involves much less work than the standard method. Numerical results are provided to demonstrate the effectiveness of our algorithm.

Keywords: two-grid; finite-volume method; nonlinear parabolic problem; error estimate

2000 Mathematics Subject Classifications: 65N12, 74S10

1. Introduction

Many processes in science and technology are described by parabolic equations, for example, the processes of fluid dynamics, hydrology and environmental protection [22,28]. There are extensive works devoted to the linear parabolic problem represented by monographs [31]. For nonlinear cases, we mention only [3,12,29] and their references. In this work, we consider the following nonlinear parabolic problem in $\mathbb{R}^2$:

$$\begin{aligned} &u_t + \nabla\cdot(a(u)\nabla u) + \mathbf{b}(u)\nabla u = f(u) && \text{in } \Omega\times(0,T], \\ &u(x,t) = 0 && \text{on } \partial\Omega\times(0,T], \\ &u(\cdot,0) = u_0 && \text{on } \Omega\times\{0\}, \end{aligned} \tag{1}$$

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where Ω is a bounded convex polygonal domain with boundary $\partial\Omega$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^{\mathrm{T}}$, and $\mathbf{b}(u) = (b_1(u), b_2(u))^{\mathrm{T}}$ is a vector function.

We define a bounded set on $\mathbb{R}^2$ as

$$G = \{u : |u| \le K_0\},$$

where K_0 is a positive constant.

Supposing the coefficients of problem (1) satisfy the following conditions:

(C_1): $a(u)$, $f(u)$ and $b_i(u)(i = 1, 2)$ are Lipschitz continuous with respect to the variable u, i.e.

$$|g(u) - g(v)| \le L|u - v|, \quad \forall u, v \in G,$$

where L is a Lipschitz constant related to K_0, $g(u)$ can take $a(u)$, $f(u)$ or $b_i(u)(i = 1, 2)$. And $||\mathbf{b}(u) - \mathbf{b}(v)||_0$ is defined by

$$||\mathbf{b}(u) - \mathbf{b}(v)||_0 = \left\{\sum_{i=1}^{2} |b_i(u) - b_i(v)|^2\right\}^{1/2}.$$

(C_2): $a(u)$, $\mathbf{b}(u)$ are bounded smooth functions with positive upper and lower bounds,

$$0 < a_* \le a(u) \le a^*, \quad 0 < b_* \le ||\mathbf{b}(u)||_{L^\infty} \le b^*, \quad -\mathrm{div}(\mathbf{b}(u)) > \sigma > 0, \quad \forall u \in G.$$

(C_3): $f(u)$ is a given real-valued function on Ω and there is a constant M such that

$$|f'(u)| + |f''(u)| \le M, \quad \forall u \in G,$$

where σ is a constant, and $f'(u) = \mathrm{d}f(u)/\mathrm{d}u$. Under the conditions above, problem (1) has a unique solution in a certain Sobolev space [31].

Finite-volume method (FVM), as one of the numerical discretization techniques, has been widely employed to solve the fluid dynamics problems [7,16,25]. It is developed as an attempt to use the finite-element idea in the finite-difference setting. The basic idea is to approximate discrete fluxes of a partial differential equation using a finite-element procedure based on volumes or control volumes, so the FVM is also called the box scheme [2,6,21]. FVM has many advantages that belong to finite-difference or finite-element method, such as, it is easy to set up and implement, conserve mass locally and it also can treat the complicated geometry and general boundary conditions flexibility [17,35]. However, the analysis of FVM lags far behind that of finite-element and finite-difference methods, for more recent developments about the FVM, readers can refer to [13–15,20,27,32].

On the other hand, the two-grid method is an efficient numerical scheme for partial differential equations based on two spaces with different mesh sizes. This kind of discretization technique for linear and nonlinear elliptic partial differential equations was first introduced by Xu [33,34]. After that, this discrete scheme has been studied by many researchers, for example, Dawson and Wheeler [10] and Dawson *et al.* [11] studied the nonlinear parabolic equations by using the finite-element or finite-difference method, respectively. Layton and Lenferink [23] for Navier–Stokes equations, Marion and Xu [26] for evolution equations and Bi and Ginting [4] have expanded a two-grid method combined with the FVM for linear and nonlinear elliptic problems.

Recently, Chen *et al.* [8] studied a class of nonlinear parabolic equations by using the two-grid FVM and presented the convergence analysis under some assumptions. With the help of the Taylor expansion and the relationship between the mesh size and time step, they obtained the optimal error estimates of the two-level FVM. Here, we continuously consider the two-grid FVM for the nonlinear parabolic problem. The difference between [8] and this work lies in some cases: (i) the equations are different, the problem investigated in [8] can be considered as a special case of our studied problem; (ii) the difficulties which encountered are different, we mainly handle with the nonlinear diffusion and convection terms; (iii) we not only present the error estimates in L^2 and H^1 norms of both the standard FVM and the two-level method, but also give the stability and existence uniqueness of discrete solution. In this work, two conforming spaces V_H and V_h with mesh sizes H and h, respectively, are chosen and $h \ll H$. On the coarse grid space, we solve the nonlinear problem (1) directly, then, use this known solution to find the fine grid solution. Error estimates indicate that the two-level finite-volume algorithm gives the same order of approximation as the standard FVM if we choose $h = \mathcal{O}(H^{3/2})$. However, in our algorithm, the nonlinear problem is only treated on the coarse grid space, in this way, a large amount of computational cost are saved.

In this work, based on some techniques that one used in [8], we study the nonlinear parabolic problem (1) systematically. An outline of this paper is as follows: In Section 2, the basic results about the FVM are stated, stability and existence uniqueness of discrete solution to nonlinear problem (1) are derived. A two-level full discrete finite-volume algorithm for nonlinear problem (1) is presented in Section 3. We, in Section 4, give the L^2-norm and H^1-norm error estimates for the standard FVM. Section 5 is devoted to the H^1-norm error estimate of the approximation solution for the two-grid algorithm. Finally, some numerical results are presented to verify the performance of our algorithm.

2. Preliminaries

In this section, firstly, we describe some notations and results which will be frequently used in this article. Standard notations are used for the Sobolev spaces $W^{s,p}(\Omega)$ with the norm $\|\cdot\|_{s,p,\Omega}$ and the seminorms $|\cdot|_{s,p,\Omega}$ [1]. We denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and skip the index $p = 2$ for simplicity. For all $T > 0$ and integer number $n \geq 0$, we define

$$H^n(0,T;W^{s,p}(\Omega)) = \left\{ v \in W^{s,p}(\Omega); \sum_{0\le i\le n} \int_0^{\mathrm{T}} \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i} \|v\|_{s,p,\Omega} \right)^2 \mathrm{d}t < \infty \right\},$$

and the corresponding norm of $H^n(0,T;W^{s,p}(\Omega))$ is denoted by

$$\|v\|_{H^n(W^{s,p}(\Omega))} = \sum_{0\le i\le n} \left[\int_0^{\mathrm{T}} \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i} \|v\|_{s,p,\Omega} \right)^2 \right]^{1/2}.$$

Especially, as $n = 0$, we denote the norm as

$$\|v\|_{L^2(W^{s,p}(\Omega))} = \left(\int_0^{\mathrm{T}} \|v\|_{s,p,\Omega}^2 \, \mathrm{d}t \right)^{1/2}.$$

Let

$$L^\infty(0,T;W^{s,p}(\Omega)) = \left\{ v \in W^{s,p}(\Omega); \operatorname{ess} \sup_{0\le t\le T} \|v\|_{s,p,\Omega} < \infty \right\}$$

and the corresponding norm is marked as

$$\|v\|_{L^\infty(W^{s,p}(\Omega))} = ess \sup_{0\le t\le T} \|v\|_{s,p,\Omega}.$$

Set T_h $(h > 0)$ denotes a regular partition of the closure $\bar{\Omega}$ of the domain Ω into a finite number of triangulations K, $h_k = \mathrm{diam}(K)$, $h = \max_{K\in T_h} h_K$. All elements of T_h will be numbered, so that $T_h = \{K_i\}_{i\in I}$, where $I \subset Z^+ = \{0, 1, 2, \ldots\}$ such that $\bar{\Omega} = \cup_{K_i \in T_h} K_i$, $\mathcal{N}_h$ denotes the set of all nodes T_h.

