Unified First Law and Thermodynamics of Dynamical Black Hole in n-dimensional Vaidya Spacetime

REN Ji-Rong, LI Ran *
Institute of Theoretical Physics, Lanzhou University, Lanzhou, 730000, Gansu, China

As a simple but important example of dynamical black hole, we analysis the dynamical black hole in n-dimensional Vaidya spacetime in detail. We investigated the thermodynamics of field equation in n-dimensional Vaidya spacetime. The unified first law was derived in terms of the methods proposed by Sean A Hayward. The first law of dynamical black hole was obtained by projecting the unified first law along the trapping horizon. At last, the second law of dynamical black hole is also discussed.

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Recently, a remarkable development in black hole theory is the proposal of dynamical black hole[1]. As opposed to the textbook theory of black holes, which mostly concerns either stationary space-times or event horizons, dynamical black hole can be defined locally without the knowledge of the whole space-time, which will be bring important applications to numerical relativity.

The vaidya metric has drawn a lot of attention as a simplified example of dynamical black hole[2]. The main purpose of this paper is to investigate the unified first law and thermodynamics of dynamical black hole in n-dimensional Vaidya spacetime. We have found the unified first law for n-dimensional Vaidya spacetime where the work term vanishes naturally because the stress energy describes the pressureless fluid. The first and second law of thermodynamics is also discussed from which the entropy is given by $S = A/4$. This coincides with the previous result[3]. The method used in this letter has been developed to discuss the thermodynamics of apparent horizon in FRW universe in different gravity model[4].

The key geometrical objects are dynamical horizons, which are hypersurfaces H (in space-time) foliated by marginal surfaces. A marginal surface is a spatial surface on which one null expansion vanishes, where the null expansions $\theta_\pm$ may be defined by

$$\theta_\pm = \frac{L_\pm \delta A}{\delta A} \quad , \tag{1}$$

where δA denotes the area element and $L_\pm$ the Lie derivatives along future-pointing null normal vectors $l_\pm$. This is a surface where outgoing or ingoing light rays are just trapped, neither converging nor diverging.

When the marginal surface has the topology of $(n-2)$-sphere, the null expansions can be explicitly expressed as

$$\theta_\pm = (n-2)\frac{\partial_\pm r}{r} \quad , \tag{2}$$

where r is used to label the marginal surface. H is generated by the vector $\partial/\partial r$, normal to the marginal surfaces.

For the author's local definition[1] of black hole as a future ($\theta_- < 0$ on H, for $\theta_+ = 0$) outer ($L_-\theta_+ < 0$) trapping horizon. If Einstein gravity and dominant energy condition are assumed, it is shown that the area of marginal surface does'nt decrease which is similar to Hawking's so-called second law for event horizons, but for a physically locatable horizon. The more detailed discussions can be found in Ref[1].

Now, we will derive the unified first law for Vaidya spacetime to make a clear identification to the method proposed by Sean A Hayward. The definition of some quantities strictly follows Ref[5].

The $n-$dimensional ($n > 3$) Vaidya spacetime[6] is described by the metric

$$ds^2 = -(1 - \frac{2m(v)}{(n-3)r^{n-3}})dv^2 + 2dvdr + r^2 d\Omega_{n-2}^2 \quad . \tag{3}$$

where $d\Omega_{n-2}^2$ stands for the metric of the unity $(n-2)$-dimensional sphere.

The stress energy tensor is determined by the derivative of $m(v)$

$$T_{\mu\nu} = \frac{1}{8\pi}\frac{(n-2)}{(n-3)r^{n-2}}\frac{dm(v)}{dv}l_\mu l_\nu \quad , \tag{4}$$

where $l_\mu = -\partial_\mu v$ is the tangent vector of ingoing null geodesics. The mass function $m(v)$ must increase to satisfy the energy condition.

This is an astrophysically unrealistic toy model, but it does serve as a very useful testing ground for dynamical black hole in numerical relativity[2]. It has been extensively used, for example, to study the formation of naked singularities[6].