Based on the partition T_h, we introduce the corresponding dual partition T_h^*. Here, we choose the circumcentre Q of an element $K \in T_h$, and the midpoints M on the edges of K, then connect Q to M by a straight line. For an arbitrary vertex $x_i \in K$, let V_i be the polygonal which is called a control volume. Then, we have $\bar{\Omega} = \cup_{x_i \in \mathcal{N}_h} V_i$, the dual mesh T_h^* is the set of these control volumes. We call the control volume mesh T_h^* is regular, i.e. there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \leq \mathrm{meas}(V_i) \leq Ch^2, \quad \forall V_i \in T_h^*.$$

We introduce a Lagrange interpolation operator I_h from $H^2(\Omega)$ into $H_0^1(\Omega)$, such that

$$\|u - I_h u\|_i \leq Ch^{2-i}\|u\|_2, \quad i = 0, 1, \ \forall u \in H^2(\Omega). \tag{2}$$

Let the trial function space $U_h \subset H_0^1(\Omega)$, whose basis functions are $\{\phi_i(x)\}$, be a linear space based on T_h and the test function space $V_h \subset L^2(\Omega)$ be a piecewise constant space on the dual partition T_h^*, whose basis functions are $\{\phi_i^*(x)\}$, defined by

$$\phi_i^*(x) = \begin{cases} 1, & x \in V_i, \\ 0, & \text{otherwise } \cup x \in \partial\Omega. \end{cases}$$

Let I_h^* be an interpolation operator from $H_0^1(\Omega)$ to V_h satisfying

$$I_h^* v = \sum_{x_i \in \mathcal{N}_h} v(x_i)\phi_i^*(x).$$

The weak form of the FVM for nonlinear problem (1) reads:

$$(u_t, I_h^* v) + a(u, u, I_h^* v) + (\mathbf{b}(u)\nabla u, I_h^* v) = (f(u), I_h^* v), \quad \forall\, v \in H_0^1(\Omega), \tag{3}$$

where $a(\cdot, \cdot, I_h^* \cdot)$ is defined by

$$a(w, u, I_h^* v) = \int_{\partial\Omega} (a(w)\nabla u) \cdot \mathbf{n} I_h^* v \, \mathrm{d}s, \quad \forall w, u, v \in H_0^1(\Omega).$$

Assume that the solution u of problem (1) satisfies the following regularities:

$$\begin{aligned} (\mathrm{C}_4): \quad & u, u_t \in L^\infty(H^3(\Omega) \cap W^{1,\infty}(\Omega)); \quad u_{tt} \in L^2(H^1(\Omega)); \\ & |u(x,t)| \leq K_0, \quad \forall (x,t) \in \Omega \times (0, T]. \end{aligned}$$

For any $v_h \in V_h$, a full discrete finite-volume formulation is defined for the solution $u_h^n \in U_h$ with time step $\triangle t$ as

$$\left(\frac{u_h^n - u_h^{n-1}}{\triangle t}, v_h\right) + a(u_h^n, u_h^n, v_h) + (\mathbf{b}(u_h^n)\nabla u_h^n, v_h) = (f(u_h^n), v_h), \tag{4}$$

where $u_h^n = u_h(t_n)$, $t_n = n\triangle t$, $n = 1, 2, \ldots, N$, $\triangle t = (T/N)$. Then Equation (4) can be rewritten as

$$(u_h^n, v_h) + \triangle t a(u_h^n, u_h^n, v_h) + \triangle t(\mathbf{b}(u_h^n)\nabla u_h^n, v_h) = (u_h^{n-1} + \triangle t f(u_h^n), v_h), \tag{4$'$}$$

where

$$a(u_h, u_h, v_h) = \sum_{x_i \in \mathcal{N}_h} \int_{\partial V_i} (a(u_h)\nabla u_h) \cdot \mathbf{n} v_h \, \mathrm{d}s = \sum_{x_i \in \mathcal{N}_h} v_h(x_i) \int_{\partial V_i} (a(u_h)\nabla u_h) \cdot \mathbf{n} \, \mathrm{d}s.$$

Define the discrete norm

$$|||u_h|||_0^2 = (u_h, I_h^* u_h), \quad \forall\, u_h \in U_h.$$

This norm is equivalent to the standard L^2-norm [25], namely, there exist two positive constants C_*, C^*, independent of h, such that

$$C_* ||u_h||_0 \le |||u_h|||_0 \le C^* ||u_h||_0, \quad \forall\, u_h \in U_h. \tag{5}$$

In order to proceed the theoretical analysis to Equation (4), the following Gronwall lemma need to be recalled [5].

LEMMA 2.1 *Let C_0 and a_k, b_k, c_k, d_k, for integers $k \ge 0$, be non-negative numbers such that*

$$a_n + \triangle t \sum_{k=0}^{n} b_k \le \triangle t \sum_{k=0}^{n} d_k a_k + \triangle t \sum_{k=0}^{n} c_k + C_0, \quad \forall n \ge 1.$$

Then,

$$a_n + \triangle t \sum_{k=0}^{n} b_k \le \left(\triangle t \sum_{k=0}^{n} c_k + C_0 \right) \exp\left(\triangle t \sum_{k=0}^{n} d_k \right), \quad \forall\, n \ge 1.$$

The following two important lemmas can be found in [9,19,24].

LEMMA 2.2 *For all $u_h, v_h \in U_h$, there exists a positive constant C, independent of h, such that*

$$(u_h, I_h^* v_h) = (v_h, I_h^* u_h), \quad (u_h, I_h^* v_h) \le C ||u_h||_0 ||v_h||_0.$$

LEMMA 2.3 *Suppose that the partition T_h is regular, and T_h^* is the corresponding dual partition. For all $w_h, u_h, v_h \in U_h$, there exist two positive constants α, C independent of h, such that*

$$\alpha ||u_h||_1^2 \le a(w_h, u_h, I_h^* u_h); \quad a(w_h, u_h, I_h^* v_h) \le C ||u_h||_1 ||v_h||_1,$$
$$|a(w_h, u_h, I_h^* v_h) - a(w_h, v_h, I_h^* u_h)| \le Ch ||u_h||_1 ||v_h||_1.$$

THEOREM 2.4 (Trace theorem) *Suppose that Ω has a Lipschitz boundary, p is a real number in the range $1 \le p \le \infty$. Then there is a constant C, such that*

$$||v||_{L^p(\partial\Omega)} \le C ||v||_{L^p(\Omega)}^{1-1/p} ||v||_{W^{1,p}(\Omega)}^{1/p}, \quad \forall v \in W^{1,p}(\Omega).$$

LEMMA 2.5 *Under the assumptions of Lemma* 2.3, $u \in H^2(\Omega)$, *and* $w \in W^{1,\infty}(\Omega)$, *there exists a positive constant C independent of h, such that*

$$|a(u - u_h, w, I_h^* v_h)| \le C(h^2 ||u||_2 + ||u - u_h||_0) ||w||_{1,\infty} ||v_h||_1, \quad \forall u_h, v_h \in U_h.$$

Proof Introducing a discrete H^1-seminorm on a triangular element K, we obtain

$$|u_h|_{1,h,K} := \left\{ \left[\left(\frac{\partial u_h}{\partial x_1}(Q) \right)^2 + \left(\frac{\partial u_h}{\partial x_2}(Q) \right)^2 \right] S_Q \right\}^{1/2},$$

where S_Q is the area of the element K with Q as its circumcentre. Chou and Li [9] have proved the equivalence of the norms $|u_h|_{1,h}$ and $|u_h|_1$. Denote by K^v the collection of vertex in the element

K, $x_l(l=1,2,3)$, $x_4=x_1$. With the definition of $a(\cdot,\cdot,I_h^*\cdot)$, we have

$$\begin{aligned}a(u,w,I_h^*v_h)-a(u_h,w,I_h^*v_h)&=\sum_{x_i\in\mathcal{N}_h}\int_{\partial V_i}(a(u)-a(u_h))\nabla w\cdot\mathbf{n}I_h^*v_h\,\mathrm{d}s\\&=\sum_{K\in\mathcal{T}_h}\sum_{x_i\in K^v}\int_{V_i\cap K}(a(u)-a(u_h))\nabla w\cdot\mathbf{n}I_h^*v_h\,\mathrm{d}s\\&=\sum_{K\in\mathcal{T}_h}\sum_{l=1}^{3}\int_{\overline{M_lQ}}(a(u)-a(u_h))\nabla w\cdot\mathbf{n}\,\mathrm{d}s[v_h(x_l)-v_h(x_{l+1})],\end{aligned}$$

where M_l is the midpoints of the edges of K. The above equalities are obtained by noticing that each line segment M_lQ is travelling twice but in opposite orientations (once as $\overline{M_lQ}$, once as $\overline{QM_l}$), and then collecting the like-terms. Using Taylor's expansion and the fact that U_h is linear in K, we have

$$|v_h(x_l)-v_h(x_{l+1})|=\left|\sum_{i=1}^{2}\frac{\partial v_h}{\partial x_i}(x_l-x_{l+1})\right|\le Ch\left(\left|\frac{\partial v_h}{\partial x_1}\right|+\left|\frac{\partial v_h}{\partial x_2}\right|\right)\le C|v_h|_{1,h,K}\le C|v_h|_{1,K}.$$

Combining the condition (C_1), Equation (2), triangular inequality and Theorem 2.4, we obtain

$$\begin{aligned}&|a(u,w,I_h^*v_h)-a(u_h,w,I_h^*v_h)|\\&\quad\le\sum_{K\in\mathcal{T}_h}\sum_{l=1}^{3}\int_{\overline{M_lQ}}|a(u)-a(u_h)||\nabla w|\,\mathrm{d}s\cdot|v_h(x_l)-v_h(x_{l+1})|\\&\quad\le C\sum_{K\in\mathcal{T}_h}|\nabla w|_{L^\infty(\bar K)}|v_h|_{1,K}\sum_{l=1}^{3}\int_{\overline{M_lQ}}|a(u)-a(u_h)|\,\mathrm{d}s\\&\quad\le C\sum_{K\in\mathcal{T}_h}|\nabla w|_{L^\infty(\bar K)}|v_h|_{1,K}\cdot\sum_{l=1}^{3}\int_{\overline{M_lQ}}|u-u_h|\,\mathrm{d}s\\&\quad\le C\sum_{K\in\mathcal{T}_h}\|v_h\|_{H^1(K)}|\nabla w|_{L^\infty(\overline{K})}\sum_{l=1}^{3}\cdot\left[\|u-I_hu\|_0^{1/2}\|u-I_hu\|_1^{1/2}\left(\int_{\overline{M_lQ}}1\,\mathrm{d}s\right)^{1/2}\right.\\&\qquad\left.+\|I_hu-u_h\|_0^{1/2}\|I_hu-u_h\|_1^{1/2}\left(\int_{\overline{M_lQ}}1\,\mathrm{d}s\right)^{1/2}\right]\\&\quad\le C\sum_{K\in\mathcal{T}_h}\|v_h\|_{H^1(K)}|\nabla w|_{L^\infty(\overline{K})}\cdot\left[h^2\|u\|_2+\|I_hu-u_h\|_0^{1/2}h^{-1/2}\|I_hu-u_h\|_0^{1/2}\left(\int_{\overline{M_lQ}}1\,\mathrm{d}s\right)^{1/2}\right]\\&\quad\le C\sum_{K\in\mathcal{T}_h}\|v_h\|_{H^1(K)}|\nabla w|_{L^\infty(\bar K)}\cdot(h^2\|u\|_2+\|I_hu-u_h\|_0)\\&\quad=C(h^2\|u\|_2+\|u-u_h\|_0)\|v_h\|_{H^1(\Omega)}\|w\|_{1,\infty}.\end{aligned}$$

■

LEMMA 2.6 [18] *Suppose $P_hu\in U_h$ satisfies*

$$a(u,P_hu-u,v_h)=0,\quad\forall v_h\in V_h,\ 0<t\le T.$$

Then there exists a constant $C > 0$, such that

$$\|\nabla P_h u\|_\infty \le C, \quad \|u - P_h u\|_1 \le Ch\|u\|_2, \quad \|u - P_h u\|_0 \le Ch^2\|u\|_3,$$
$$\|(u - P_h u)_t\|_1 \le Ch\{\|u\|_2 + \|u_t\|_2\}, \quad \|(u - P_h u)_t\|_0 \le Ch^2\{\|u\|_3 + \|u_t\|_3\}.$$

LEMMA 2.7 *Under the conditions of Lemma* 2.3 *and* (C_2), *suppose $u_h^n \in U_h$, then problem* (4) *admits a uniqueness solution.*

Proof Choosing $v_h = I_h^* u_h^n$ in Equation (4′), combining with Equation (5) and Lemma 2.3, we get

$$\begin{aligned}\frac{1}{\triangle t}&(u_h^n, I_h^* u_h^n) + a(u_h^n, u_h^n, I_h^* u_h^n) + (\mathbf{b}(u_h^n)\nabla u_h^n, I_h^* u_h^n)\\ &\ge \frac{C_*}{\triangle t}\|u_h^n\|_0^2 + \alpha\|u_h^n\|_1^2 + \sum_{x_i\in\mathcal{N}_h}\int_{\partial V_i}\mathbf{b}(u_h^n)\cdot\mathbf{n} I_h^* u_h^n \cdot u_h^n \,\mathrm{d}s - \int_\Omega u_h^n \cdot \mathrm{div}(b(u_h^n) I_h^* u_h^n)\,\mathrm{d}x.\end{aligned} \tag{6}$$

Denote $\Gamma_{ij} = \partial V_i \cap \partial V_j$, n_{ij} is the unit outer normal direction of Γ_{ij}, and define

$$\beta_{ij} = \int_{\Gamma_{ij}} \mathbf{b}(u)\cdot n_{ij}\,\mathrm{d}s.$$

Then, ∂V_i can be divided into a flow in part and a flow out part according to the sign of β_{ij}:

$$\begin{cases}(\partial V_i)_- = \cup_{\beta_{ij}\le 0}\Gamma_{ij}, & \text{(flow in)},\\ (\partial V_i)_+ = \cup_{\beta_{ij}\ge 0}\Gamma_{ij}, & \text{(flow out)}.\end{cases}$$

Write

$$\begin{cases}\beta_{ij}^+ = \max(\beta_{ij}, 0),\\ \beta_{ij}^- = \max(-\beta_{ij}, 0).\end{cases}$$

$$u_h^+ = \begin{cases}\lim_{x'\to x, x'\notin V_i} u_h(x'), & \text{when } x \in (\partial V_i)_-,\\ \lim_{x'\to x, x'\in V_i} u_h(x'), & \text{when } x \in (\partial V_i)_+.\end{cases}$$

$$u_h^- = \begin{cases}\lim_{x'\to x, x'\in V_i} u_h(x'), & \text{when } x \in (\partial V_i)_-,\\ \lim_{x'\to x, x'\notin V_i} u_h(x'), & \text{when } x \in (\partial V_i)_+.\end{cases}$$

The above values lead to the following approximation:

$$\begin{aligned}\int_{\partial V_i}(\mathbf{b}(u_h^n)\cdot\mathbf{n}) I_h^* u_h^n \cdot u_h^n\,\mathrm{d}s &\approx (\beta_{ij}^+ u_h^n(x_i) - \beta_{ij}^- u_h^n(x_j)) I_h^* u_h^n\\ &= \int_{\partial V_i}(I_h^* u_h^n)^+ I_h^* u_h^n (\mathbf{b}(u_h^n)\cdot n_{ij})\,\mathrm{d}s.\end{aligned} \tag{7}$$

It follows from the Green formulation that

$$\begin{aligned}&-\sum_{x_i\in\mathcal{N}_h}\int_{V_i} u_h^n \cdot \mathrm{div}(\mathbf{b}(u_h^n) I_h^* u_h^n)\,\mathrm{d}x\\ &\quad= \sum_{x_i\in\mathcal{N}_h}\left[\int_{V_i}(\mathbf{b}(u_h^n)\cdot\nabla u_h^n) I_h^* u_h^n\,\mathrm{d}x - \int_{\partial V_i}(\mathbf{b}(u_h^n)\cdot\mathbf{n}) u_h^n I_h^* u_h\,\mathrm{d}s\right]\end{aligned}$$

$$= \sum_{x_i \in \mathcal{N}_h} \left[\int_{V_i} (\mathbf{b}(u_h^n) \cdot \nabla u_h^n) I_h^* u_h^n \, \mathrm{d}x - \int_{(\partial V_i)_+} (\mathbf{b}(u_h^n) \cdot \mathbf{n})(I_h^* u_h^n)^+ I_h^* u_h \, \mathrm{d}s \right.$$

$$\left. - \int_{(\partial V_i)_-} (\mathbf{b}(u_h^n) \cdot \mathbf{n})(I_h^* u_h^n)^- I_h^* u_h \, \mathrm{d}s \right]. \tag{8}$$

Combining Equations (7) and (8), we arrive at

$$\sum_{x_i \in \mathcal{N}_h} \int_{\partial V_i} \mathbf{b}(u_h^n) \cdot \mathbf{n} I_h^* u_h^n \cdot u_h^n \, \mathrm{d}s - \int_{\Omega} u_h^n \cdot \mathrm{div}(b(u_h^n) I_h^* u_h^n) \, \mathrm{d}x$$

$$= \sum_{x_i \in \mathcal{N}_h} \int_{V_i} (b(u_h^n) \cdot \nabla u_h^n) I_h^* u_h^n \, \mathrm{d}x + \int_{(\partial V_i)_-} (b(u_h^n) \cdot \mathbf{n})[I_h^* u_h^n] I_h^* u_h \, \mathrm{d}s, \tag{9}$$

where $[I_h^* u_h^n] = I_h^* u_h^{n+} - I_h^* u_h^{n-}$ is the jump of u_h^n across $(\partial V_i)_-$. Applying the results presented in Sections 6.2 and 7.2 of the book [25] about Equation (9), we have

$$(\mathbf{b}(u_h^n) \nabla u_h^n, I_h^* u_h^n) \geq \gamma_0 (\|u_h^n\|_0^2 + \|u_h^n\|_{\partial\Omega}^2), \quad \gamma_0 = \min\left(\sigma, \frac{1}{2}\right), \tag{10}$$

where $\sigma > 0$ is a constant, which is defined in condition (C_2).