The dual-null coordinate can be introduced as

$$d\xi^+ = \frac{1}{2}dv \tag{5}$$

$$d\xi^- = (1 - \frac{2m(v)}{(n-3)r^{n-3}})dv - 2dr \tag{6}$$

Then, the Vaidya metric can be put into the dual-null form

$$ds^2 = -2d\xi^+ d\xi^- + r^2 d\Omega_{n-2}^2 \quad . \tag{7}$$

*Corresponding author. Electronic mail: liran05@lzu.cn

It is easy to find the relation

$$\frac{\partial}{\partial\xi^+} = 2\frac{\partial}{\partial v} + (1 - \frac{2m(v)}{(n-3)r^{n-3}})\frac{\partial}{\partial r} \ , \qquad (8)$$

$$\frac{\partial}{\partial\xi^-} = -\frac{1}{2}\frac{\partial}{\partial r} \ . \qquad (9)$$

The null expansions is given by

$$\theta_+ = (n-2)(\frac{1}{r} - \frac{2m(v)}{(n-3)r^{n-2}}) \ , \qquad (10)$$

$$\theta_- = -\frac{(n-2)}{2r} \ , \qquad (11)$$

from which one can see that the dynamical horizon is located at $r_{\rm DH} = (2m(v)/(n-3))^{1/(n-3)}$, i.e. the null expansion $\theta_+ = 0$.

One can also find

$$\partial_-\theta^+ = -\frac{(n-2)}{2r^2}(\frac{(n-2)}{(n-3)}\frac{2m(v)}{r^{n-3}} - 1) \ . \qquad (12)$$

So, on the dynamical horizon $r_{\rm DH}$, one have

$$\partial_-\theta^+ = -\frac{(n-2)(n-3)}{2r_{\rm DH}} < 0 \ . \qquad (13)$$

The dynamical horizon can be expressed as

$$\Phi = r - r_{\rm DH} = 0 \ , \qquad (14)$$

from which one can calculate that

$$g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = -4(\frac{1}{n-3})^{(\frac{n-2}{n-3})} \times(2m(v))^{-(\frac{n-4}{n-3})}\frac{dm(v)}{dv} \ . \qquad (15)$$

So the dynamical horizon here is a spacelike hypersurface since $dm/dv > 0$(so that the energy condition is satisfied). In vaidya spacetime, the dynamical horizon is not consistent with the event horizon because event horizon must be a null hypersurface from the definition of event horizon.

The Misner-Sharp energy[7−9] is defined as

$$E = \frac{(n-1)(n-2)V_{n-1}}{16\pi}r^{n-3}(1 - \nabla^a r\nabla_a r) \ , \qquad (16)$$

where a is the index of transverse space orthogonal to the tangent space of marginal surface, i.e. $a = \{+,-\}$. $V_{n-1} = \pi^{(\frac{n-1}{2})}/\Gamma(\frac{n+1}{2})$ denotes the volume of the $(n-1)$ dimensional unit ball.

The Misner-Sharp energy at $r = r_{\rm DH}$ is just the mass of dynamical black hole in Vaidya spacetime. Various properties of the Misner-Sharp energy in the spherically symmetric spacetime is established by Sean A. Hayward in Ref[8]. This energy is the total energy inside the sphere with radius r. The important advantage of Misner-Sharp energy which has the relation to the field equation will be shown below.

In the dual-null coordinate, the non-vanishing component of stress energy tensor is

$$T_{++} = \frac{1}{2\pi}\frac{(n-2)}{(n-3)r^{n-2}}\frac{dm(v)}{dv} \ . \qquad (17)$$

According to the dual-null form of Vaidya metric, Einstein equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ can be written as

$$\partial_+\partial_+ r = -\frac{4}{(n-3)}\frac{1}{r^{n-3}}\frac{dm}{dv} \ , \qquad (18)$$

$$\partial_-\partial_- r = 0 \ , \qquad (19)$$

$$(n-3)\partial_+ r\partial_- r + r\partial_+\partial_- r + \frac{1}{2}(n-3) = 0 \ . \qquad (20)$$

Energy density ω is defined as

$$\omega = -\frac{1}{2}{\rm Tr}\, T \ , \qquad (21)$$

where the trace is performed also in the transverse space orthogonal to the tangent space of marginal surface. It is easy to calculate that ω vanishes in the Vaidya spacetime.