It follows from Equations (6) and (10) and condition (C_2) that

$$\frac{1}{\triangle t}(u_h^n, I_h^* u_h^n) + a(u_h^n, u_h^n, I_h^* u_h^n) + (\mathbf{b}(u_h^n) \nabla u_h^n, I_h^* u_h^n)$$

$$\geq \left(\frac{C_*}{\triangle t} + \gamma_0\right) \|u_h^n\|_0^2 + \alpha \|u_h^n\|_1^2 + \gamma_0 \|u_h^n\|_{\partial\Omega}^2.$$

This guarantees the unique existence of the solution to problem (4′) for a given u_h^{n-1}, i.e. the problem (4) has a unique solution u_h^n for a given u_h^{n-1}.

Now, we state the stability of the standard finite-volume formulation (4) for nonlinear problem (1). ■

THEOREM 2.8 *Let u_h^n be the solution of Equation* (4), *and $u_h^0 = u_0$, $f \in L^2(\Omega)$ are valid, then we have*

$$||u_h^l||_0^2 + \alpha \sum_{n=1}^{l} ||u_h^n||_1^2 \Delta t \leq ||u_0||_0^2 + C \sum_{n=1}^{l} ||f(u_h^n)||_0^2 \Delta t, \quad 1 \leq l \leq N,$$

where $\alpha > 0$ is a constant.

Proof Taking $v_h = I_h^* u_h^n$ in Equation (4), we have

$$\left(\frac{u_h^n - u_h^{n-1}}{\triangle t}, I_h^* u_h^n\right) + a(u_h^n, u_h^n, I_h^* u_h^n) + (\mathbf{b}(u_h^n) \nabla u_h^n, I_h^* u_h^n) = (f(u_h^n), I_h^* u_h^n). \tag{11}$$

With the help of the definition of $(\cdot, I_h^* \cdot)$ and Lemma 2.2, we get

$$\left(\frac{u_h^n - u_h^{n-1}}{\triangle t}, I_h^* u_h^n\right) = \frac{1}{2\triangle t}[(u_h^n - u_h^{n-1}, I_h^*(u_h^n + u_h^{n-1})) + (u_h^n - u_h^{n-1}, I_h^*(u_h^n - u_h^{n-1}))]$$

$$\geq \frac{1}{2\triangle t}(u_h^n - u_h^{n-1}, I_h^*(u_h^n + u_h^{n-1})) = \frac{1}{2\triangle t}(|||u_h^n|||_0^2 - |||u_h^{n-1}|||_0^2). \tag{12}$$

Substituting Equation (12) into (11), multiplying by $2\triangle t$, summing Equation (11) from $n=1$ to $n=l$ $(1 \le l \le N)$ and employing the condition (C_2), Lemma 2.3, Hölder inequality and Equation (5), we have

$$\begin{aligned}||u_h^l||_0^2 - ||u_h^0||_0^2 + 2\alpha \sum_{n=1}^{l} ||u_h^n||_1^2 \triangle t &\le 2\sum_{n=1}^{l} (||f(u_h^n)||_0 + \|b(u_h^n)\|_{L^\infty} \|\nabla u_h^n\|_0)\|u_h^n\|_0 \triangle t \\ &\le C_1 \sum_{n=1}^{l} ||f(u_h^n)||_0^2 \triangle t + C_2 \sum_{n=1}^{l} ||u_h^n||_0^2 \triangle t + \alpha \sum_{n=1}^{l} ||u_h^n||_1^2 \triangle t.\end{aligned}$$

Applying Lemma 2.1, we complete the proof. ■

From Theorem 2.8, we can see that the fully discrete finite-volume scheme (4) is stable.

3. Two-grid finite-volume algorithm

In this section, we present a two-level finite-volume algorithm for the nonlinear parabolic problem (1). First of all, we introduce two regular triangulations of Ω, denoting as T_H and T_h with mesh sizes H and h $(h \ll H)$. Based on the partitions T_H and T_h, we define two finite-element spaces U_H and U_h which satisfy $U_H \subset U_h$, they are called the coarse-grid and fine-grid spaces, respectively. The idea to solve problem (1) using the two-grid finite-volume algorithm is presented as follows:

ALGORITHM *Step I. Find $u_H^n \in U_H$ $(n = 1, 2, \ldots)$ on coarse grid T_H, such that, for all $v_H \in V_H$*

$$\begin{cases}(\partial_t u_H^n, v_H) + a(u_H^n, u_H^n, v_H) + (\mathbf{b}(u_H^n)\nabla u_H^n, v_H) = (f(u_H^n), v_H),\\ u_H^0 = u_0.\end{cases}$$

Step II. On the fine grid T_h, $\forall\, v_h \in V_h$, find $u_h^n \in U_h$ $(n = 1, 2, \ldots)$, such that

$$\begin{cases}(\partial_t u_h^n, v_h) + a(u_H^n, u_h^n, v_h) + (\mathbf{b}(u_H^n)\nabla u_h^n, v_h) = (f(u_H^n) + f'(u_H^n)(u_h^n - u_H^n), v_h),\\ u_h^0 = u_0,\end{cases} \tag{13}$$

where $\partial_t u_k^n = (u_k^n - u_k^{n-1})/\triangle t$ (k takes h or H). By using the known solution, which is obtained in Step I and by the Taylor expansion, nonlinear problem (4) *transforms into a linear problem in Step II, which is much easier to solve than solving problem* (4) *on a fine grid space directly.*

4. Error estimates of the standard FVM

This section is devoted to the error estimates of the approximate solution for the standard finite-volume scheme (4). As usual, we write the error $e(t) = u^n(t) - u_h^n(t)$ as a sum of two terms

$$u^n - u_h^n = (u^n - P_h u^n) + (P_h u^n - u_h^n) = \eta^n + \xi^n,$$

where $u^n = u(t_n)$ and P_h is defined in Lemma 2.6. Firstly, we present the L^2-norm error estimate of standard FVM for problem (1).

THEOREM 4.1 *Under the conditions* (C_1)–(C_4) *and* $u_h^0 = P_h u_0$, *the numerical solution* u_h^n *of problem* (4) *satisfies the following error estimate for* $0 \le t_n \le T$:

$$||u - u_h^n||_0 \le C(h^{3/2} + \triangle t).$$

where C is a constant, which depends on $||u||_{L^\infty(H^3(\Omega))}$, $||u_t||_{L^2(H^3(\Omega))}$, $||u_{tt}||_{L^2(L^2(\Omega))}$, but is independent of h and $\triangle t$.

Proof Denote $\partial_t \xi^n = (\xi^n - \xi^{n-1})/\triangle t$, subtracting Equation (4) from Equation (3), choosing $v = \xi^n$ in Equation (3) and $v_h = I_h^* \xi^n$ in Equation (4), we have

$$\begin{aligned}(\partial_t \xi^n, I_h^* \xi^n) + a(u_h^n, \xi^n, I_h^* \xi^n) &= (\partial_t u^n - u_t^n, I_h^* \xi^n) - (\partial_t \eta^n, I_h^* \xi^n) - a(u^n - u_h^n, P_h u^n, I_h^* \xi^n) \\ &\quad - ((\mathbf{b}(u^n) - \mathbf{b}(u_h^n))\nabla u^n, I_h^* \xi^n) - (\mathbf{b}(u_h^n)\nabla(u^n - u_h^n), I_h^* \xi^n) \\ &\quad + (f(u^n) - f(u_h^n), I_h^* \xi^n). \qquad (14)\end{aligned}$$

By virtue of the definition of $|||\cdot|||_0$, Lemma 2.2 and the estimate skill of (12), we get

$$(\partial_t \xi^n, I_h^* \xi^n) \geq \frac{1}{2\triangle t}(|||\xi^n|||_0^2 - |||\xi^{n-1}|||_0^2). \qquad (15)$$

Combining Equation (15) with Equation (14), multiplying by $2\triangle t$ and summing Equation (14) from $n = 1$ to $n = l$ $(1 \leq l \leq N)$, thanks to Equation (5) and Lemma 2.3, we have

$$\begin{aligned}||\xi^l||_0^2 + 2\alpha \sum_{n=1}^{l} ||\xi^n||_1^2 \triangle t &\leq 2\sum_{n=1}^{l}(\partial_t u^n - u_t^n, I_h^* \xi^n)\triangle t - 2\sum_{n=1}^{l}(\partial_t \eta^n, I_h^* \xi^n)\triangle t \\ &\quad - 2\sum_{n=1}^{l}((\mathbf{b}(u^n) - \mathbf{b}(u_h^n))\nabla u^n, I_h^* \xi^n)\triangle t \\ &\quad - 2\sum_{n=1}^{l}(\mathbf{b}(u_h^n)\nabla(u^n - u_h^n), I_h^* \xi^n)\triangle t \\ &\quad + 2\sum_{n=1}^{l}(f(u^n) - f(u_h^n), I_h^* \xi^n)\triangle t \\ &\quad - 2\sum_{n=1}^{l} a(u^n - u_h^n, P_h u^n, I_h^* \xi^n)\triangle t = \sum_{i=1}^{6} E_i. \qquad (16)\end{aligned}$$

Now, we estimate the right-hand side terms of Equation (16). For E_1 and E_2, with Poincáre inequality, Lemma 2.6 and the results given in [30], we have

$$\begin{aligned}|E_1| &\leq C_1 \sum_{n=1}^{l}\left(\int_{t^{n-1}}^{t^n} ||u_{tt}^n||_0 \,\mathrm{d}t\right)^2 \triangle t + C_2 \sum_{n=1}^{l} ||\xi^n||_0^2 \triangle t \\ &\leq C_1 \left(\int_0^{t^l} ||u_{tt}^n||_0^2 \,\mathrm{d}t\right)(\triangle t)^2 + C_2 \sum_{n=1}^{l} ||\xi^n||_0^2 \triangle t. \\ |E_2| &\leq C_1 \sum_{n=1}^{l} \int_{t^{n-1}}^{t^n} ||\eta_t^n||_0^2 \,\mathrm{d}t + C_2 \sum_{n=1}^{l} ||\xi^n||_0^2 \triangle t \\ &\leq C_1 h^4 \left(\int_0^{t^l} (\|u^n\|_3 + ||u_t^n||_3)^2 \,\mathrm{d}t\right) + C_2 \sum_{n=1}^{l} ||\xi^n||_0^2 \triangle t.\end{aligned}$$

For E_3 and E_4, under the conditions of (C_1)–(C_2), by Lemma 2.6 and Theorem 2.4, we have

$$\begin{aligned}
|E_3| &= \left|\sum_{n=1}^{l}((\mathbf{b}(u^n)-\mathbf{b}(u_h^n))\nabla u^n, I_h^*\xi^n)\triangle t\right| \\
&\le \sum_{n=1}^{l}||\mathbf{b}(u^n)-\mathbf{b}(u_h^n)||_0||\nabla u^n||_{L^\infty}||I_h^*\xi^n||_0\triangle t \\
&\le \sum_{n=1}^{l}L||u^n-u_h^n||_0||\nabla u^n||_{L^\infty}||\xi^n||_0\triangle t \\
&\le C\sum_{n=1}^{l}||\xi^n+\eta^n||_0||\nabla u^n||_{L^\infty}||\xi^n||_0\triangle t \\
&\le C_1\sum_{n=1}^{l}||\xi^n||_0^2||u^n||_{1,\infty}\triangle t + C_2\sum_{n=1}^{l}||\eta^n||_0||u^n||_{1,\infty}||\xi^n||_0\triangle t \\
&\le C_1\sum_{n=1}^{l}||\xi^n||_0^2\triangle t + C_2\sum_{n=1}^{l}h^4(||u^n||_3^2+||u_t^n||_3^2)\triangle t. \\
|E_4| &= \left|\sum_{n=1}^{l}(\mathbf{b}(u_h^n)\nabla(u^n-u_h^n), I_h^*\xi^n)\triangle t\right| = \left|\sum_{n=1}^{l}(\mathbf{b}(u_h^n)\nabla(\xi^n+\eta^n), I_h^*\xi^n)\triangle t\right| \\
&\le \sum_{n=1}^{l}||\mathbf{b}(u_h^n)||_{L^\infty}||\nabla\xi^n||_0||I_h^*\xi^n||_0\triangle t + \sum_{n=1}^{l}||\mathbf{b}(u_h^n)||_{L^\infty}|(\operatorname{div}\eta^n, I_h^*\xi^n)|\triangle t \\
&\le \sum_{n=1}^{l}b^*||\xi^n||_1||\xi^n||_0\triangle t + \sum_{n=1}^{l}b^*\left|\int_{\partial\Omega}(u^n-P_hu^n)\cdot I_h^*\xi^n\mathbf{n}\,\mathrm{d}s\right|\triangle t \\
&\le \sum_{n=1}^{l}b^*||\xi^n||_1||\xi^n||_0\triangle t + \sum_{n=1}^{l}b^*||u^n-P_hu^n||_{L^2(\partial\Omega)}||\xi^n||_{L^2(\partial\Omega)}\triangle t \\
&\le \sum_{n=1}^{l}b^*||\xi^n||_1||\xi^n||_0\triangle t + \sum_{n=1}^{l}C||u^n-P_hu^n||_0^{1/2}||u^n-P_hu^n||_1^{1/2}||\xi^n||_0^{1/2}||\xi^n||_1^{1/2}\triangle t \\
&\le \sum_{n=1}^{l}b^*||\xi^n||_1||\xi^n||_0\triangle t + \sum_{n=1}^{l}Ch^{3/2}(||u^n||_3+\|u_t^n\|_3)||\xi^n||_1\triangle t \\
&\le \epsilon\sum_{n=1}^{l}||\xi^n||_1^2\triangle t + C(\epsilon)\sum_{n=1}^{l}||\xi^n||_0^2\triangle t + Ch^3\sum_{n=1}^{l}(||u^n||_3+\|u_t^n\|_3)^2\triangle t.
\end{aligned}$$

For (E_5)–(E_6), by (C_1) and Lemma 2.5, we get

$$\begin{aligned}
|E_5| &= \left|\sum_{n=1}^{l}(f(u^n)-f(u_h^n), I_h^*\xi^n)\triangle t\right| \\
&\le C_1\sum_{n=1}^{l}(||\xi^n||_0^2+||\eta^n||_0^2)\triangle t + C_2\sum_{n=1}^{l}||\xi^n||_0^2\triangle t
\end{aligned}$$

$$\leq C_1 h^4 \sum_{n=1}^{l} (||u^n||_3 + ||u_t^n||_3)^2 \triangle t + C_2 \sum_{n=1}^{l} ||\xi^n||_0^2 \triangle t.$$

$$\begin{aligned}
|T_6| &= |\sum_{n=1}^{l} a(u^n - u_h^n, P_h u^n, I_h^* \xi^n) \triangle t| \\
&\leq \sum_{n=1}^{l} C(h^2 ||u^n||_2 + ||u^n - u_h^n||_0) ||u^n||_{1,\infty} ||\xi^n||_1 \triangle t \\
&\leq C_1 \sum_{n=1}^{l} h^2 (\|u^n\|_3 + \|u_t^n\|_3) ||u^n||_{1,\infty} ||\xi^n||_1 \triangle t + C_2 \sum_{n=1}^{l} ||\xi^n||_0 ||u^n||_{1,\infty} ||\xi^n||_1 \triangle t \\
&\leq C(\epsilon) h^4 \sum_{n=1}^{l} \triangle t + \epsilon \sum_{n=1}^{l} ||\xi^n||_1^2 \triangle t + C \sum_{n=1}^{l} ||\xi^n||_0^2 \triangle t.
\end{aligned}$$