Energy flux or momentum density ψ is defined as

$$\psi = T\nabla r + \omega\nabla r \ . \qquad (22)$$

A direct calculation shows that

$$\psi_+ = -\frac{1}{2\pi}\frac{(n-2)}{(n-3)r^{n-2}}\frac{dm}{dv}\partial_- r \ , \qquad (23)$$

$$\psi_- = 0 \ . \qquad (24)$$

From the definition of Misner-Sharp energy, using Einstein equation and the definition of energy flux, one can show that

$$\partial_+ E = A\psi_+ \ , \qquad (25)$$

$$\partial_- E = 0 \ , \qquad (26)$$

where $A = A_{n-2}r^{n-2}$ is the area of marginal $(n-2)$-sphere with r being the radius and $A_{n-2} = (n-1)V_{n-1}$ being the area of the $(n-2)$-dimensional sphere.

The above two equations can be uniformly written as

$$dE = A\psi \ . \qquad (27)$$

This is just the unified first law in Vaidya spacetime. Because the stress energy $T_{\mu\nu}$ describes the null dust, a pressureless fluid with energy density $\frac{1}{8\pi}\frac{(n-2)}{(n-3)r^{n-2}}\frac{dm(v)}{dv}$ and velocity l_μ, the work term vanishes naturally and only the energy supply term is remained.

Now, we turn to the discussion of the first law of dynamical black hole in Vaudya spacetime. The dynamical surface gravity is defined as

$$\kappa = \frac{1}{2}\nabla^a\nabla_a r \ . \qquad (28)$$

For Vaidya spacetime, using the Einstein equation one can obtain

$$\kappa = \frac{1}{2r_{\rm DH}} \ . \tag{29}$$

Introduce a vector z that is tangent to the dynamical horizon. In the dual-null coordinate, z can be expressed as

$$z = z^{+}\partial_{+} + z^{-}\partial_{-} \ . \tag{30}$$

Since $\partial_{+}r$ always vanishes on the dynamical horizon, the projection of $\partial_{+}r$ to the dynamical horizon also vanishes. So, we have

$$z^{+}\partial_{+}\partial_{+}r + z^{-}\partial_{-}\partial_{+}r = 0 \ . \tag{31}$$

One can obtain the first law of dynamical black hole through projecting the unified first law to the dynamical horizon. The key procedure is to prove the relation

$$\langle A\psi, z\rangle = \frac{\kappa}{8\pi}\langle dA, z\rangle \ . \tag{32}$$

where $\langle \ , \ \rangle$ denotes the inner product.

In our situation, this relation is indeed easy to prove. From the above relation, if we identify the temperature T of dynamical black hole with $T = \frac{\kappa}{2\pi}$ and the left hand side of above equation with δQ, one can get the Clausius relation

$$\delta Q = TdS \ , \tag{33}$$

where $S = \frac{A}{4}$ is the entropy of dynamical black hole with $A = A_{n-2}r_{\rm DH}^{n-2}$ being the area of dynamical horizon.

Ted Jacobson[10] have been able to derive the Einstein equation from the proportionality of entropy and horizon area together with the Clausius relation $\delta Q = TdS$ connecting heat, entropy, and temperature. Recently, further research[11] dedicates that it is impossible to derive the field equation of $f(R)$ gravity from the Clausius relation $\delta Q = TdS$ in terms of the viewpoint of equilibrium thermodynamics. This problem can be canceled by adding a entropy production term d_iS to the Clausius relation which is ultimately changed into $dS = \frac{\delta Q}{T} + d_iS$. Now, we have found the thermodynamics intrinsic in the Einstein equation. This implies that there will be deep relationship between the thermodynamics and gravity theory.

By projecting the unified first law to the horizon, the first law of dynamical black hole thermodynamics can be obtained

$$\langle dE, z\rangle = \frac{\kappa}{8\pi}\langle dA, z\rangle \ . \tag{34}$$

This equation is straightforward from the equation (27) and (32).

At last, we briefly discuss the second law of dynamical black hole in Vaidya spacetime. As have been discussed, the area of dynamical horizon is $A = A_{n-2}(\frac{2m(v)}{n-3})^{\frac{n-2}{n-3}}$. Since the mass $m(v)$ function increase to satisfy the energy condition, the area A of dynamical black hole can not decrease, i.e. the entropy of dynamical black hole must increase. This is just the second law of dynamical black hole thermodynamics.

In conclusion, we have discussed the thermodynamics of dynamical black hole in $n-$dimensional Vaidya spacetime. In this paper, Einstein' gravity is assumed. If the unified first law is also valid for other gravity theory will be studied in the future.

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