Combining the above inequalities with Equation (16), we get

$$\begin{aligned}
||\xi^l||_0^2 + \sum_{n=1}^{l} ||\xi^n||_1^2 \triangle t \leq{}& C_1 h^3 \sum_{n=1}^{l} (||u^n||_3^2 + ||u_t^n||_3^2) \triangle t + C_2 (\triangle t)^2 \int_0^{t^l} ||u_{tt}^n||_0^2 \, \mathrm{d}t \\
&+ C_3 h^4 \left[\int_0^{t^l} (||u^n||_3 + ||u_t^n||_3)^2 \, \mathrm{d}t + T + \sum_{n=1}^{l} (||u^n||_3 + ||u_t^n||_3)^2 \triangle t \right] \\
&+ C_4 \sum_{n=1}^{l} ||\xi^n||_0^2 \triangle t + \epsilon \sum_{n=1}^{l} ||\xi^n||_1^2 \triangle t. \qquad (17)
\end{aligned}$$

Choosing proper ϵ and kicking the last term to the left side of Equation (17), applying Lemma 2.1, we arrive at

$$\begin{aligned}
||\xi^l||_0^2 + \sum_{n=1}^{l} ||\xi^n||_1^2 \triangle t \leq{}& C_1 h^3 \sum_{n=1}^{l} (||u^n||_3^2 + ||u_t^n||_3^2) \triangle t + C_2 (\triangle t)^2 \int_0^{t^l} ||u_{tt}^n||_0^2 \, \mathrm{d}t \\
&+ C_3 h^4 \left[\int_0^{t^l} (||u^n||_3 + ||u_t^n||_3)^2 \, \mathrm{d}t + T + \sum_{n=1}^{l} (||u^n||_3 + ||u_t^n||_3)^2 \triangle t \right].
\end{aligned}$$

This, along with triangular inequality and Lemma 2.6, yields the desired result. ■

Remark 4.1 From Theorem 4.1, we can see that the backward Euler scheme is only first order in $\triangle t$. To balance the spatial and temporal errors, one would choose $\triangle t = \mathcal{O}(h^{3/2})$, which is a restriction to the backward Euler method. Therefore, in the proof of the following theorems, we choose $\triangle t = \mathcal{O}(h^{3/2})$ is reasonable.

Next, we present the H^1-norm error estimate for problem (1) in the standard FVM.

THEOREM 4.2 *Under the conditions of Theorem 4.1 and* $u_h^0 = P_h u_0, \triangle t = \mathcal{O}(h^{3/2})$, *for* $0 \leq t^n \leq T$, *the solution* u_h^n *of problem* (4) *satisfies*

$$||u - u_h^n||_1 \leq C(h^{1/2} + \triangle t),$$

where C *is a constant, independent of* h *and* $\triangle t$.

Proof We get the following error equation by choosing $v = \partial_t \xi^n$ in Equation (3) and $v_h = I_h^* \partial_t \xi^n$ in Equation (4), respectively:

$$\begin{aligned}
&(\partial_t \xi^n, I_h^* \partial_t \xi^n) + a(u_h^n, \xi^n, I_h^* \partial_t \xi^n) \\
&\quad = (\partial_t u^n - u_t^n, I_h^* \partial_t \xi^n) - (\partial_t \eta^n, I_h^* \partial_t \xi^n) \\
&\qquad - ((\mathbf{b}(u^n) - \mathbf{b}(u_h^n))\nabla u^n, I_h^* \partial_t \xi^n) - (\mathbf{b}(u_h^n)\nabla(u^n - u_h^n), I_h^* \partial_t \xi^n) \\
&\qquad - a(u^n - u_h^n, P_h u^n, I_h^* \partial_t \xi^n) + (f(u^n) - f(u_h^n), I_h^* \partial_t \xi^n). \qquad (18)
\end{aligned}$$

Thanks to Lemma 2.3, using the trick used in [8] yields

$$\begin{aligned}
a(u_h^n, \xi^n, I_h^* \partial_t \xi^n) \geq\ & \frac{1}{2\triangle t}[a(u_h^n, \xi^n, I_h^* \xi^n) - a(u_h^n, \xi^{n-1}, I_h^* \xi^{n-1})] \\
& - \frac{1}{2\triangle t}[a(u_h^n, \partial_t \xi^n, I_h^* \xi^n) - a(u_h^n, \xi^n, I_h^* \partial_t \xi^n)]. \qquad (19)
\end{aligned}$$

Combining Equation (18) with Equation (19), testing Equation (18) against $\triangle t$ and summing over n from 1 to l $(1 \leq l \leq N)$, with the help of Equation (5) and Lemma 2.3, we have

$$\begin{aligned}
& C \sum_{n=1}^{l} ||\partial_t \xi^n||_0^2 \triangle t + \frac{\alpha}{2} ||\xi^l||_1^2 \\
& \leq \sum_{n=1}^{l} (\partial_t u^n - u_t^n, I_h^* \partial_t \xi^n) \triangle t - \sum_{n=1}^{l} (\partial_t \eta^n, I_h^* \partial_t \xi^n) \triangle t \\
& \quad - \sum_{n=1}^{l} a(u^n - u_h^n, P_h u^n, I_h^* \partial_t \xi^n) \triangle t + \frac{1}{2} \sum_{n=1}^{l} [a(u_h^n, \partial_t \xi^n, I_h^* \xi^n) - a(u_h^n, \xi^n, I_h^* \partial_t \xi^n)] \triangle t \\
& \quad - \sum_{n=1}^{l} ((\mathbf{b}(u^n) - \mathbf{b}(u_h^n))\nabla u^n, I_h^* \partial_t \xi^n) \triangle t - \sum_{n=1}^{l} (\mathbf{b}(u_h^n)\nabla(u^n - u_h^n), I_h^* \partial_t \xi^n) \triangle t \\
& \quad + \sum_{n=1}^{l} (f(u^n) - f(u_h^n), I_h^* \partial_t \xi^n) \triangle t = \sum_{i=1}^{7} E_i. \qquad (20)
\end{aligned}$$

Now, we estimate the terms of the right-hand side of Equation (20), for E_1, E_2, similar estimates of $E_i (i = 1, 2)$ in Theorem 4.1, we have

$$|E_1| \leq C(\epsilon) \left(\int_0^{t^l} ||u_{tt}^n||_0^2 \,\mathrm{d}t \right) (\triangle t)^2 + \epsilon \sum_{n=1}^{l} ||\partial_t \xi^n||_0^2 \triangle t.$$

$$|E_2| \leq C(\epsilon) h^4 \left(\int_0^{t^l} (||u_t^n||_3 + \|u^n\|_3)^2 \,\mathrm{d}t \right) + \epsilon \sum_{n=1}^{l} ||\partial_t \xi^n||_0^2 \triangle t.$$

For E_3 and E_4, by Lemmas 2.3, 2.5, 2.6, Theorem 4.1, inverse and Cauchy inequalities, we deduce that

$$\begin{aligned}
|E_3| & \leq C \sum_{n=1}^{l} (h^2 ||u^n||_2 + ||u^n - u_h^n||_0) ||P_h u^n||_{1,\infty} ||\partial_t \xi^n||_1 \triangle t \\
& \leq C(h^2 ||u^n||_2 + h^{3/2} + \triangle t) \sum_{n=1}^{l} ||P_h u^n||_{1,\infty} ||\partial_t \xi^n||_1 \triangle t
\end{aligned}$$

$$\le C(h^{1/2}+h^{-1}\triangle t)\sum_{n=1}^{l}||P_hu^n||_{1,\infty}||\partial_t\xi^n||_0\triangle t$$

$$\le C(\epsilon)(h^{1/2}+h^{-1}\triangle t)^2\sum_{n=1}^{l}||P_hu^n||^2_{1,\infty}\triangle t+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t.$$

$$|E_4|\le\frac{Ch}{2}\sum_{n=1}^{l}||\xi^n||_1||\partial_t\xi^n||_1\triangle t\le C(\epsilon)\sum_{n=1}^{l}||\xi^n||_1^2\triangle t+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t.$$

For E_5 and E_6, Theorem 4.1 and conditions (C_1) and (C_2) yield

$$\begin{aligned}|E_5|&=\left|\sum_{n=1}^{l}((\mathbf{b}(u^n)-\mathbf{b}(u_h^n))\nabla u^n,I_h^*\partial_t\xi^n)\triangle t\right|\\
&\le\sum_{n=1}^{l}||\mathbf{b}(u^n)-\mathbf{b}(u_h^n)||_0||\nabla u^n||_{L^\infty}||\partial_t\xi^n||_0\triangle t\\
&\le\sum_{n=1}^{l}L||u^n-u_h^n||_0||\nabla u^n||_{L^\infty}||\partial_t\xi^n||_0\triangle t\\
&\le C(h^{3/2}+\triangle t)\sum_{n=1}^{l}||u^n||_{1,\infty}||\partial_t\xi^n||_0\triangle t\\
&\le C(\epsilon)(h^3+\triangle t^2)\sum_{n=1}^{l}||u^n||^2_{1,\infty}\triangle t+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t.\\
|E_6|&=|\sum_{n=1}^{l}(\mathbf{b}(u_h^n)\nabla(u^n-u_h^n),I_h^*\partial_t\xi^n)\triangle t|\\
&\le\sum_{n=1}^{l}||\mathbf{b}(u_h^n)||_{L^\infty}||\nabla(u^n-u_h^n)||_0||I_h^*\partial_t\xi^n||_0\triangle t\\
&\le b^*\sum_{n=1}^{l}||\nabla(\xi^n+\eta^n)||_0||I_h^*\partial_t\xi^n||_0\triangle t\\
&\le b^*\sum_{n=1}^{l}||\xi^n||_1||\partial_t\xi^n||_0\triangle t+b^*\sum_{n=1}^{l}||u^n-P_hu^n||_1||\partial_t\xi^n||_0\triangle t\\
&\le\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t+C_1(\epsilon)\sum_{n=1}^{l}||\xi^n||_1^2\triangle t+C_2(\epsilon)h^2\sum_{n=1}^{l}||u^n||_2^2\triangle t.\end{aligned}$$

For E_7, thanks to Theorem 4.1, condition (C_1) and Young inequality, we have

$$|E_7|\le C(\epsilon)\sum_{n=1}^{l}(||u^n-u_h^n||_0^2)\triangle t+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t\le C(\epsilon)T(h^3+\triangle t^2)+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t.$$

Combining the above inequalities with Equation (20), one gets

$$\begin{aligned}
C_1 \sum_{n=1}^{l} ||\partial_t \xi^n||_0^2 \triangle t + ||\xi^l||_1^2 \le C_2 h^2 &\left[\int_0^{t^l} (||u_t||_3 + \|u\|_3)^2 \, \mathrm{d}t + \sum_{n=1}^{l} ||u^n||_2^2 \triangle t + T \right] \\
&+ C_3(\epsilon)(\triangle t)^2 \left(\int_0^{t^l} ||u_{tt}||_0^2 \, \mathrm{d}t + T \right) + C_4(\epsilon) \\
&\times \sum_{n=1}^{l} ||\xi^n||_1^2 \triangle t + C_5(\epsilon) \sum_{n=1}^{l} (h + h^{-2} \triangle t^2) \triangle t + \epsilon \sum_{n=1}^{l} ||\partial_t \xi^n||_0^2 \triangle t.
\end{aligned} \tag{21}$$

Under the condition $\triangle t = \mathcal{O}(h^{3/2})$, choosing proper ϵ and kicking the last term to the left side of Equation (21), applying Lemma 2.1, we have

$$\begin{aligned}
C_1 \sum_{n=1}^{l} ||\partial_t \xi^n||_0^2 \triangle t + ||\xi^l||_1^2 \le C_2 h^2 &\left[\int_0^{t^l} (||u_t||_3 + \|u\|_3)^2 \, \mathrm{d}t + \sum_{n=1}^{l} ||u^n||_2^2 \triangle t \right] \\
&+ C_1(\epsilon)(\triangle t)^2 \int_0^{t^l} ||u_{tt}||_0^2 \, \mathrm{d}t + C_2(\epsilon) T h.
\end{aligned}$$

Combining triangular inequality and Lemma 2.6, we complete the proof. ■

5. Error estimate of the two-level finite-volume algorithm

This section is devoted to the convergence analysis of the approximate solution in the H^1-norm for the two-level finite-volume algorithm.

THEOREM 5.1 *Under the conditions* (C_1)–(C_4) *and* $u_h^0 = P_h u_0$, u_h^n *be the solution of the two-grid finite-volume algorithm* (13)*, for* $0 \le t^n < T$*, we have*

$$||u - u_h^n||_1 \le C[h^{1/2} + H^{3/4}(1 + (H^{3/2} h^{-1})^{1/2}) + \triangle t], \tag{22}$$

where C *is a constant, independent of* h *and* $\triangle t$.

Proof We get the following error equation by choosing $v = \partial_t \xi^n$ in Equation (3) and $v_h = I_h^* \partial_t \xi^n$ in Equation (13), respectively:

$$\begin{aligned}
&(\partial_t \xi^n, I_h^* \partial_t \xi^n) + a(u_H^n, \xi^n, I_h^* \partial_t \xi^n) \\
&\quad = (\partial_t u^n - u_t^n, I_h^* \partial_t \xi^n) - (\partial_t \eta^n, I_h^* \partial_t \xi^n) - ((\mathbf{b}(u^n) - \mathbf{b}(u_H^n)) \nabla u^n, I_h^* \partial_t \xi^n) \\
&\qquad - (\mathbf{b}(u_H^n) \nabla (u^n - u_h^n), I_h^* \partial_t \xi^n) - a(u^n - u_H^n, P_h u^n, I_h^* \partial_t \xi^n) \\
&\qquad + (f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n), I_h^* \partial_t \xi^n).
\end{aligned} \tag{23}$$

Using the proof as Equation (19), we have

$$\begin{aligned}
&(\partial_t \xi^n, I_h^* \partial_t \xi^n) + \frac{1}{2\triangle t} (a(u_H^n, \xi^n, I_h^* \xi^n) - a(u_H^n, \xi^{n-1}, I_h^* \xi^{n-1})) \\
&\quad \le (\partial_t u^n - u_t^n, I_h^* \partial_t \xi^n) - (\partial_t \eta^n, I_h^* \partial_t \xi^n) + \frac{1}{2\triangle t} [a(u_H^n, \partial_t \xi^n, I_h^* \xi^n) - a(u_H^n, \xi^n, I_h^* \partial_t \xi^n)]
\end{aligned}$$

$$
\begin{aligned}
&-((\mathbf{b}(u^n)-\mathbf{b}(u_H^n))\nabla u^n, I_h^*\partial_t\xi^n)-(\mathbf{b}(u_H^n)\nabla(u^n-u_h^n), I_h^*\partial_t\xi^n)\\
&+(f(u^n)-f(u_H^n)-f'(u_H^n)(u_h^n-u_H^n), I_h^*\partial_t\xi^n)\\
&-a(u^n-u_H^n, P_hu^n, I_h^*\partial_t\xi^n)=\sum_{i=1}^{7}E_i.
\end{aligned}\tag{24}
$$

Multiplying (24) with $\triangle t$ and summing from $n=1$ to $n=l,\ 1\le l\le N$. For E_1–E_3 and E_5, under the condition of (C_2), by Lemma 2.3, 2.5, 2.6 and the Cauchy inequality, we can estimate them as in Theorem 4.2. For E_4 and E_7, under the conditions of (C_1), we have

$$
\begin{aligned}
|E_4| &= |\sum_{n=1}^{l}((\mathbf{b}(u^n)-\mathbf{b}(u_H^n))\nabla u^n, I_h^*\partial_t\xi^n)\triangle t|\\
&\le \sum_{n=1}^{l} L||u^n-u_H^n||_{L^2}||\nabla u^n||_{L^\infty}||I_h^*\partial_t\xi^n||_{L^2}\triangle t\\
&\le C(\epsilon)\sum_{n=1}^{l}||u^n-u_H^n||_0^2||u^n||_{1,\infty}^2\triangle t+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t.\\
|E_7| &\le \sum_{n=1}^{l} C(H^2||u^n||_2+||u^n-u_H^n||_0)||P_hu^n||_{1,\infty}||\partial_t\xi^n||_1\triangle t\\
&\le \sum_{n=1}^{l} C(H^2||u^n||_2+||u^n-u_H^n||_0)||P_hu^n||_{1,\infty}||\partial_t\xi^n||_0h^{1/2}\\
&\le C(H^{3/2}+\triangle t)\sum_{n=1}^{l}||\partial_t\xi^n||_0h^{1/2}\le C(\epsilon)(H^{3/2}+\triangle t)^2\frac{h}{\triangle t}+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t\\
&\le C(\epsilon)(H^{3/2}+h)(h^{-1}H^{3/2}+1)+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t.
\end{aligned}
$$

For T_6, thanks to the condition (C_3), using the proofs presented in [8] and Lemma 2.6, we have

$$
\begin{aligned}
&(f(u^n)-f(u_H^n)-f'(u_H^n)(u_h^n-u_H^n), I_h^*\partial_t\xi^n)\triangle t\\
&\quad\le C(\epsilon)(||\xi^n||_0^2+||\eta^n||_0^2)\triangle t+C(\epsilon)(H^{3/2}+\triangle t)^2\triangle t+\epsilon||\partial_t\xi^n||_0^2\triangle t.
\end{aligned}
$$

With the above estimates and Equation (24), we arrive at

$$
\begin{aligned}
C\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t+||\xi^l||_1^2 &\le Ch^2\left[\int_0^{t^l}(||u_t||_3+||u||_3)^2\,\mathrm{d}t+\sum_{n=1}^{l}||u^n||_2^2\triangle t\right]+C(\epsilon)(\triangle t)^2\\
&\quad\times\int_0^{t^l}||u_{tt}||_0^2\,\mathrm{d}t+C(\epsilon)\sum_{n=1}^{l}||\xi^n||_1^2\triangle t+\epsilon\sum_{n=1}^{l}||\partial_t\xi^n||_0^2\triangle t\\
&\quad+C(\epsilon)\sum_{n=1}^{l}(H^{3/2}+\triangle t)^2\triangle t+C(\epsilon)(H^{3/2}+h)(h^{-1}H^{3/2}+1).
\end{aligned}\tag{25}
$$

Applying the Lemma 2.1, we get that

$$||\xi^l||_1 \leq C[h^{1/2} + H^{3/4}(1 + (H^{3/2}h^{-1})^{1/2}) + \triangle t],$$

where C is a constant, dependent on $||u||_{L^\infty(H^3(\Omega))}$, $||u_t||_{L^2(H^3(\Omega))}$, $||u_{tt}||_{L^2(L^2(\Omega))}$, but independent of h and $\triangle t$. We finish the proof by combining the triangular inequality with Lemma 2.6. ■

Remark 5.1 Comparing with the H^1-norm error estimates between the standard FVM and the two-level method for the nonlinear parabolic problem (1), the theoretical rate of convergence for u are

$$\begin{aligned} ||u(t) - u_{\text{one}}(t)||_1 &\leq C(h^{1/2} + \triangle t), \\ ||u(t) - u_{\text{two}}(t)||_1 &\leq C\left[h^{1/2} + H^{3/4}(1 + (H^{3/2}h^{-1})^{1/2}) + \triangle t\right], \end{aligned} \tag{26}$$

where $u_{\text{one}}(t)$ and $u_{\text{two}}(t)$ are the approximation solutions which were obtained by using the standard FVM and the two-level method, respectively. From the expression of Equation (26), one should choose $h = \mathcal{O}(H^{3/2})$, then Equation (22) can be rewritten as

$$||u(t_n) - u_{\text{two}}(t_n)||_1 \leq C(h^{1/2} + H^{3/4} + \triangle t) \approx C(h^{1/2} + \triangle t). \tag{27}$$

From Equation (27), we can see that the two-level method has the same order of approximation as that of the standard method. However, the nonlinear problem is only treated on the coarse grid space in the two-level algorithm, in this way, a large amount of computational cost can be saved.

6. Numerical validation

In order to gain insights on the theoretical results established in previous sections, we present some numerical experiments in this section. Our main interest is to verify the performances of the two-level finite-volume algorithm. In all experiments, $\Omega \times [0, T] = [0, 1]^2 \times [0, 1]$. The mesh consists of triangular elements and the backward Euler scheme is used for the time discretization. In order to show the prominent features of the two-level method, we compare this method with the standard FVM for nonlinear parabolic equations. In each time iterative interval $[t_{m-1}, t_m]$, the stopping criterion

$$\left[\sum_{i=1}^{(N+1)^2} (u_{h,i}^m - u_{h,i}^{m-1})^2\right]^{1/2} \leq 10^{-4}$$

is employed, where N is the number of nodes in each orientation, m is the step of the iterative and initial value $u_h^0 = u_h(0)$. Let $u_h^{\text{one}}(t_n)$ and $u_h^{\text{two}}(t_n)$ be the numerical solutions which were obtained by using the standard FVM and the two-level method at t_n. The experimental rates of convergence with respect to the mesh size h are calculated by the formula $\log(E_i/E_{i+1})/\log(h_i/h_{i+1})$, where E_i and E_{i+1} are the relative errors corresponding to the mesh of sizes h_i and h_{i+1}, respectively.

Firstly, We choose the coefficients of nonlinear problem (1) are $a(u) = 1$, $b_1(u) = b_2(u) = u$. The initial-boundary values and $f(u)$ are determined by the exact solution $u = \mathrm{e}^{-t}x(1 - x)y(1 - y)$. The CPU time, H^1-norm relative errors and convergence rates at $t = t_n$ are listed in Tables 1 and 2 for the standard FVM and two-level method with some h, $\triangle t$ and H values.

From Tables 1 and 2, we can see that the numerical results coincide with the theoretical analysis, and the two-level method spends less time than the standard method, that is to say, our algorithm is effective for saving a large amount of computational time and still keeping good precise.

Table 1. CPU(s), relative errors and convergence rates on [0,T] for standard FVM.

$\frac{1}{h}$	$\frac{1}{\triangle t}$	CPU (s)	$\frac{\|u(t_n) - u_h^{\text{one}}(t_n)\|_1}{\|u(t_n)\|_1}$	u_{H^1} rate
8	$\sqrt[2]{8}$	0.156	0.252386	–
16	4	0.953	0.134916	0.936
27	$\sqrt[2]{27}$	4.453	0.0898215	0.7775
36	6	9.969	0.0751895	0.6181
49	7	27.937	0.0649899	0.4728

Table 2. CPU(s), relative errors and convergence rates on [0,T] for two-level FVM.

$\frac{1}{h}$	$\frac{1}{H}$	$\frac{1}{\triangle t}$	CPU (s)	$\frac{\|u(t_n) - u_h^{\text{one}}(t_n)\|_1}{\|u(t_n)\|_1}$	u_{H^1} rate
8	4	$\sqrt[2]{8}$	0.063	0.249555	–
16	7	4	0.437	0.129116	1.0670
27	9	$\sqrt[2]{27}$	1.141	0.0870843	0.7527
36	11	6	2.469	0.0724002	0.6419
49	14	7	6.672	0.0611606	0.5472

Table 3. CPU(s), relative errors and convergence rates on [0,T] for standard FVM.

$\frac{1}{h}$	$\frac{1}{\triangle t}$	CPU (s)	$\frac{\|u(t_n) - u_h^{\text{one}}(t_n)\|_1}{\|u(t_n)\|_1}$	u_{H^1} rate
8	$\sqrt[2]{8}$	0.788	0.338136	–
16	4	4.694	0.170966	0.9839
27	$\sqrt[2]{27}$	16.252	0.112245	0.8044
36	6	33.431	0.092745	0.6633
49	7	71.887	0.0794112	0.5035

Table 4. CPU(s), relative errors and convergence rates on [0,T] for two-level method.

$\frac{1}{h}$	$\frac{1}{H}$	$\frac{1}{\triangle t}$	CPU (s)	$\frac{\|u(t_n) - u_h^{\text{one}}(t_n)\|_1}{\|u(t_n)\|_1}$	u_{H^1} rate
8	4	$\sqrt[2]{8}$	0.163	0.409151	–
16	7	4	0.940	0.191356	1.0964
27	9	$\sqrt[2]{27}$	1.882	0.129294	0.7493
36	11	6	3.754	0.1048564	0.7282
49	14	7	9.411	0.0862658	0.6330

On the other hand, we present some numerical results by setting the exact solution $u = \mathrm{e}^{-t}x(1-x)y(1-y)$ and the coefficients are $a(u) = u$, $b_1(u) = b_2(u) = u$. We compare the CPU time, H^1-norm relative errors and convergence rates at $t = t_n$ between the standard FVM and the two-level method with the some parameter values.

From Tables 3 and 4, we can see that although the H^1-norm relative errors between two methods are closed, the two-level method spends less time than the standard method. That is to say, our algorithm is effective to solve the nonlinear parabolic problem.

7. Conclusion

In this work, we have provided the convergence analysis for nonlinear parabolic equations (1) by using the two-level FVM. The analysis has extended the work in [8] to a more interesting and meaningful case. In fact, the nonlinear convection and diffusion terms are contained in many fluid dynamics problems, such as the nonlinear convection–diffusion-reaction equations, conduction–convection problem and Navier–Stokes equations, and how to deal with them effectively is a research focus. Here, we have provided a generalized theoretical analysis for the nonlinear parabolic problem, which contains those nonlinear terms, by using a two-level FVM. In this sense, we have pushed forward the existing knowledge. Numerical tests have revealed that the two-level FVM is highly efficient for the nonlinear parabolic problem.

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